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Invited Paper

ON QUASI EINSTEIN AND GENERALIZED QUASI EINSTEIN MANIFOLDS

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Abstract. *Recently, Prof. M. C. Chaki introduced the notion of a quasi Einstein manifold [1], denoted by* $(QE)_n$ *, whose Ricci tensor S of type (0,2) is not identically zero and satisfies the condition :*

$$S(X,Y) = a\, g(X,Y) + b\, A(X)A(Y)$$

where a, b are scalars of which $b \neq 0$ *and A is a non-zero 1-form* such *that*

$$g(X,U) = A(X)$$

for all vector fields X, U being a unit vector field.

If the existence of a 4-dimensional Lorentz manifold is established whose Ricci tensor is of the form given above, then it is found that such a space-time represents a perfect fluid space-time in cosmology.

Investigations by Karcher [2] and others have revealed that a conformally flat perfect fluid space-time has the geometric structure of quasi-constant curvature. It is found that a manifold of quasi-constant curvature is a natural sub-class of quasi Einstein manifold. Investigations on quasi Einstein manifolds help us to have a deeper understanding of the global character of the universe [3] including the topology. Consequently, we can study the nature of the singularities defined from a differential geometric standpoint.

In a subsequent paper [4], Prof. Chaki introduced the generalized quasi Einstein manifolds denoted by $G(QE)_n$*. Chen and Yano [5] had introduced the notion of a manifold of quasi-constant curvature denoted by* $(QC)_n$*. a generalization of a manifold of quasi-constant curvature, called a manifold of generalized quasi-constant curvature, denoted by* $G(QC)_n$*, has been done by Prof. Chaki [4]. This is necessary for the study of* $G(QE)_n$*. It is found that every* $G(QC)_n$ $(n \geq 3)$ *is a* $G(QE)_n$*, while every* $G(QC)_n$ $(n>3)$ *is a conformally flat* $G(QE)_n$*. The importance of a* $G(QE)_n$ *lies in the fact that such a 4-dimensional semi-Riemannian manifold is relevant to the study of a general relativistic fluid space-time admitting heat flux [6]. The global properties of such a space-time is under investigation. Study of space-times admitting fluid viscosity and electromagnetic fields require further generalization of the Ricci tensor and is under process.*

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INTRODUCTION

Recently, Prof. M. C. Chaki along with R. K. Maity [1], introduced the notion of a quasi Einstein manifold denoted by $(QE)_n$, whose Ricci tensor S of type (0,2) is not identically zero and satisfies the condition:

$$S(X,Y) = a\, g(X,Y) + b\, A(X)A(Y)$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X)$$

for all vector fields X, U being a unit vector field. The eigen values corresponding to the Ricci tensor S were obtained and the eigen vectors corresponding to the eigen values were identified. A necessary and suffient condition was obtained in order that the relation $R(X,Y).S = 0$ might hold. The authors also considered two sufficient conditions in order that a $(QE)_n$ $(n>3)$ might be conformally conservative.

Investigations in a subsequent paper [2] revealed that a conformally flat $(QE)n$ has the geometric structure of quasi-constant curvature. It was found that a manifold of quasi-constant curvature is a natural sub-class of quasi Einstein manifold. It was further shown that if the generator of a conformally flat $(QE)_n$ $(n>3)$ is an irrotational vector field, then the $(QE)_n$ is a product manifold of quasi constant curvature. Some results relating to sectional curvatures of this kind of product manifold were also obtained. Investigations on $(QE)_n$ help us to have a deeper understanding of the global character of $(QE)_n$ [3] including the topology.

In a subsequent paper [4], Prof. Chaki introduced the generalized quasi Einstein manifolds denoted by $G(QE)_n$. Chen and Yano [5] had introduced the notion of a manifold of quasi-constant curvature denoted by $(QC)_n$. A generalization of a manifold of quasi-constant curvature, called a manifold of generalized quasi-constant curvature, denoted by $G(QC)_n$, has been done by Prof. Chaki [4]. This was necessary for the study of $G(QE)_n$. It was found that every $G(QC)_n$ $(n\geq 3)$ is a $G(QE)_n$, while every $G(QC)_n$ $(n>3)$ is a conformally flat $G(QE)_n$.

The study of $(QE)_n$ and $G(QE)_n$ becomes meaningful due to its application in the general theory of relativity and cosmology.

It is found that a perfect fluid space-time of general relativity is a four-dimensional semi-Riemannian quasi Einstein manifold whose associated scalars are $[(r/2) + p]$ and $(\rho + p)$ respectively where ρ and p are the energy density and the isotropic pressure of the fluid and r is the scalar curvature, the generator of the manifold being the unit timelike velocity vector field of the fluid.

The importance of a $G(QE)_n$ lies in the fact that such a 4-dimensional semi-Riemannian manifold is relevant to the study of a general relativistic fluid space-time admitting heat flux [6]. The global properties of such a space-time is under investigation.

1. QUASI EINSTEIN MANIFOLDS

The notion of a quasi Einstein manifold was introduced and studied by M. C. Chaki and R. K. Maity [1] in 2000. A non-flat Riemannian manifold $(M^n,g)(n>2)$ was defined to be a quasi Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y); X,Y,.....\in \chi(M) \tag{I.1}$$

where a, b are scalars of which $b\neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X)\forall X, \tag{I.2}$$

X being a vector field and U being a unit vector field.

In such a case, a and b were called the associated scalars, A was called the associated 1-form and U was called the generator of the manifold. An n-dimensional manifold of this kind was denoted by the symbol $(QE)_n$. It is obvious that if $b = 0$, then this $(QE)_n$ reduces to the well-known Einstein manifold if $a = r/n$. This justifies the name 'Quasi Einstein' given to this type of manifold. The above paper deals with $(QE)_n(n>3)$ which are not conformally flat. The following results were obtained in this paper:

In a $(QE)_n(n>3)$, the Ricci tensor S has only two distinct eigen values $(a+b)$ and a of which the former is simple and the latter is of multiplicity $n - 1$, the generator U being an eigen vector corresponding to the eigen value $(a+b)$.

In a $(QE)_n$, the relation $R(X,Y).S = 0$ does not in general hold. A necessary and sufficient condition was obtained in order that this relation might hold.

A Riemannian manifold (M^n,g) is said to be conformally conservative [7] if the divergence of its conformal curvature tensor is zero. Thus every conformally flat Riemannian manifold is conformally conservative, but the converse is not, in general, true. Since this paper dealt with $(QE)_n(n>3)$ which are not conformally flat, the authors considered two sufficient conditions in order that a $(QE)_n(n>3)$ might be conformally conservative.

IA. Preliminaries:

Since U is a unit vector field, we have

$$A(U) = g(U,U) = 1. \tag{I.A.1}$$

Contracting (I.1) over X and Y gave the result

$$r = na + b \tag{I.A.2}$$

where r denotes the scalar curvature of the manifold. Substitution of $Y=U$ in (I.1) leads to the relation

$$S(X,U) = (a+b)A(X) = (a+b)g(X,U). \tag{I.A.3}$$

If L denotes the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, then

$$g(LX,Y) = S(X,Y)\forall X,Y. \tag{I.A.4}$$

If R be the curvature tensor of $(QE)_n$, then R(X,Y) may be regarded as a derivation of the tensor algebra at each point of the tangent space. Hence

$$[R(X,Y)\circ L](Z) = R(X,Y)LZ - L(R(X,Y)Z). \tag{I.A.5}$$

The transformations $R(X,Y)$ and L are called the curvature transformation and the Ricci transformation respectively. It is found that if $R(X,Y)\circ L=0,$ then $R(X,Y)$ and L commute, that is

$$R(X,Y)\circ L = L\circ R(X,Y). \tag{I.A.6}$$

IB. Significance of the associated scalars in a $(QE)_n(n>3)$:

From (I.A.3) it is evident that $(a+b)$ is an eigen value of the Ricci tensor S and U is an eigen vector corresponding to this eigen value. If V be any other vector orthogonal to U, then $g(U,V)=0$ *i.e.* $A(V)=0$. From (I.1) it follows that a is an eigen value of the Ricci tensor and V is an eigen vector corresponding to this eigen value. For this n-dimensional manifold, V is any vector orthogonal to U. Hence it follows from a known result in linear algebra [8], that the eigen value a is of multiplicity n-1 and the multiplicity of the eigen value $(a+b)$ must be unity.

This leads to the theorem:

Theorem I.1: ***In a $(QE)_n(n>3)$, the Ricci tensor S has only two distinct eigen values $(a+b)$ and 'a' of which the former is simple and the latter is of multiplicity n-1.***

Note: If $(a+b)=0$ then $a\neq 0$ because $b\neq 0$.

IC. $(QE)_n(n>3)$ satisfying the relation $R(X,Y)\cdot S=0$:

It is known that

$$\begin{aligned}[R(X,Y)\cdot S](Z,W) &= -S[R(X,Y)Z,W]-S[Z,R(X,Y)W]\\ &= -[ag(R(X,Y)Z,W)+bA(R(X,Y)Z)A(W)] \qquad \text{by (I.1)}\\ &\quad -[ag(R(X,Y)W,Z)+bA(R(X,Y)W)A(Z)]\\ &= -b[A(R(X,Y,Z))A(W)+A(R(X,Y,W))A(Z)].\end{aligned} \tag{I.C.1}$$

Since $b\neq 0$, it follows from (I.C.1) that in a $(QE)_n(n>3)$ the relation $R(X,Y)\cdot S=0$ does not hold in general. However, if $A(R(X,Y,Z))=0$ then $[R(X,Y)\cdot S](Z,W)=0 \forall Z,W.$ In that case, $R(X,Y)\cdot S=0.$ It is now supposed that $R(X,Y)\cdot S=0.$ Then (I.C.1) yields

$$A(R(X,Y,Z))A(W)+A(R(X,Y,W))A(Z)=0. \tag{I.C.2}$$

Substitution of $W=U$ gives in view of (I.A.1), $A(R(X,Y,Z))+g(R(X,Y,U),U)A(Z)=0.$

Since $g(R(X,Y,U),U)=0,$ it follows that

$$A(R(X,Y,Z))=0. \tag{I.C.3}$$

This leads to the theorem:

Theorem I.2: ***In a $(QE)_n(n>3)$, the relation $R(X,Y)\cdot S=0$ holds if and only if $A(R(X,Y,Z))=0$***

If $R(X,Y)\cdot S=0$, then $S(R(X,Y,Z),W)+S(R(X,Y,W),Z)=0.$

Simplification with the help of (I.A.4) yields the relation

$$g[\{LR(X,Y,Z)-R(X,Y,LZ)\},W]=0. \tag{I.C.4}$$

Hence

$$LR(X,Y,Z)-R(X,Y,LZ)=0. \tag{I.C.5}$$

Substitution of this in (I.A.5) leads to

$$[R(X,Y)\circ L](Z)=0\forall Z. \tag{I.C.6}$$

Hence

$$R(X,Y)\circ L=0. \tag{I.C.7}$$

In that case $R(X,Y)\cdot S=0.$ In virtue of (I.C.7) it follows that *the curvature and the Ricci transformations commute* [by (I.A.6)].

The converse is also true, that is, if the curvature and the Ricci transformations commute, then (I.C.7) holds and therefore $R(X,Y)\cdot S=0.$ This leads to the following theorem:

Theorem I.3: ***In a $(QE)_n(n>3)$, the curvature and the Ricci transformations commute if and only if the relation $A(R(X,Y,Z))=0$ holds.***

ID. $(QE)_n(n>3)$ with divergence-free conformal curvature tensor:

The conformal curvature tensor C of a Riemannian manifold (M^n,*g*) is said to be conservative [7] if the divergence of C is zero i.e. $divC=0.$ In such a case the manifold is said to be conformally conservative. In this section, the authors have obtained two sufficient conditions for a $(QE)_n(n>3)$ to be conformally conservative.

It is assumed that

$$\begin{aligned}&H(X,Y,Z)\\&=(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)-\frac{1}{2(n-1)}[dr(X)g(Y,Z)-dr(Z)g(Y,X)]\end{aligned} \tag{I.D.1}$$

where *r* is the scalar curvature of $(QE)_n$. Then it is known [9] that $divC=0.$ if and only if $H(X,Y,Z)=0.$ Two types of $(QE)_n(n>3)$ has been considered in the following sequence:

Type I: The associated scalars a and b are constants and therefore r is constant.

Type II: a and b are not constants but $a+b=0$.

Type I: For this type $da(X)=0$ and $db(X)=0.$ Therefore $dr(X)=0.$ Use of (I.1) leads to

$$\begin{aligned}&(\nabla_X S)(Y,Z)\\&=da(X)g(Y,Z)+db(X)A(Y)A(Z)+b[(\nabla_X A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)]\\&=b[(\nabla_X A)(Y)A(Z)+(\nabla_X A)A(Z)A(Y)].\end{aligned}$$

Hence

$$\begin{aligned}&(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X)\\&=b[(\nabla_X A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)-(\nabla_Z A)(Y)A(X)-(\nabla_Z A)(X)A(Y)].\end{aligned} \tag{I.D.2}$$

Therefore, (I.D.1) takes the following form

$$\begin{aligned}&H(X,Y,Z)\\&=b[(\nabla_x A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)-(\nabla_Z A)(Y)A(X)-(\nabla_Z A)(X)A(Y)]\end{aligned} \tag{I.D.3}$$

since $dr(X) = 0.$

Imposition of the condition that the generator U of the manifold is a recurrent vector field [10] with the associated 1-form A not being the 1-form of recurrence, gives $\nabla_X U = B(X)U$, where B is the 1-form of recurrence. Hence $g(\nabla_X U, Y) = g(B(X)U, Y)$, that is

$$(\nabla_X A)(Y) = B(X)A(Y). \tag{I.D.4}$$

In virtue of (I.D.4), (I.D.3) is expressed as follows

$$\begin{aligned} &H(X,Y,Z) \\ &= b[B(X)A(Y)A(Z) + B(X)A(Z)A(Y) - B(Z)A(Y)A(X) - B(Z)A(X)A(Y)]. \end{aligned} \tag{I.D.5}$$

Since $(\nabla_X A)(U) = 0,$ it follows from (I.D.4) that $B(X) = 0.$ Therefore, $H(X,Y,Z) = 0.$ Consequently we have the theorem:

Theorem I.4: ***If in a $(QE)_n(n>3)$ the associated scalars are constants and the generator U of the manifold is a recurrent vector field with the associated 1-form A not being the 1-form of recurrence, then the manifold is conformally conservative.***

Type II: For this type

$$r = (n-1)a \tag{I.D.6}$$

by (I.A.2). Hence r is neither zero nor a constant. From (I.D.6) it follows that

$$dr(X) = (n-1)da(X). \tag{I.D.7}$$

Since $a+b = 0$ i.e. $b = -a$, using this and equations (I.1) and (I.D.7) in the equation (I.D.1) leads to

$$\begin{aligned} &H(X,Y,Z) \\ &= \frac{1}{2}da(X)[g(Y,Z) - 2A(Y)A(Z)] - \frac{1}{2}da(Z)[g(Y,X) - 2A(Y)A(X)] \\ &+ a[(\nabla_Z A)(Y)A(X) - (\nabla_X A)(Y)A(Z) + A(Y)\{(\nabla_Z A)(X) - (\nabla_X A)(Z)\}] \end{aligned} \tag{I.D.8}$$

rom which it is evident that $H(X,Y,Z)$ is not in general zero. Next, the following conditions are imposed:

(i) $U = \frac{1}{2a} grada$ and (ii) $\nabla_X U = -X + A(X)U.$

In view of (i) it is obvious that $g(X,U) = g(\frac{1}{2a} grada, X)$, that is

$$da(X) = 2a(A(X)). \tag{I.D.9}$$

From (ii) we get

$$(\nabla_X A)(Y) = -g(X,Y) + A(X)A(Y). \tag{I.D.10}$$

In virtue of (I.D.9) and (I.D.10), (I.D.8) reduces to $H(X,Y,Z) = 0.$

This leads to the following:

Theorem I.5: ***If in a $(QE)_n(n>3)$ the associated scalars are not constants but their sum is zero and the generator satisfies the conditions (i) and (ii), then the manifold is conformally conservative.***

Geometric significance of the condition $\nabla_X U = -X + A(X)U$:

To understand the geometric significance of the above condition, the $(n-1)$-dimensional distribution $U^{\perp}$ in $(QE)_n$ orthogonal to U, is considered. If X and Y belong to $U^{\perp}$ where $Y \neq \lambda X$, then

$$g(X,U) = 0 \quad \text{and} \quad g(Y,U) = 0. \tag{I.D.11}$$

Since $(\nabla_X g)(Y,U) = 0 = (\nabla_Y g)(X,U)$ it follows from (I.D.11) and the above condition that

$$\begin{aligned} &g(\nabla_X Y,U) = g(\nabla_X U,Y) = g(X,Y) - A(X)A(Y) \\ &= g(\nabla_Y U,X) = g(\nabla_Y X,U). \end{aligned} \tag{I.D.12}$$

Now, $[X,Y] = \nabla_X Y - \nabla_Y X$. Therefore

$$g([X,Y],U) = g(\nabla_X Y - \nabla_Y X,U) = g(\nabla_X Y,U) - g(\nabla_Y X,U) = 0$$

by (I.D.12). Hence $[X,Y]$ is orthogonal to U. In other words, $[X,Y] \in U^{\perp}$.

Thus the distribution $U^{\perp}$ is involutive [11]. Hence from Frobenius' theorem [11] it follows that $U^{\perp}$ is integrable. This implies that $(QE)_n$ is a product manifold.

Theorem I.6: ***If in a $(QE)_n(n>3)$ the associated scalars are not constants but their sum is zero and the generator of the manifold satisfies the conditions (i) and (ii) then this $(QE)_n$ is a product manifold.***

2. ON QUASI EINSTEIN MANIFOLDS:

In the same year, 2000, in a separate paper, M. C. Chaki and M. L. Ghosh [2] studied some more features of $(QE)_n$ $(n \geq 3)$.

In 1956, S. S. Chern studied a type of Riemannian manifold [12] whose curvature tensor ' R of type (0,4) satisfies the condition

$$'R(X,Y,Z,W) = B(X,W)B(Y,Z) - B(Y,W)B(X,Z) \tag{II.1}$$

where B is a symmetric tensor of type (0,2). Such an n-dimensional manifold was called a special manifold with the associated symmetric tensor B and has been denoted by the symbol $(\psi B)_n$.

Such a manifold is important for the following reasons:

Firstly, for possessing some remarkable properties relating to curvature and characteristic classes and **secondly** for containing a manifold of quasi-constant curvature [5] as a subclass.

The following results were proved in this paper [2]:

A $(QE)_3$ is a $(\psi B)_3$ whose associated symmetric tensor is given by

$$B(X,Y)=(\frac{a-b}{2})^{\frac{1}{2}}g(X,Y)+b(\frac{2}{a-b})^{\frac{1}{2}}A(X)A(Y). \quad \text{(II.2)}$$

A $(QE)_n$ is not, in general, a $(\psi B)_n$. If, however, a $(QE)_n(n>3)$ is conformally flat, then it is shown that the $(QE)_n$ is a $(\psi B)_n$ whose associated symmetric tensor B is given by

$$B(X,Y)=[\frac{a(n-2)-b}{(n-1)(n-2)}]^{\frac{1}{2}}g(X,Y)+\frac{b}{(n-2)}[\frac{(n-1)(n-2)}{a(n-2)-b}]^{\frac{1}{2}}A(X)A(Y). \quad \text{(II.3)}$$

Further a conformally flat $(QE)_n(n>3)$ is a manifold of quasi-constant curvature.

It is further shown that if the generator of a conformally flat $(QE)_n(n>3)$ is an irrotational vector field, then the $(QE)_n$ is a product manifold $(\psi B)_n$ of quasi-constant curvature. Moreover some results relating to sectional curvatures of this kind of product manifold $(\psi B)_n$ are obtained.

IIA. Three-dimensional Quasi Einstein Manifold:

Here a $(QE)_3$ with associated scalars a, b, associated 1-form A and generator U has been considered. Contracting (I.1) we get for $n = 3$

$$r=3a+b. \quad \text{(II.A.1)}$$

It is known [13] that the curvature tensor $'R$ of type (0,4) in a three-dimensional manifold (M^3,g) has the following form

$$\begin{aligned}'R(X,Y,Z,W)&=[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\\&+S(X,W)g(Y,Z)-S(Y,W)g(X,Z)]\\&+\frac{r}{2}[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)]\end{aligned} \quad \text{(II.A.2)}$$

By virtue of (I.1), the equation (II.A.2) can be written as follows:

$$\begin{aligned}'R(X,Y,Z,W)&=(2a-\frac{r}{2})[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\&+b[A(Y)A(Z)g(X,W)-A(X)A(Z)g(Y,W)\\&+A(X)A(W)g(Y,Z)-A(Y)A(W)g(X,Z)]\end{aligned} \quad \text{(II.A.3)}$$

Using (II.A.1) we obtain

$$\begin{aligned}'R(X,Y,Z,W)&=q^2[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]\\&+qq'[A(Y)A(Z)g(X,W)-A(X)A(Z)g(Y,W)\\&+A(X)A(W)g(Y,Z)-A(Y)A(W)g(X,Z)]\end{aligned} \quad \text{where} \quad \text{(II.A.4)}$$

$$q=(\frac{a-b}{2})^{\frac{1}{2}} \text{ and} \quad \text{(II.A.5)}$$

$$q' = b(\frac{2}{a-b})^{\frac{1}{2}}. \quad \text{(II.A.6)}$$

Substitution of

$$B(X,Y) = qg(X,Y) + q' A(X)A(Y) \quad \text{(II.A.7)}$$

in (II.A.4) leads to

$$'R(X,Y,Z,W) = B(X,W)B(Y,Z) - B(Y,W)B(X,Z) \,. \quad \text{(II.A.8)}$$

By virtue of (II.A.5) to (II.A.7) the following theorem is stated:

Theorem II.1: ***Every (QE)$_3$ is a (ψB)$_3$ whose associated symmetric tensor B is given by (II.2).***

Remark: From (I.1) and (II.A.7) it is obtained that

$$B(X,Y) = \frac{1}{a}(\frac{a-b}{2})^{\frac{1}{2}} S(X,Y) + (\frac{b}{2})\frac{a+b}{\sqrt{2(a-b)}} A(X)A(Y). \quad \text{(II.A.9)}$$

From this it follows that in general, B is different from S, but when $(a+b) = 0$, then $B(X,Y) = \frac{1}{\sqrt{a}} S(X,Y)$.

IIB. Conformally flat (QE)$_n$(n>3):

In this case, contracting (I.1) one obtains (I.A.2). It is known [14] that in case of a conformally flat Riemannian manifold $(M^n,g)(n>3)$, the curvature tensor $'R$ of type (0,4) has the following form:

$$\begin{aligned} 'R(X,Y,Z,W) &= \frac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] \\ &+ \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \end{aligned} \quad \text{(II.B.1)}$$

Using (I.1) and (I.A.2) the equation (II.B.1) can be written as follows:

$$\begin{aligned} 'R(X,Y,Z,W) &= \frac{a(n-2)-b}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \frac{b}{n-2}[A(Y)A(Z)g(X,W) - A(X)A(Z)g(Y,W) \\ &+ A(X)A(W)g(Y,Z) - A(Y)A(W)g(X,Z)] \end{aligned} \quad \text{(II.B.2)}$$

The following substitution

$$B(X,Y) = pg(X,Y) + p'A(X)A(Y) \text{ with} \quad \text{(II.B.3)}$$

$$p = [\frac{a(n-2)-b}{(n-1)(n-2)}]^{\frac{1}{2}} \text{ and} \quad \text{(II.B.4)}$$

$$p' = \frac{b}{n-2}\left[\frac{(n-1)(n-2)}{a(n-2)-b}\right]^{\frac{1}{2}}, \quad \text{(II.B.5)}$$

leads to

$$'R(X,Y,Z,W) = B(X,W)B(Y,Z) - B(Y,W)B(X,Z). \quad \text{(II.B.6)}$$

The above results are stated in the following theorem:

Theorem II.2: ***Every conformally flat $(QE)_n(n>3)$ is a $(\psi B)_n$ whose associated symmetric tensor B is given by (II.B.3).***

Remark 1: In general, B is different from S, but when $(a+b)=0$, $B(X,Y)= \frac{1}{\sqrt{a(n-2)}}S(X,Y)$.

Remark 2: In the following it is established that a $(\psi B)_n$ a manifold of quasi-constant curvature as a subclass.

In 1972, Chen and Yano [5] introduced the notion of a manifold of quasi-constant curvature as follows:

A non-flat Riemannian manifold $(M^n,g)(n>3)$ is said to be of quasi-constant curvature if its curvature tensor $'R$ of type (0,4) satisfies the following condition:

$$\begin{aligned} 'R(X,Y,Z,W) &= a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ b[A(Y)A(Z)g(X,W) - A(X)A(Z)g(Y,W) \\ &+ A(X)A(W)g(Y,Z) - A(Y)A(W)g(X,Z)] \end{aligned} \quad \text{(II.B.7)}$$

where a and b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that $g(X,U) = A(X)\forall X$, U being a unit vector field. In such a case a and b were called associated scalars, A was called the associated 1-form and U was called the *generator* of the manifold. Such an n-dimensional manifold was denoted by the symbol $(QC)_n$ [2]. Putting

$$B(X,Y) = \sqrt{a}g(X,Y) + \frac{b}{\sqrt{a}}A(X)A(Y) \quad \text{(II.B.8)}$$

it followed from (II.B.7) that

$$'R(X,Y,Z,W) = B(X,W)B(Y,Z) - B(Y,W)B(X,Z). \quad \text{(II.B.9)}$$

From (II.B.9) it could be seen that a $(QC)_n$ is a $(\psi B)_n$.

Remark 3: Comparing (II.B.2) with (II.B.7) it can be concluded that a conformally flat $(QE)_n(n>3)$ is a manifold of quasi-constant curvature.

IIC. Conformally flat $(QE)_n(n>3)$ whose generator is an Irrotational vector field:

It is known [16, p.358] that a vector field U in a Riemannian manifold (M^n,g) is said to be irrotational if

$$g(\nabla_X U, Y) = g(\nabla_Y U, X) \text{ for every } X,Y. \quad \text{(II.C.1)}$$

In this section, the authors considered a conformally flat $(QE)_n(n>3)$, whose generator U satisfies the condition (II.C.1). As in section (I.D), the $(n-1)$-dimensional distribution

$U^{\perp}$ in $(QE)_n$ orthogonal to U, is considered. This means that $X\in U^{\perp}$ if $g(X,U)=0$ and if $X\in U^{\perp}$ then $g(X,U)=0$. If X and Y belong to $U^{\perp}$ where $Y\neq \lambda X$, then condition (I.D.11) will be valid in this case also. In that case, in view of the condition (II.C.1) and because the Riemannian connection ∇ is of vanishing torsion, i.e. $[X,Y]=\nabla_X Y-\nabla_Y X$ it follows that if $X,Y\in U^{\perp}$ then $[X,Y]$ also belongs to $U^{\perp}$ and the same conclusions as in section (I.D) follow i.e. the distribution $U^{\perp}$ is involutive, $U^{\perp}$ is integrable and that $(QE)_n$ is a product manifold.

In view of this result and Remark 3 of section IIB above the following theorem is stated:

Theorem II.3: ***If in a conformally flat $(QE)_n(n>3)$, the generator U is an irrotational vector field, then the $(QE)_n$ is a product manifold $(\psi B)_n$ of quasi-constant curvature.***

IID. Sectional curvatures at a point of a conformally flat $(QE)_n(n>3)$ whose generator is an Irrotational vector field:

From (II.B.2) we have

$$R(X,Y,Z)=\frac{a(n-2)-b}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y] + \frac{b}{n-2}[A(Y)A(Z)X-A(X)A(Z)Y+g(Y,Z)A(X)U-g(X,Z)A(Y)U] \tag{II.D.1}$$

where R is the curvature tensor of type (1,3). From (II.D.1) it is obtained that

$$R(X,Y,Z)=\frac{a(n-2)-b}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y] \text{ when } X,Y,Z\in U^{\perp}, \text{ and} \tag{II.D.2}$$

$$R(X,U,U)=(\frac{a+b}{n-1})X \text{ when } X\in U^{\perp}. \tag{II.D.3}$$

If $k(X,Y)$ denotes the sectional curvature of $(QE)_n$ at a point corresponding to the section-plane spanned by the vectors X and Y, then it is known that

$$k(X,Y)=\frac{g[R(X,Y,Y),X]}{g(X,X)g(Y,Y)-[g(X,Y)]^2}. \tag{II.D.4}$$

From (II.D.2) it follows that

$$R(X,Y,Y)=[\frac{a(n-2)-b}{(n-1)(n-2)}][g(Y,Y)X-g(X,Y)Y] \text{ when } X,Y\in U^{\perp}. \tag{II.D.5}$$

Hence from (II.D.4) it is obtained that

$$k(X,Y)=\frac{a(n-2)-b}{(n-1)(n-2)} \text{ when } X,Y\in U^{\perp}. \tag{II.D.6}$$

Substitution of $Y=U$ yields

$$k(X,U)=\frac{a+b}{n-1} \text{ when } X\in U^{\perp}. \quad \text{(II.D.7)}$$

This leads to the following theorem:

Theorem II.4: ***If in a conformally flat (QE)$_n$(n>3), the generator U is an irrotational vector field and U$^{\perp}$ is the (n-1)-dimensional distribution orthogonal to U, then the scalar*** $\frac{a(n-2)-b}{(n-1)(n-2)}$ ***is the sectional curvature at a point corresponding to the section-plane spanned by X,Y belonging to U$^{\perp}$, and the scalar*** $\frac{a+b}{n-1}$ ***is the sectional curvature at a point corresponding to the section-plane spanned by X and U where X∈U$^{\perp}$.***

3. SOME GLOBAL PROPERTIES OF QUASI EINSTEIN MANIFOLDS:

In this paper, M. C. Chaki and P. K. Ghoshal [3] studied some global properties of a compact orientable (QE)$_n$ without boundary. The following results have been used in the sequel.

From (I.1)

$$S(X,X)=ag(X,X)+bA(X)A(X)=a|X|^2+b[g(X,U)]^2\forall X. \quad \text{(III.1)}$$

If θ be the angle between U and any vector X, then

$$\cos\theta=\frac{g(X,U)}{\sqrt{g(X,X)}\sqrt{g(U,U)}}=\frac{g(X,U)}{g(X,X)}$$ by (I.A.1). Hence

$$g(X,X)=|X|^2\geq[g(X,U)]^2. \quad \text{(III.2)}$$

It follows that

$$S(X,X)\geq(a+b)[g(X,U)]^2 \text{ when } a>0 \text{ and} \quad \text{(III.3)}$$

$$S(X,X)\leq(a+b)|X|^2 \text{ when } b>0\,. \quad \text{(III.4)}$$

IIIA. Sufficient condition for a compact orientable (*QE*)$_n$ to be conformal to a sphere in E_{n+1}:

By definition, an n-dimensional Riemannian manifold (M,g) is said to be conformal to another n-dimensional manifold (M′,g′) if there exists a one-one differentiable mapping (M,g)→ (M′,g′) such that the angle between any two vectors at a point p of M is always equal to that of the corresponding two vectors at the corresponding point p′ of M′.

Y. Watanabe [16] has given a sufficient condition of conformality of an n-dimensional Riemannian manifold to an n-dimensional sphere immersed in E_{n+1}. Its statement is as follows:

If in an n-dimensional Riemannian manifold M, there exists a non-parallel vector field X such that the condition

$$\int_M S(X,X)dv = \tfrac{1}{2}\int_M |dX|^2 dv + \frac{n-1}{n}\int_M (\partial X)^2 dv \qquad \text{(III.A.1)}$$

holds, then M is conformal to a sphere in E_{n+1}, where dv is the volume element of M and dX and ∂X are the curl and divergence of X respectively.

In this section the authors considered a compact orientable quasi Einstein manifold $(QE)_n=M$ without boundary, with associated scalars a,b and generator U. It satisfies (I.1) and (I.2). Hence $S(U,U) = a+b$.

In virtue of this and putting $X = U$, condition (III.A.1) takes the following form

$$\int_M (a+b)dv = \tfrac{1}{2}\int_M |dU|^2 dv + \frac{(n-1)}{n}\int_M (\partial U)^2 dv. \qquad \text{(III.A.2)}$$

Next, it is supposed that $U = gradf$.

Then U cannot be a parallel vector field, for otherwise $\nabla U = 0$ or $\nabla(gradf) = 0$ i.e. $\Delta f = 0$, where Δ denotes the Laplacian operator and ∇ denotes the covariant differentiation with respect to the metric of m. Hence by Bochner's lemma [17, p.39], f is constant, which implies that $U = 0$ which is not admissible.

Since by assumption, $U = gradf$, thus $|dU|^2 = 0.$ Hence (III.A.2) takes the form

$$\int_M (a+b)dv = \frac{(n-1)}{n}\int_M (\partial U)^2 dv. \qquad \text{(III.A.3)}$$

In that case, by Watanabe's condition (III.A.1), the manifold M is conformal to a sphere in E_{n+1}. This is stated in the following theorem:

Theorem III.1: ***If in a compact, orientable quasi Einstein manifold M=(QE)$_n$(n≥3) without boundary, having a, b as associated scalars, the generator U is the gradient of a scalar function and satisfies the condition (III.A.3), then the manifold (QE)$_n$ is conformal to a sphere immersed in E$_{n+1}$.***

IIIB. Killing vector field in a compact orientable $(QE)_n$(n≥3) without boundary:

In this section, a compact, orientable $(QE)_n=M$, (n≥3) without boundary having a, b as associated scalars and U as the generator, has been considered.

It is known [16, 17(p.43)] that in such a manifold M, the following relation holds:

$$\int_M [S(X,X) - |\nabla X|^2 - (divX)^2]dv = 0 \forall X. \qquad \text{(III.B.1)}$$

If X is a killing vector field, then $div\, X = 0$ [18, p.43]. Hence (III.B.1) takes the form

$$\int_M [S(X,X) - |\nabla X|^2]dv = 0. \qquad \text{(III.B.2)}$$

If b>0, then by (III.4), $(a+b)|X|^2 \geq S(X,X).$

Therefore, $(a+b)|X|^2 - |\nabla X|^2 \geq S(X,X) - |\nabla X|^2$.

Consequently, $\int_M [(a+b)|X|^2 - |\nabla X|^2]dv \geq \int_M [S(X,X) - |\nabla X|^2]dv$

and by (III.B.2) $\int_M [(a+b)|X|^2 - |\nabla X|^2]dv \geq 0$.

If $a + b = 0$, then $\int_M [(a+b)|X|^2 - |\nabla X|^2]dv = 0$.

Therefore $X = 0$. This leads to the following result:

Theorem III.2: ***If in a compact, orientable quasi Einstein manifold $M=(QE)_n(n\geq 3)$ without boundary the associated scalars are such that b>0 and a+b<0, then there exists no non- zero Killing vector field in this manifold.***

Two corollaries of the above theorem follow easily.

Corollary 1: ***If in a compact, orientable quasi Einstein manifold $M=(QE)_n(n\geq 3)$ without boundary, the generator U is a Killing vector field, then the following relation holds***

$$\int_M [(a+b) - |\nabla U|^2]dv \geq 0.$$

Corollary 2: ***If in a compact, orientable quasi Einstein manifold $M=(QE)_n(n\geq 3)$ without boundary, b>0 and a+b<0, then the generator U of the manifold cannot be a Killing vector field.***

IIIC. Killing p-form in a compact, orientable conformally flat $M=(QE)_n(n\geq 3)$ without boundary:

It is assumed that w is a p-form in a compact, orientable conformally flat $M=(QE)_n(n>3)$ without boundary and $F_p(w,w)$ is the well-known [17] quadratic form given by

$$F_p(w,w) = S_{ij} w^{ii_2 \ldots\ldots i_p} w^{j}_{i_2 \ldots\ldots i_p} + \frac{(p-1)}{2} R_{ijkl} w^{iji_3 \ldots\ldots i_p} w^{kl}_{i_3 \ldots\ldots i_p} \tag{III.C.1}$$

where R_{ijkl} and S_{ij} are the components of the curvature tensor R of type (0,4) and the Ricci tensor of type (0,2) respectively of the $(QE)_n$.

Since the $(QE)_n$ is chosen to be conformally flat, the Riemann tensor R can be expressed as follows [18(p.234); 14(p.40)] :

$$\begin{aligned} R(X,Y,Z,W) = & \frac{1}{(n-2)}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)] \\ & - \frac{r}{(n-1)(n-2)}[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]. \end{aligned} \tag{III.C.2}$$

By (I.1) it can be written that

$$R(X,Y,Z,W) = a'[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$$
$$+b'[A(Y)A(Z)g(X,W) - A(X)A(Z)g(Y,W)$$
$$+A(X)A(W)g(Y,Z) - A(Y)A(W)g(X,Z)]$$

where

$$a' = \frac{a}{(n-1)} - \frac{b}{(n-1)(n-2)} \text{ and } b' = \frac{b}{(n-2)}. \quad \text{(III.C.3)}$$

In view of (I.1) and (III.C.2), equation (III.C.1) can be expressed as follows:

$$F_p(w,w) = [(n-p)a' + b']|w|^2 + (n-2p)b'[U.w]^2 \quad \text{(III.C.4)}$$

where the components of w are $w_{i_1 i_2 \ldots\ldots i_p}$, those of U are U^i and $U.w$ is a tensor of type $(0, p-1)$ with components $U^j w_{j i_1 \ldots\ldots i_{p-1}}$ and $|w|^2 = w_{i_1 i_2 \ldots\ldots i_p} w^{i_1 i_2 \ldots\ldots i_p}$.

Using (III.C.3) the relation (III.C.4) can be expressed as follows:

$$F_p(w,w) = \frac{[(n-p)(n-2)a + b(p-1)]}{(n-1)(n-2)}|w|^2 + (n-2p)\frac{b}{(n-2)}[U.w]^2. \quad \text{(III.C.5)}$$

It is now supposed that w is a Killing p-form. Then it is known [17] that

$$\int_{(QE)_n} [F_p(w,w) - |\nabla w|^2]dv = 0. \quad \text{(III.C.6)}$$

In virtue of (III.C.5) equation (III.C.6) can be expressed as follows:

$$\int_{(QE)_n} [\frac{(n-p)(n-2)a + b(p-1)}{(n-1)(n-2)}|w|^2 + \frac{(n-2p)}{(n-2)}b(U.w)^2 - |\nabla w|^2]dv = 0. \quad \text{(III.C.7)}$$

If $(n-2p)b < 0$ and $(n-p)(n-2)a + b(p-1) < 0$ then from (III.C.7) it follows that $w = 0$.

This leads to the following result.

Theorem III.3: ***If in a compact, orientable conformally flat $(QE)_n(n>3)$ without boundary, (n-2p)b<0 and (n-p)(n-2)a + b(p-1)<0, where p>1 but <n, then there exists no non-zero Killing p-form in such a manifold.***

4. GENERALIZED QUASI EINSTEIN MANIFOLDS:

The notions of a generalized quasi Einstein manifold and a manifold of generalized quasi constant curvature were introduced by M. C. Chaki [4] in 2001.

A non-flat Riemannian manifold (M^n, g)($n\geq3$) is called a generalized quasi Einstein manifold $G(QE)_n$ if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \quad \text{(IV.1)}$$

where a, b, c are scalars of which $b\neq0$, $c\neq0$, A, B are non-zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X) \quad \text{(IV.2)}$$

for all X and U, V are two unit vector fields perpendicular to each other.

In such a case a, b, c were called the associated scalars, A and B the associated 1-forms and U and V the generators of the manifold. Such an n-dimensional manifold has been denoted by the symbol $G(QE)_n$.

For c=0, (IV.1) takes the form (I.1). This justifies the name "generalized quasi Einstein manifold" for this new type of manifold.

As already mentioned in section IIB, in 1972, Chen and Yano [5] introduced the notion of a manifold of quasi-constant curvature, which is defined by the relation (II.B.7).

A generalization of a manifold of quasi-constant curvature, called a manifold of *generalized quasi-constant curvature*, denoted by $G(QC)_n$, has been done by Chaki [4]. This was necessary for the study of $G(QE)_n$.

A manifold of generalized quasi-constant curvature has been defined as follows:

A non-flat Riemannian manifold (M^n, g)($n\geq 3$) is called a manifold of generalized quasi constant curvature if its curvature tensor ' R of type (0,4) satisfies the condition:

$$\begin{aligned}
&'R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&+ b[A(Y)A(Z)g(X,W) - A(X)A(Z)g(Y,W) \\
&+ A(X)A(W)g(Y,Z) - A(Y)A(W)g(X,Z)] \qquad \text{(IV.3)} \\
&+ c[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} - g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\
&+ g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} - g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}]
\end{aligned}$$

where a, b, c are scalars of which $b\neq 0$, A, B are two non-zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X) \quad \text{(IV.4)}$$

for all X and U, V are two mutually perpendicular unit vector fields.

In such a case a, b, c were called the associated scalars, A and B the associated 1-forms and U, V were called the generators of the manifold. For $c = 0$, (IV.3) takes the form (II.B.7) and the manifold becomes a manifold of quasi-constant curvature. The following results were established in this paper:

In a $G(QE)_n$, the scalars a and $a + b$ are the Ricci curvatures in the directions of the vector fields V and U respectively and the scalar c is less than $\frac{1}{\sqrt{2}}l$, where l is the length of the Ricci tensor S.

Every $G(QE)_3$ is a $G(QC)_3$ but a $G(QE)_n(n>3)$ is not, in general, a $G(QC)_n$.

A $G(QE)_n(n>3)$ is a $G(QC)_n$ if it is conformally flat.

A $G(QC)_n$ (n>3) is a conformally flat $G(QE)_n$.

If $U^\perp$ denotes the ($n-1$)-dimensional distribution of a $G(QE)_n(n>3)$ orthogonal to the generator U, then the sectional curvature of the plane determined by the vectors X, Y is $\frac{(3n-2)a+b}{(n-1)(n-2)}$ when $X,Y\in U^\perp$, while the sectional curvature of the plane determined by the vectors X, U is $\frac{(3n-2)a+nb}{(n-1)(n-2)}$ when $X\in U^\perp$.

IVA. Scalar curvature and the associated scalars of a $G(QE)_n(n>3)$:

In this section, a $G(QE)_n$ with associated scalars a, b, c, associated 1-forms A, B and generators U and V corresponding to the 1-forms A and B, respectively, was considered. In virtue of (IV.4) we must have

$$g(U,U)=1,\ \ g(V,V)=1 \text{ and } g(U,V)=A(V)=B(U)=0 \qquad \text{(IV.A.1)}$$

Contraction of (IV.1) over *X* and *Y* gave

$$S(U,U)=a+b, \qquad \text{(IV.A.3)}$$

$$S(V,V)=a \text{ and } \qquad \text{(IV.A.4)}$$

$$S(U,V)=c. \qquad \text{(IV.A.5)}$$

If *X* is a unit vector field, then *S*(*X*,*X*) is the Ricci curvature in the direction of *X*. Hence, from (IV.A.4) and (IV.A.5), it can be stated that $(a+b)$ and a are the Ricci curvatures in the directions of *U* and *V*, respectively. It is assumed that

$$g(LX,Y)=S(X,Y). \qquad \text{(IV.A.6)}$$

Also, the square of the length of the Ricci tensor S is denoted by l^2 Then,

$$l^2=S(Le_i,e_i) \qquad \text{(IV.A.7)}$$

where $\{e_i\}$, $i=1,2,\ldots,n$ is an orthonormal basis of the tangent space of $G(QE)_n$. Equation (IV.1) gives $S(Le_i,e_i)=(n-1)a^2+(a+b)^2+2c^2$. Hence $l^2-2c^2=(n-1)a^2+(a+b)^2\geq 0.$ Therefore

$$l^2>2c^2 \text{ since } l^2-2c^2\neq 0 \qquad \text{(IV.A.8)}$$

which means that

$$c<\frac{1}{\sqrt{2}}l. \qquad \text{(IV.A.9)}$$

The following theorem sums up the above deduction;

Theorem IV.1: ***In a $G(QE)_n(n>3)$ the scalar (a + b) and a are the Ricci curvatures in the directions of the generators U and V, respectively and the associated scalar c is less than $\frac{1}{\sqrt{2}}l$, where l is the length of the Ricci tensor S.***

IVB. Three-dimensional generalized quasi Einstein manifold:

In this section, a $G(QE)_3$ has been considered. In this case

(IV.B.1) $r=3a+b.$

It is known [13] that in a 3-dimensional Riemannian manifold (M^n,*g*) the curvature tensor $'R$ of type (0,4) has the following form:

$$\begin{aligned}'R(X,Y,Z,W)&=[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)\\&+S(X,W)g(Y,Z)-S(Y,W)g(X,Z)]\\&+\frac{r}{2}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)].\end{aligned} \qquad \text{(IV.B.2)}$$

Using (IV.1), equation (IV.B.2) can be written as

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= (2a+\frac{r}{2})[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
&+ b[A(Y)A(Z)g(X,W)-A(X)A(Z)g(Y,W) \\
&+ A(X)A(W)g(Y,Z)-A(Y)A(W)g(X,Z)] \qquad \text{(IV.B.3)} \\
&+ c[g(Y,Z)\{A(X)B(W)+A(W)B(X)\}-g(X,Z)\{A(Y)B(W)+A(W)B(Y)\} \\
&+ g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\}-g(Y,W)\{A(X)B(Z)+A(Z)B(X)\}].
\end{aligned}
$$

In virtue of (IV.3),it follows from (IV.B.3) that a $G(QE)_3$ is a $G(QC)_3$. this leads to the following result:

Theorem IV.2: ***Every G(QE)$_3$ is a G(QC)$_3$.***

IVC. Conformally flat $G(QE)_n(n>3)$:

In general, a $G(QE)_n(n>3)$ is not a $G(QC)_n$. In this section, a conformally flat $G(QE)_n$ has been considered. It is found that in such a case the $G(QE)_n$ becomes a $G(QC)_n$.

It is known [14] that in a conformally flat Riemannian manifold $(M^n,g)(n>3)$, the curvature tensor 'R of type (0,4) has the following form:

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= \frac{1}{(n-2)}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W) \\
&+ S(X,W)g(Y,Z)-S(Y,W)g(X,Z)] \qquad \text{(IV.C.1)} \\
&+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)].
\end{aligned}
$$

Using (IV.1), equation (IV.C.1) can be expressed as follows:

$$
\begin{aligned}
{}'R(X,Y,Z,W) &= \frac{2a(n-1)+r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
&+ \frac{b}{(n-2)}[A(Y)A(Z)g(X,W)-A(X)A(Z)g(Y,W) \\
&+ A(X)A(W)g(Y,Z)-A(Y)A(W)g(X,Z)] \\
&+ \frac{c}{(n-2)}[g(Y,Z)\{A(X)B(W)+A(W)B(X)\} \qquad \text{(IV.C.2)} \\
&- g(X,Z)\{A(Y)B(W)+A(W)B(Y)\} \\
&+ g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\} \\
&- g(Y,W)\{A(X)B(Z)+A(Z)B(X)\}].
\end{aligned}
$$

In virtue of (IV.3) it follows from (IV.C.2) that a conformally flat $G(QE)_n$ is a $G(QC)_n$ since $b\neq 0$. This leads to the following theorem:

Theorem IV.3: ***Every conformally flat G(QE)$_n$ (n>3) is a G(QC)$_n$.***

To find whether every $G(QC)_n$ ($n\geq 3$) is a $G(QE)_n$, equation (IV.3) is contracted over Y and Z. This yields

$$S(X,W) = [a(n-1)+b]g(X,W) + b(n-2)A(X)A(W) + c(n-2)[A(X)B(W) + A(W)B(X)]. \tag{IV.C.3}$$

In virtue of (IV.1) it follows from (IV.C.3) that a $G(QC)_n$($n\geq 3$) is a $G(QE)_n$ since $b\neq 0$.

In a Riemannian manifold (M^n,g)($n>3$), the conformal curvature tensor $'C$ of type (0,4) has the following form:

$$'C(X,Y,Z,W) = {}'R(X,Y,Z,W) - \frac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{IV.C.4}$$

Using (IV.3) and (IV.C.3) it follows from (IV.C.4) that

$$'C(X,Y,Z,W) = 0 \tag{IV.C.5}$$

i.e. the manifold under consideration is conformally flat. The following theorem can therefore be stated:

Theorem IV.4: ***Every G(QC)$_n$ (n≥3) is a G(QE)$_n$ while every G(QC)$_n$(n>3) is a conformally flat G(QE)$_n$.***

IVD. Sectional curvatures at a point of a conformally flat $G(QE)_n$(n>3):

Once again the (n−1)-dimensional distribution $U^{\perp}$ in a conformally flat $G(QE)_n$(n>3) orthogonal to U, is considered as in sections ID, IIC and IID. If X and Y belong to $U^{\perp}$ where $Y\neq \lambda X$, then condition (I.D.11) will be valid in this case also.

The sectional curvature K of the planes determined by the vectors $X,Y\in U^{\perp}$ and by the vectors X,U where $X\in U^{\perp}$ has been determined in this section. Substitution of Z=Y and W=X in (IV.C.1) yields

$$'R(X,Y,Y,X) = \frac{2a(n-1)+r}{(n-1)(n-2)}[g(X,X)g(Y,Y) - \{g(X,Y)\}^2]. \tag{IV.D.1}$$

The sectional curvature of the plane determined by X,Y is given by (II.D.4) which using (IV.D.1)in this case becomes

$$K(X,Y) = \frac{2a(n-1)+r}{(n-1)(n-2)} = \frac{(3n-2)a+b}{(n-1)(n-2)}. \tag{IV.D.2}$$

The sectional curvature $K(X,U)$ of the plane determined by X and U is given by

$$K(X,U) = \frac{'R(X,U,U,X)}{g(X,X)g(U,U) - \{g(X,U)\}^2}. \tag{IV.D.3}$$

But

$$'R(X,U,U,X)=[\frac{2a(n-1)+r}{(n-1)(n-2)}+\frac{b}{(n-2)}]g(X,X). \qquad \text{(IV.D.4)}$$

Hence from (IV.D.3) it is obtained that

$$K(X,U)=\frac{2a(n-1)+r}{(n-1)(n-2)}+\frac{b}{(n-2)}=\frac{(3n-2)a+nb}{(n-1)(n-2)}. \qquad \text{(IV.D.5)}$$

To sum up, the following theorem has been stated:

Theorem IV.5: ***In a conformally flat $G(QE)_n$ (n>3) the sectional curvature of the plane determined by two vectors $X,Y \in U^{\perp}$ is $\frac{(3n-2)a+b}{(n-1)(n-2)}$ while the sectional curvature of the plane determined by two vectors X,U where $X \in U^{\perp}$ is $\frac{(3n-2)a+nb}{(n-1)(n-2)}$.***

5. PHYSICAL INTERPRETATION

It is known [15, p.339] that for a perfect fluid space-time of general relativity, the Einstein field equation without cosmological constant is of the form

$$R_{ij}-\tfrac{r}{2}g_{ij}=[(\rho+p)u_iu_j+pg_{ij}] \qquad \text{(V.1)}$$

where ρ and p are the energy density and the isotropic pressure of the fluid respectively, *R* is the Ricci tensor and r is the scalar curvature of the space-time.

The above equation can be recast into the form

$$R_{ij}=\alpha g_{ij}+\beta u_iu_j \qquad \text{(V.2)}$$

with $\alpha=(r/2+p)$ and $\beta=\rho+p$. Here both α and β are scalars, of which $\beta\neq 0$ i.e. $\rho+p\neq 0$. If $\beta = 0$, then the space-time will reduce to the well-known Einstein manifold.

In the index-free notation equation (V.2) becomes

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y) \qquad \text{(V.3)}$$

with *S*(*X*,*Y*), *A*(*X*) and *A*(*Y*) has the same meaning as in equation (I.1). Comparing (V.3) with (I.1) we can say that a perfect fluid space-time of general relativity is a four-dimensional semi-Riemannnian quasi Einstein manifold of Lorentz signature (+,+,+,-) and whose associated scalars are $r/2+p$ and $\rho+p$ respectively. Further, if *U* represents the unit timelike velocity vector field of the flow, then U will be an eigen vector of the field with a simple eigen value $r/2+\rho+2p$. All eigen vectors orthogonal to *U* will have eigen value $r/2+p$ with multiplicity $(n-1)$ i.e. of multiplicity 3 for the space-time.

The importance of a generalized quasi Einstein manifold lies in the fact that such a four-dimensional semi-Riemannian manifold is relevant to the study of a general relativistic fluid space-time admitting heat flux [6], where the vectors *U* and *V* referred to in sections IV and IVA are now identified as the velocity vector field and the heat flux vector field of the fluid respectively.

Einstein field equation without cosmological constant for a fluid admitting heat flux is written as

$$S(X,Y) - \tfrac{r}{2} g(X,Y) = (\rho + p)A(X)A(Y) + pg(X,Y) + A(X)B(Y) + A(Y)B(X) \tag{V.4}$$

where the symbols r, ρ, p, g, S and A have the same meanings as before.

Rearranging terms in the above equation we get

$$S(X,Y) = (\tfrac{r}{2} + p)g(X,Y) + (\rho + p)A(X)A(Y) + A(X)B(Y) + A(Y)B(X). \tag{V.5}$$

Comparing (V.5) with (IV.1) we can say that a fluid space-time of general relativity admitting heat flux is a four-dimensional semi-Riemannnian generalized quasi Einstein manifold of Lorentz signature (+,+,+,-) whose associated scalars are $a = (r/2 + p)$ $b = (\rho + p)$ and $c = 1$ respectively. The Ricci curvatures in the directions of the velocity vector U and the heat flux vector V are $(r/2 + \rho + 2p)$ and $(r/2 + p)$ respectively. The length l of the Ricci tensor S is greater than √2.

For a conformally flat generalized quasi Einstein space-time admitting heat flux, the sectional curvature of the plane containing U and V will be given by

$$K(V,U) = \frac{'R(V,U,U,V)}{g(V,V)g(U,U) - \{g(U,V)\}^2} = 'R(V,U,U,V).$$

Since from equation (IV.3) we find that $'R(V,U,U,V)$ is determined by a and b which in turn depends on the energy density ρ and the pressure p of the fluid, the sectional curvature of the plane containing U and V also depends on ρ and p.

6. CONCLUDING REMARKS

In Cosmology, the reason for studying various types of space-time models is mainly for the purpose of representing the different phases in the evolution of the Universe. Broadly speaking, the evolution of the universe to its present state can be divided into three phases:

The **initial phase** just after the big bang when the effects of both viscosity and heat flux were quite pronounced.

The **intermediate phase** when the effect of viscosity was no longer significant but the heat flux was still not negligible.

The **final phase**, which extends to the present state of the Universe when both the effects of viscosity and the heat flux have become negligible and the matter content of the Universe may be assumed to be a perfect fluid.

The importance of the study of the $G(QE)_n$ and $(QE)_n$ lies in the fact that these space-time manifolds represent the second and the third phase respectively in the evolution of the Universe. What remains now is the representation of the first phase in the evolution of the Universe. Significant progresses have been made in that direction and the results are expected to be published soon.

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KVAZI-EINSTEIN-OVE I GENERALISANE KVAZI-EINSTEIN-OVE MNOGOSTRUKOSTI

Sarbari Guha

Profesor M.C. Chaki je uveo pojam kvazi-Eustein ova mnogostrukost [1], obeleživši je sa $(QE)_n$*, čiji je RICCI-jev tenzor S tipa (0,2) i nije identički jednak nulii zadovoljava sledeći uslov:*

$$S(X,Y) = a\, g(X,Y) + b\, A(X)A(Y)$$

gde su a, b *skalari od kojih je* $b \neq 0$ *and A je nenulta 1-forma takva da je*

$$g(X,U) = A(X)$$

za sva vektorska polja X, U je jedinični vektor polja.

Ako postoji 4-dimenzionalna Lorencova mnogostrukost čiji je Ricci-jev tenzor u obliku prethodne forme, tada postoji takva prostor-vreme, koja predstavlja prostor-vreme idealnog fluida u kosmologiji.

Istraživanja Karcher-a [2] i drugih su pokazala konformni ravni prostor-vreme idealnog fluida imaju geometrijsku strukturu kvazi-koonstantne krivine. Utvrdjeno je da da mnogostrukost kvazi-konstantne strukture je prirodna podklasa kvazi-Eunstein-ove mnogostrukosti. Istraživanja kvazi-Eunstein-ovih mnogostrukosti nam pomažu da dublje razumemo globalni karakter univerzuma [3] uključujući i topologiju. Kao posledica toga, u ovom radu je studirana priroda singulariteta definitnih formi sa stanovišta diferencijalne geometrije. Istraživanja mnogostrukosti prostor-vreme omogućavaju uvodjenje viskoznog fluida i eletromagnetskog polja, kao i buduće generalizacije Ricci-jevih tenzora, koji su u toku.