On Quasi-Stationary Models of Mixtures of Compressible Fluids

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ON QUASI-STATIONARY MODELS OF MIXTURES OF COMPRESSIBLE FLUIDS

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ABSTRACT. We consider mixtures of compressible viscous fluids consisting of two miscible species. In contrast to mixture - models considered by the french school where one has only one velocity field the mixture equation considered here have densities and velocity fields assigned to each species of the fluid.

1. Statement of the Problem and Main Results

In this article we deal with mixtures of compressible viscous fluids consisting of two miscible species. In contrast to some mixture - models considered by the french school where one has only one velocity field the mixture equation considered here have densities and velocity fields assigned to each species of the fluid. For the derivation of the constitutive equations from the physical model see the books of Rajagopal and Haupt [12], [7]. We consider here the quasi-stationary model which is a reasonable approximation of the general case if the accelerations are small.

The one component quasi-stationary model as an approximation to the Navier-Stokes-system has been considered in the works [1], [9], [11], [10].
The stationary Stokes-like-case with two components has been considered in [2], [3] and [5]. In these papers the exictence of weak solutions with additional L^p - properties of the densities has been achieved.

We establish in the present article existence of global *classical* solutions to the initial - boundary value problem for the system of equations which follow immediately by using new a priori estimates. This is remarkable since the system is nonlinear and of first order with respect to the densities.

The partial differential equations of the quasi-stationary model which describe the motion of the mixture in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, read:
Balance of momentum for the i-th species $(i = 1, 2)$:

$$\sum_{j=1}^{2} \left(\mu_{ij}\Delta u^{(j)} + (\mu_{ij} + \lambda_{ij})\nabla \operatorname{div} u^{(j)}\right) + (-1)^{i+1}(u^{(2)} - u^{(1)})g - \nabla p^{(i)} = 0\,. \tag{1}$$

Conservation of mass for the i-th species $(i = 1, 2)$*:*

$$\frac{\partial}{\partial t}\rho^{(i)} + \operatorname{div}(\rho^{(i)} u^{(i)}) = 0\,. \tag{2}$$

The equations (1) and (2) have to hold in $Q_T = \Omega \times (0, T)$ *,* $T = \text{const} > 0$.

The quantities in equation (1) and (2) have the following meaning:

- $\rho^{(i)}(x, t)$ mass density for the i-th component of the mixture, $i = 1, 2$;
- $p^{(i)}(\rho^{(1)}, \rho^{(2)})$ pressure for the i-th component of the mixture, $i = 1, 2$;
- $u^{(i)}_j(x, t)$ the j-th component of the i-th velocity field, $j = 1, \ldots, N$;

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- $u^{(i)} = \big(u_1^{(i)}, \dots, u_N^{(i)}\big)\,, \quad x = (x_1, \dots, x_N)\,, \quad t-$time;
- μ_{ij} , λ_{ij} - viscosity constants;
- Δ - Laplacian in $\mathbb{R}^N$, $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\big)$ - gradient operator, $\operatorname{div} u = \sum\limits_{i=1}^{N} \frac{\partial u_i}{\partial x_i}$.

For simplicity we start with the case where the flow domain is taken to be the N-dimensional parallelepiped Ω:

$$\Omega = \prod_{k=1}^{N} (0, d_k) = \{x \in \mathbb{R}^n | 0 < x_k < d_k\,,\ k = 1, \dots, N\}\,.$$

The boundary $\partial\Omega$ of Ω consists of the parts

$$S_k = \big(\{x_k = 0\} \cup \{x_k = d_k\}\big) \cap \bar{\Omega}\,,\ \partial\Omega = \bigcup_{k=1}^{N} S_k\,.$$

For the *viscosity constants* we require

$$\text{(3)} \qquad \begin{cases} \mu_{11} > 0\,,\ \mu_{22} > 0\,,\ \nu_{11} > 0\,,\ \nu_{22} > 0\,,\ \nu_{ij} = 2\mu_{ij} + \lambda_{ij}\,,\ 1 \le i\,,\ j \le 2\,, \\ 4\mu_{11}\mu_{22} - (\mu_{12} + \mu_{21})^2 > 0\,,\ 4\nu_{11}\nu_{22} - (\nu_{12} + \nu_{21})^2 > 0\,. \end{cases}$$

The *pressure law* has the form

$$\text{(4)} \qquad \begin{cases} p^{(i)}(\rho^{(1)}, \rho^{(2)}) = k^{(i)} \rho^{(i)} p(\rho^{(1)}, \rho^{(2)})\,,\ k^{(i)} = \text{const} > 0\,,\ i = 1, 2\,, \\ p(\rho^{(1)}, \rho^{(2)}) = \big(\rho^{(1)}/\rho^{(1)}_{\text{ref}} + \rho^{(2)}/\rho^{(2)}_{\text{ref}}\big)^{\gamma - 1}\,,\ \gamma = \text{const.} > 1\,. \end{cases}$$

The factor g in the interaction term is assumed to satisfy

$$\text{(5)} \qquad \begin{cases} g(t, x, \rho^{(1)}, \rho^{(2)}, u^{(2)} - u^{(1)}) = a_0 + a_1(\rho^{(1)} + \rho^{(2)})^{\theta_1} + a_2(1 + |u^{(2)} - u^{(1)}|^2)^{\theta_2}\,, \\ \text{a) } a_0 = \text{const.} \ge 0\,,\ a_1 = \text{const.} \ge 0\,,\ a_2 = \text{const.} \ge 0\,, \\ \text{b) } \theta_1 = \text{const.} > 0\,,\ \theta_1 \in \big[0, \dfrac{2}{N}\big)\,, \\ \text{c) } \theta_2 = \text{const.} > 0\,,\ \theta_2 \in \big[0, \frac{1}{N\gamma - 1}\big) \text{ for } N \ge 2\,,\ \theta_2 \in \big[0, \frac{1}{2\gamma - 2}\big) \text{ for } N = 1\,. \end{cases}$$

We have the initial condition

$$\text{(6)} \qquad \rho^{(i)}(x, t)|_{t=0} = \rho_0^{(i)}(x)\,, \qquad x \in \bar{\Omega}\,,\ i = 1, 2\,.$$

We consider the following mixed boundary condition for the velocity fields $u^{(i)}$:

$$\text{(7)} \qquad \begin{cases} u_k^{(i)} = 0 \text{ on } S_k \times [0, T]\,, & k = 1, \dots, N \\ \dfrac{\partial u_m^{(i)}}{\partial x_k} = 0 \text{ on } S_k \times [0, T]\,, & m \ne k\,,\ m, k = 1, 2, \dots, N\,. \end{cases}$$

Remark 1.1. a) If $N = 1$, then $k = 1$, $S_1 = \partial\Omega$, and we have just the Dirichlet boundary condition

$$u^{(i)} = 0 \text{ on } \partial\Omega \times [0, T]\,.$$

b) If $N = 2$ the boundary conditions (7) have the form

$$\begin{cases} u^{(i)} \cdot \vec{n} & = 0 \text{ on } \partial\Omega \times [0, T]\,, \\ \operatorname{curl} u^{(i)} & = 0 \text{ on } \partial\Omega \times [0, T]\,. \end{cases}$$

c) If $N = 3$ the boundary conditions (7) read

$$\begin{cases} u^{(i)} \cdot \vec{n} & = 0 \text{ on } \partial\Omega \times [0, T]\,, \\ \vec{n} \times \operatorname{curl} u^{(i)} & = 0 \text{ on } \partial\Omega \times [0, T]\,. \end{cases}$$

Here $\vec{n}$ is the outer normal vector at the boundary.

Remark 1.2. We treat all dimensions $N \ge 1$. The results and methods of proof hold and work analogously in the case of periodic boundary conditions and can be simply extended to the case of a mixture of l species, $l \ge 3$.

Definition 1.1. *A classical solution to problem (1)–(7) is a quadrupel of functions* $(u^{(1)}$, $u^{(2)}$, $\rho^{(1)}$, $\rho^{(2)})$ *such that*

$$u^{(1)}, u^{(2)} \in C^{2,1}\big(\bar{\Omega} \times [0,T]\big)\,; \quad \rho^{(1)}, \rho^{(2)} \in C^{1}\big(\bar{\Omega} \times [0,T]\big)\,;$$
$$\rho^{(1)}(x,t) > 0\,,\ \rho^{(2)}(x,t) > 0 \ \text{in } \bar{\Omega} \times [0,T]\,.$$

The main results of the article are contained in the following

Theorem 1.1. *Let the initial data* $\rho_0^{(1)}$, $\rho_0^{(2)}$ *satisfy* $\rho_0^{(1)}$, $\rho_0^{(2)} \in W^{l,r}(\Omega)$, $r > 1$, $l > 1$, $r(l-1) > N$, $0 < m_0 \le \rho_0^{(i)} \le M_0$, $i = 1, 2$, *where* m_0 , M_0 *are constants. Then there exists a global unique classical solution* $(u^{(1)}, u^{(2)}, \rho^{(1)}, \rho^{(2)})$ *of the boundary-initial-value problem (1)–(7), and there holds*

$$a)\ \frac{\partial^k \rho^{(i)}}{\partial t^k} \in L^{\infty}\big(0, T\,; W^{l-k,r}(\Omega)\big)\,,\ i = 1, 2$$
$$b)\ \frac{\partial^k u^{(i)}}{\partial t^k} \in L^{\infty}\big(0, T\,; W^{l+1-k,r}(\Omega)\big)\,,\ i = 1, 2$$

for $0 \le k \le l$.

Furthermore, there exist numbers m_1 *and* M_1 *such that*

$$0 < m_1 \le \rho^{(i)}(x,t) \le M_1 < \infty\,,\ (x,t) \in \bar{\Omega} \times [0,T]\,, \qquad i = 1, 2\,.$$

Strategy of Proof

The existence and uniqueness for classical solutions in a sufficiently small time interval is well-known and follows from the theory of [13],[14],[15]. Therefore, the main difficulty in studying the "global in time" problem is connected in obtaining *a priori estimates* with constants depending only on the data of the problem and the duration T of the time interval, but independent of the interval of existence of a local solution. Then a local solution can be extended to the whole interval $[0,T]$.

In section 2 the system for the effective viscous fluxes is established. Section 3 contains first estimates for the velocities and the densities. In section 4 we prove a global L^∞ - bound for the densities from above and from below. In the last section we establish $W^{2,p}$ - estimates for the velocities and $W^{1,p}$ - estimates for the densities, using an approach for obtaining $W^{1,\infty}$ - estimates for linear elliptic systems due to Yudovich [17], [18].

2. Auxiliary Results

We state some assertions that are used later. The lemmata (2.2) - (2.5) are simple inequalities for real numbers which are used for the proof of the boundedness assertions in section 4. The consideration concerning the effective viscous fluxes start with (15).

Lemma 2.1. *Let* $\Omega \subset \mathbb{R}^N (N \ge 1)$ *be an arbitrary bounded domain satisfying the cone condition.*
1) Then the following inequality is valid for every function $u \in W^{l,p}(\Omega) \cap L^q(\Omega)$, $l \ge 1$, $p > 1$, $q > 1$:

$$\|u\|_{W^{k,r}(\Omega)} \le c_1 \|u\|^{\alpha}_{W^{l,p}(\Omega} \cdot \|u\|^{1-\alpha}_{L^q(\Omega)}\,, \tag{8}$$

where $\frac{1}{r} = \frac{k}{N} + \alpha \cdot (\frac{1}{p} - \frac{l}{N}) + \frac{1-\alpha}{q}$, $\frac{k}{L} \le \alpha \le 1$.
If $l - k - \frac{N}{p}$ *integer,* $l - k - \frac{N}{p} \ge 0$ *and* $1 < p < \infty$, *then* $0 \le \alpha < 1$.

2) Furthermore, the following inequality is valid for every function $u \in \overset{\circ}{W}{}^{1,m}(\Omega)$ *or* $u \in W^{1,m}(\Omega)$, $\int\limits_{\Omega} u\,dx = 0$ *or* $u \in W^{1,m}(\Omega)$, $u|_{S_0} = 0$, $S_0 \subset \partial\Omega$, $mes_{\partial\Omega} S_0 > 0$

$$\|u\|_{L^q(\Omega)} \le C_2 \cdot \|\nabla u\|^{\alpha}_{L^m(\Omega)} \cdot \|u\|^{1-\alpha}_{L^r(\Omega)} , \tag{9}$$

where $\alpha = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{m} + \frac{1}{N})^{-1}$*; moreover, if* $m < N$ *then* $q \in [r, \frac{mN}{N-m}]$ *for* $r \le \frac{mN}{N-m}$, *and* $q \in [\frac{mN}{N-m}, r]$ *for* $r > \frac{Nm}{N-m}$*. If* $m \ge N$ *then* $q \in [r, \infty)$ *is arbitrary; moreover, if* $m > N$ *then inequality (9) is also valid for* $q = \infty$.

The positive constants C_1 , C_2 in inequalities (8), (9) are independent of the function $u(x)$. Inequalities (8) and (9) are particular case of the more general multiplicative inequalities proven in [4], [8], [6].

Lemma 2.2. *Let* $\nu_{ij}(i,j = 1,2)$ *be constants such that*

$$\nu_{11} > 0\,,\ \nu_{22} > 0\,,\ 4\nu_{11}\nu_{22} - (\nu_{12} + \nu_{21})^2 > 0\,.$$

Then there exists a number $\nu_{00} > 0$ *such that*

$$1 - \frac{\nu_{12}\,\nu_{00}}{\nu_{22}} > 0\,,\ 1 - \frac{\nu_{21}}{\nu_{00}\,\nu_{11}} > 0\,.$$

Proof: We consider the following four cases:

(i) If $\nu_{12}, \nu_{21} \le 0$, then choose $\nu_{00} = 1$.
(ii) If $\nu_{12} \le 0$, $\nu_{21} > 0$, choose $\nu_{00} = 2\frac{\nu_{21}}{\nu_{11}}$.
(iii) If $\nu_{12} > 0$, $\nu_{21} \le 0$, choose $\nu_{00} = \frac{1}{2}\frac{\nu_{22}}{\nu_{12}}$.
(iv) If $\nu_{12} > 0$, $\nu_{21} > 0$, choose $\nu_{00} = \frac{1}{2}\Big(\frac{\nu_{21}}{\nu_{11}} + \frac{\nu_{22}}{\nu_{12}}\Big)$.

With these choices the statement of Lemma 2.2 is satisfied in all cases. □

For further use we define

$$M = \min\{1, 1 - \frac{\nu_{12}\,\nu_{00}}{\nu_{22}}, 1 - \frac{\nu_{21}}{\nu_{00}\,\nu_{11}}\}, \qquad M \in (0,1]\,. \tag{10}$$

Lemma 2.3. *Let* ν_{ij} , $i,j = 1,2$, *be constants such that*

$$\nu_{11} > 0\,,\ \nu_{22} > 0\,,\ 4\nu_{11}\nu_{22} - (\nu_{12} + \nu_{21})^2 > 0\,,$$

and let $m > 1$, $k^{(1)} > 0$, $k^{(2)} > 0$, *be constants. Then there exist numbers* $D^{(1)} > 0$, $D^{(2)} > 0$ *such that for all* $x \ge 0$, $y \ge 0$ *the inequality*

$$\begin{aligned} &D^{(1)}k^{(1)}\nu_{22}x^{m+1} + D^{(2)}k^{(2)}\nu_{11}y^{m+1} - D^{(1)}k^{(2)}\nu_{12}x^m y - D^{(2)}k^{(1)}\nu_{21}y^m x \\ &\ge M\big(D^{(1)}k^{(1)}\nu_{22}x^{m+1} + D^{(2)}k^{(2)}\nu_{11}y^{m+1}\big) \end{aligned}$$

holds, where M *is the constant from (10).*

Proof: To prove the Lemma we define

$$\begin{cases} D^{(1)} &= \frac{\nu_{11}}{\nu_{22}}\Big(\frac{k^{(1)}}{k^{(2)}}\Big)^m \nu_{00}^{m+1}\,, \quad D^{(2)} = 1\,, \\ a &= x\,\big(D^{(1)}k^{(1)}\nu_{22}\big)^{\frac{1}{m+1}}\,, \quad b = y\,\big(D^{(2)}k^{(2)}\nu_{11}\big)^{\frac{1}{m+1}} \end{cases} \tag{11}$$

where ν_{00} comes from Lemma 2.2.

The left hand side of the inequality stated in Lemma 2.3 has the form

$$\begin{aligned} F(a,b) &= a^{m+1} + b^{m+1} - \frac{\nu_{00}\,\nu_{12}}{\nu_{22}}a^m b - \frac{\nu_{21}}{\nu_{00}\,\nu_{11}}ab^m \\ &= a^{m+1} + b^{m+1} - a^m b - ab^m + \Big(1 - \frac{\nu_{00}\,\nu_{12}}{\nu_{22}}\Big)a^m b + \Big(1 - \frac{\nu_{21}}{\nu_{00}\nu_{11}}\Big)ab^m \\ &\ge M(a^{m+1} + b^{m+1}) + (1 - M)(a^m - b^m)(a - b)\,, \end{aligned}$$

where M has been defined in (10). Thus

$$F(a,b) \geq M(a^{m+1}+b^{m+1})\,,$$

and the lemma is proved. □

Lemma 2.4. *Let $\gamma>1$, $m>1$ and $D^{(1)}$, $D^{(2)}>0$ be constants. Then there holds, for all $x,y\geq 0$, the inequality*

$$(x+y)^{\gamma} \leq K_{00}\big[(D^{(1)}x^{m+1}+D^{(2)}y^{m+1})(x+y)^{\gamma-1}\big]^{\delta_1}(x+y)^{1-\delta_1}\,,$$

where

$$\delta_1=\frac{\gamma-1}{m+\gamma-1}\ ,\ K_{00}=\left[\frac{\big((D^{(1)})^{1/m}+(D^{(2)})^{1/m}\big)^m}{D^{(1)}D^{(2)}}\right]^{\delta_1}.$$

Proof: It is easy to see that the statement follows from the inequality

$$(x+y)^{m+1} \leq K_{00}^{1/\delta_1}(D^{(1)}x^{m+1}+D^{(2)}y^{m+1})\,, \qquad x\geq 0\,,\ y\geq 0\,.$$

For proving this, one considers the minimization problem: Find (a,b) such that

$$D^{(1)}a^{m+1}+D^{(2)}b^{m+1}=\min!\,,\quad a+b=1\,,\quad a\geq 0\,,\quad b\geq 0\,.$$

One finds that

$$\min_{\substack{a+b=1\\ a\geq 0,b\geq 0}}(D^{(1)}a^{m+1}+D^{(2)}b^{m+1})=\frac{D^{(1)}D^{(2)}}{\big((D^{(1)})^{\frac{1}{m}}+(D^{(2)})^{\frac{1}{m}}\big)^m}$$

and Lemma 2.4 follows. □

Lemma 2.5. *Let $\gamma>1$, $m>1$, and $D^{(1)}>0$, $D^{(2)}>0$ be constants. Then there holds for all $x\geq 0$, $y\geq 0$ the inequality*

$$D^{(1)}x^m+D^{(2)}y^m \leq K_{01}\big[(D^{(1)}x^{m+1}+D^{(2)}y^{m+1})(x+y)^{\gamma-1}\big]^{\delta_3}(x+y)^{\delta_4}\,. \tag{12}$$

Here δ_3 , δ_4 are positive constants, $0<\delta_3<\frac{m}{m+\gamma}$, $\delta_4=m-(m+\gamma)\delta_3$ and

$$K_{01}=2^{1/(m+1)}(D^{(1)}+D^{(1)})^{1-\delta_3}\,.$$

Proof: By homogenity, it suffices to prove (12) for all $x\geq 0$, $y\geq 0$ such that $x+y=1$. By a convexity argument we have

$$\Big(\frac{r_1^m+r_2^m}{2}\Big) \leq \Big(\frac{r_1^{m+1}+r_2^{m+1}}{2}\Big)^{1/(m+1)}\,, \qquad r_1\geq 0\,,\ r_2\geq 0\,,\ m>1\,.$$

Hence we conclude

$$\Big(\frac{D^{(1)}a^m+D^{(2)}b^m}{2}\Big)^{1/m} \leq \Big(\frac{(D^{(1)})^{1+1/m}a^{m+1}+(D^{(2)})^{1+1/m}b^{m+1}}{2}\Big)^{1/(m+1)}.$$

and we continue to estimate

$$\begin{aligned} D^{(1)}a^m+D^{(2)}b^m &\leq 2^{\frac{1}{m+1}}(D^{(1)}+D^{(2)})^{\frac{1}{m+1}}(D^{(1)}a^{m+1}+D^{(2)}b^{m+1})^{\frac{m}{m+1}} \\ &\leq 2^{\frac{1}{m+1}}(D^{(1)}+D^{(2)})^{\frac{1}{m+1}}(D^{(1)}a^{m+1}+D^{(2)}b^{m+1})^{\delta_3}(D^{(1)}a^{m+1}+D^{(2)}b^{m+1})^{\frac{m}{m+1}-\delta_3} \\ &\leq 2^{\frac{1}{m+1}}(D^{(1)}+D^{(2)})^{1-\delta_3}(D^{(1)}a^{m+1}+D^{(2)}b^{m+1})^{\delta_3}\,. \end{aligned}$$

Here we have used, that $\frac{m}{m+1}-\delta_3\geq 0$, $a\geq 0$, $b\geq 0$, $a+b=1$. Thus (12) is proved and the Lemma follows. □

Remark 2.1. *¿From Lemma 2.5 there follows*

$$(D^{(1)}x^m+D^{(2)}y^m) \leq K_{02}\big[(D^{(1)}x^{m+1}+D^{(2)}y^{m+1})(x+y)^{\gamma-1}\big]^{\delta_2}(x+y)^{1-\delta_2}\,, \tag{13}$$

where $\delta_2=\frac{m-1}{m+\gamma-1}$ and $K_{02}=2^{1/(m+1)}(D^{(1)}+D^{(2)})^{\gamma/(m+\gamma-1)}$.

Remark 2.2. *During our consideration we use that the differential equations (1) and the boundary conditions (7) imply the additional natural boundary conditions (in the generalized sense)*

$$(14)\qquad \begin{cases} \frac{\partial}{\partial \vec{n}}\big(\nu_{11}\,\mathrm{div}\,u^{(1)} + \nu_{12}\,\mathrm{div}\,u^{(2)} - p^{(1)} + p_1^{(1)}\big)|_{\partial\Omega} = 0\,, \\ \frac{\partial}{\partial \vec{n}}\big(\nu_{21}\,\mathrm{div}\,u^{(1)} + \nu_{22}\,\mathrm{div}\,u^{(2)} - p^{(2)} + p_1^{(2)}\big)|_{\partial\Omega} = 0\,, \\ p_1^{(i)} = \frac{1}{mes\Omega}\int\limits_\Omega p^{(i)}(\rho^{(1)},\rho^{(2)})\,dx\,, \qquad i=1,2\,, \quad \textit{for all } t\in[0,T]\,. \end{cases}$$

We now derive an "algebraic" equation between the quantities $\mathrm{div}\,u^{(i)}$, $\rho^{(i)}$ which corresponds to the equation of effective viscous flux in the one component case.

We introduce the following function φ defined by

$$(15)\qquad \begin{cases} \Delta\varphi &= \mathrm{div}\,\big((u^{(2)} - u^{(1)})g\big)\,, \\ \frac{\partial\varphi}{\partial\vec{n}}|_{\partial\Omega} &= 0\,, \qquad \int\limits_\Omega \varphi\,dx = 0\,, \qquad \text{for all } t\in[0,T]. \end{cases}$$

¿From (1) and (15) we deduce

$$-\Delta\Big(\sum_{j=1}^{2}\nu_{ij}\,\mathrm{div}\,u^{(j)}\Big) + \Delta\big((-1)^i\varphi + p^{(i)} - p_1^{(i)}\big) = 0\,, \qquad i=1,2\,.$$

Then we find from (14) and (15) after some calculation

$$(16)\qquad \begin{cases} \nu_{11}\,\mathrm{div}\,u^{(1)} + \nu_{12}\,\mathrm{div}\,u^{(2)} &= -\varphi + p^{(1)} - p_1^{(1)} \\ \nu_{21}\,\mathrm{div}\,u^{(1)} + \nu_{22}\,\mathrm{div}\,u^{(2)} &= \;\;\varphi + p^{(2)} - p_1^{(2)} \end{cases}$$

¿From (16) we eliminate $\mathrm{div}\,u^{(i)}$ using the number $D^{(0)} = \nu_{11}\nu_{22} - \nu_{12}\nu_{21} > 0$. Then we find the equations for the *effective viscous fluxes*:

$$(17)\qquad \begin{cases} D^{(0)}\,\mathrm{div}\,u^{(1)} &= -(\nu_{22}+\nu_{12})\varphi + \nu_{22}(p^{(1)} - p_1^{(1)}) - \nu_{12}(p^{(2)} - p_1^{(2)})\,, \\ D^{(0)}\,\mathrm{div}\,u^{(2)} &= \;\;(\nu_{11}+\nu_{21})\varphi + \nu_{11}(p^{(2)} - p_1^{(2)}) - \nu_{21}(p^{(1)} - p_1^{(1)})\,. \end{cases}$$

3. The First A Priori Estimate for the Velocities and the Densities

Differently to the usual procedure in compressible flow theory, we do not start with the usual energy estimate coming from the momentum equation by testing with $u^{(i)}$, $i=1,2$, but we establish in the first step , L^q - bounds for the densities via the equation of the effective viscous fluxes.

Let $(u^{(1)},u^{(2)},\rho^{(1)},\rho^{(2)})$ be a classical solution of the problem under consideration.
1) From (2) and (7) there follows

$$(18)\qquad \int\limits_\Omega \rho^{(i)}(x,t)\,dx = \int\limits_\Omega \rho_0^{(i)}(x)\,dx\,, \quad i=1,2,\,, \text{ for all } t\in[0,T]\,.$$

2) Let $m = \text{const.} \geq \gamma > 1$. From (2) and (7) we obtain the equations $(i=1,2)$:

$$(19)\qquad \frac{1}{m-1}\cdot\frac{d}{dt}\int\limits_\Omega \big(\rho^{(i)}\big)^m\,dx + \int\limits_\Omega \big(\rho^{(i)}\big)^m\cdot\mathrm{div}\,u^{(i)}\,dx = 0\,, \qquad \text{for all } t\in[0,T]\,.$$

This will be used for a certain sequence of numbers $m\to\infty$; the aim is to obtain an L^∞ - bound $\rho^{(i)}$ (which reminds us to Moser's iteration technique).

Let $D^{(2)} = 1$ and $D^{(1)} = \frac{\nu_{11}\cdot\nu_{00}}{\nu_{22}}\cdot\Big(\frac{k^{(1)}\nu_{00}}{k^{(2)}}\Big)^m$ (see Lemma 2.3 where these constants have been introduced.).

Now, we substitute $\operatorname{div} u^{(i)} (i=1,2)$ in formula (19) by the terms in (17). Then we obtain the following identities:

$$
(20)\quad \begin{cases}
\frac{1}{m-1}\cdot\frac{d}{dt}\int\limits_\Omega D^{(1)}(\rho^{(1)})^m + D^{(2)}(\rho^{(2)})^m\,dx \\
\quad +\frac{1}{D^{(0)}}\int\limits_\Omega D^{(1)}(\rho^{(1)})^m\big[\nu_{22}(p^{(1)}-p_1^{(1)})-\nu_{12}(p^{(2)}-p_1^{(2)})-(\nu_{22}+\nu_{12})\varphi\big]\,dx \\
\quad +\frac{1}{D^{(0)}}\int\limits_\Omega D^{(2)}(\rho^{(2)})^m\big[\nu_{11}(p^{(2)}-p_1^{(2)})-\nu_{21}(p^{(1)}-p_1^{(1)})+(\nu_{11}+\nu_{21})\varphi\big]\,dx = 0\,, \\
D^{(0)}=\nu_{11}\nu_{22}-\nu_{12}\nu_{21}>0\,,\ \ p_1^{(1)}=\frac{1}{mes\Omega}\cdot\int\limits_\Omega p^{(1)}\,dx\,,\quad p_1^{(2)}=\frac{1}{mes\Omega}\cdot\int\limits_\Omega p^{(2)}\,dx\,.
\end{cases}
$$

Let us define (for all $t\in[0,T]$):

$$
(21)\quad \begin{cases}
I_1 &= \frac{1}{D^{(0)}}\int\limits_\Omega D^{(1)}(\rho^{(1)})^m\big[\nu_{22}p_1^{(1)}-\nu_{12}p_1^{(2)}\big]+D^{(2)}(\rho^{(2)})^m\big[p_1^{(2)}\nu_{11}-p_1^{(1)}\nu_{21}\big]\,dx\,, \\
I_2 &= \frac{1}{D^{(0)}}\int\limits_\Omega \big[D^{(1)}(\rho^{(1)})^m(\nu_{22}+\nu_{12})-D^{(2)}(\rho^{(2)})^m(\nu_{11}+\nu_{21})\big]\cdot\varphi\,dx\,, \\
I_3 &= \frac{1}{D^{(0)}}\int\limits_\Omega D^{(1)}(\rho^{(1)})^m\big[\nu_{22}p^{(1)}-\nu_{12}p^{(2)}\big]+D^{(2)}(\rho^{(2)})^m\big[p^{(2)}\nu_{11}-p^{(1)}\nu_{21}\big]\,dx\,, \\
y(t) &= \int\limits_\Omega D^{(1)}(\rho^{(1)})^m+D^{(2)}(\rho^{(2)})^m\,dx\,, \\
A(t) &= \int\limits_\Omega \big[D^{(1)}(\rho^{(1)})^{m+1}+D^{(2)}(\rho^{(2)})^{m+1}\big](\rho^{(1)}+\rho^{(2)})^{\gamma-1}\,dx\,.
\end{cases}
$$

In the rest of this section we confine ourselves to the case $m=\gamma$.

2a) By (4) we have the estimate

$$
I_1\le Cy(t)\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})^\gamma\,dx\,.
$$

Furthermore, from Lemma 2.4, Lemma 2.5 and Hölder's inequality:

$$
\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})^\gamma\,dx\le C\big(A(t)\big)^{\frac{\gamma-1}{2\gamma-1}}\Big[\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})\,dx\Big]^{1-\frac{\gamma-1}{2\gamma-1}}\,,
$$

$$
y(t)\le C\big(A(t)\big)^{\frac{\gamma-1}{2\gamma-1}}\Big[\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})\,dx\Big]^{1-\frac{\gamma-1}{2\gamma-1}}
$$

Hence we obtain the inequality

$$
(22)\qquad I_1\le C\big(A(t)\big)^{\frac{2\gamma-2}{2\gamma-1}}\,,
$$

with positive constants C.

2b) Now, we look at the term which can be estimated in the following way

$$
(23)\quad \begin{aligned}
I_2 &\le C\int\limits_\Omega\big(D^{(1)}\cdot(\rho^{(1)})^\gamma+D^{(2)}\cdot(\rho^{(2)})^\gamma\big)\cdot|\varphi|\,dx \\
&\le C\Big[\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})^{\gamma\cdot\frac{q_1}{q_1-1}}\,dx\Big]^{1-\frac{1}{q_1}}\cdot\|\varphi_1\|_{L^{q_1}(\Omega)} \\
&\quad +C\Big[\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})^\gamma\cdot\frac{q_2}{q_2-1}\,dx\Big]^{1-\frac{1}{q_2}}\cdot\|\varphi_2\|_{L^{q_2}(\Omega)} \\
&\le C\Big[\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})^{2\gamma}\,dx\Big]^{\frac{\gamma-1+\frac{1}{q_1}}{2\gamma-1}}\cdot\|\varphi_1\|_{L^{q_1}(\Omega)} \\
&\quad +C\Big[\int\limits_\Omega(\rho^{(1)}+\rho^{(2)})^{2\gamma}\,dx\Big]^{\frac{\gamma-1+\frac{1}{q_2}}{2\gamma-1}}\cdot\|\varphi_2\|_{L^{q_2}(\Omega)}\,,
\end{aligned}
$$

with positive constants C, $q_1 = \text{const.} > 2$, $q_2 = \text{const.} > 2$.

The functions φ_1 and φ_2 are defined in the following way: We write $\varphi = \varphi_1 + \varphi_2$ and define φ_1, φ_2 as solutions to the problems:

$$\begin{cases} \Delta\varphi_1 &= \operatorname{div}\left((a_0 + a_1(\rho^{(1)} + \rho^{(2)})^{\theta_1})(u^{(2)} - u^{(1)})\right), \\ \frac{\partial\varphi_1}{\partial \vec{n}}\Big|_{\partial\Omega} &= 0, \qquad \int\limits_\Omega \varphi_1\, dx = 0, \quad \forall t \in [0,T], \end{cases} \tag{24}$$

$$\begin{cases} \Delta\varphi_2 &= \operatorname{div}\left(a_2(1 + |u^{(2)} - u^{(1)}|^2)^{\theta_2} \cdot (U^{(2)} - U^{(1)})\right), \\ \frac{\partial\varphi_2}{\partial \vec{n}}\Big|_{\partial\Omega} &= 0, \qquad \int\limits_\Omega \varphi_2\, dx = 0, \quad \forall t \in [0,T]. \end{cases} \tag{25}$$

On account of Lemma 2.3 we find:

$$\begin{aligned} I_3 &\geq \frac{M}{D^{(0)}} \int\limits_\Omega \left[D^{(1)}k^{(1)}\nu_{22}(\rho^{(1)})^{\gamma+1} + D^{(2)}k^{(2)}\nu_{11}(\rho^{(2)})^{\gamma+1}\right]\Big(\frac{\rho^{(1)}}{\rho^{(1)}_{ref}} + \frac{\rho^{(2)}}{\rho^{(2)}_{ref}}\Big)^{\gamma-1} dx \\ &\geq C \int\limits_\Omega \left[D^{(1)}(\rho^{(1)})^{\gamma+1} + D^{(2)}(\rho^{(2)})^{\gamma+1}\right](\rho^{(1)} + \rho^{(2)})^{\gamma-1}\, dx \end{aligned} \tag{26}$$

where $M = \text{const.} > 0$ comes from Lemma 2.3, $C = \text{const.} > 0$. So, we conclude from (20) in the case $m = \gamma$ the inequality

$$\begin{aligned} &\frac{1}{\gamma - 1} \cdot \frac{d}{dt} y(t) + C\, A(t) \leq C\left(A(t)\right)^{\frac{2\gamma-2}{2\gamma-1}} + \\ &\quad + C\, \|\rho^{(1)} + \rho^{(2)}\|_{L^{2\gamma}(\Omega)}^{\frac{2\gamma(\gamma-1+\frac{1}{q_1})}{2\gamma-1}} \cdot \|\varphi_1\|_{L^{q_1}(\Omega)} + C\, \|\rho^{(1)} + \rho^{(2)}\|_{L^{2\gamma}(\Omega)}^{\frac{2\gamma(\gamma-1+\frac{1}{q_2})}{2\gamma-1}} \cdot \|\varphi_2\|_{L^{q_2}(\Omega)}. \end{aligned}$$

¿From this, there follows

$$\frac{d}{dt} y(t) + C\, A(t) \leq C\left(1 + \|\varphi_1\|_{L^{q_1}(\Omega)}^{\frac{2\gamma-1}{\gamma-\frac{1}{q_1}}} + \|\varphi_2\|_{L^{q_2}(\Omega)}^{\frac{2\gamma-1}{\gamma-\frac{1}{q_2}}}\right), \tag{27}$$

with C being a positive constant.

3) From (1) and the boundary condition (7) one obtains via (3):

$$\begin{aligned} &\|\nabla u^{(1)}\|^2_{L^2(\Omega)} + \|\nabla u^{(2)}\|^2_{L^2(\Omega)} + \int\limits_\Omega g\, |u^{(2)} - u^{(1)}|^2\, dx \\ &\quad \leq C\left(\|p^{(1)} - p^{(1)}_1\|^2_{L^2(\Omega)} + \|p^{(2)} - p^{(2)}_1\|^2_{L^2(\Omega)}\right) \leq C\left(1 + \|\rho^{(1)} + \rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right). \end{aligned} \tag{28}$$

4) Let $q_1 > 2$, $q_2 > 2$. Then the problems (24), (25) are solvable and by the usual L^p-theory for elliptic operators (see also Lemma 2.1) we have the following estimates:

$$\|\varphi_1\|_{L^{q_1}(\Omega)} \leq C\|\nabla\varphi_1\|_{L^{r_1}(\Omega)} \leq C\,\|\left(a_0 + a_1 \cdot (\rho^{(1)} + \rho^{(2)})^{\theta_1}\right) \cdot |u^{(2)} - u^{(1)}|\,\|_{L^{r_1}(\Omega)}, \tag{29}$$

$$\|\varphi_2\|_{L^{q_2}(\Omega)} \leq C\|\nabla\varphi_2\|_{L^{r_2}(\Omega)} \leq C\,\|a_2 \cdot (1 + |u^{(2)} - u^{(1)}|^2)^{\theta_2} \cdot |u^{(2)} - u^{(1)}|\,\|_{L^{r_2}(\Omega)}, \tag{30}$$

where r_1 , $r_2 \in (1, +\infty)$ for $N = 1$ and $r_1 \in [\frac{Nq_1}{N+q_1}\,,\ +\infty)$, $r_2 \in [\frac{Nq_2}{N+q_2}\,,\ +\infty)$ for $N \geq 2$. The numbers satisfy $\frac{Nq_1}{N+q_1} > 1$, $\frac{Nq_2}{N+q_2} > 1$ for $N \geq 2$, since $q_1, q_2 \in (2, +\infty)$.

Now, we estimate the terms at the right hand side of (29), (30):

4a) In the case $N = 1$ we set (observing that here $0 < \theta_1 < 2$):

$$\begin{cases} q_1 &= \frac{4(6-\theta_1)}{(2-\theta_1)(4-\theta_1)} > 2, \qquad \text{since } 0 < \theta_1 < 2; \\ r_1 &= \frac{2}{1+\theta_1}, \qquad \text{if} \quad 0 < \theta_1 < 1; \\ r_1 &= \frac{6-\theta_1}{4}, \qquad \text{if} \quad 1 \leq \theta_1 < 2. \end{cases}$$

¿From this there follows:

$$
(31) \qquad \begin{cases} 1) & 1 < r_1 < 2\,; \\ 2) & \theta_1 r_1 \le 2 - r_1\,, \qquad\qquad\qquad \text{if } 0 < \theta_1 < 1\,; \\ 3) & 2 - r_1 < \theta_1 r_1 < 2\gamma(2 - r_1)\,, \ \text{if } 1 \le \theta_1 < 2\,; \\ 4) & \frac{2}{q_1} + \theta_1 < \frac{2}{r_1}\,, \qquad\qquad\qquad \text{if } 1 \le \theta_1 < 2\,. \end{cases}
$$

In the case $N \ge 2$, we set (observe $0 < \theta_1 < \frac{2}{N}$):

$$
\begin{cases} r_1 & = \frac{2N}{N+2} + \varepsilon\,,\ 0 < \varepsilon \le \frac{2N}{N+2} \cdot \frac{2/N - \theta_1}{1+\theta_1}\,; \\ q_1 & = 2 + \delta\,,\ \delta = \varepsilon \cdot \left(1 + \frac{2}{N}\right)^2 . \end{cases}
$$

¿From this we get

$$
(32) \qquad \begin{cases} 5)\ 1 \le r_1 \cdot \left(\frac{1}{N} + \frac{1}{q_1}\right)\,, \qquad r_1 < 1\,; \\ 6)\ \theta_1 r_1 \le 2 - r_1\,. \end{cases}
$$

Furthermore, from (29) and Hölder's inequality ($1 < r_1 < 2$) we have

$$
\|\varphi_1\|_{L^{q_1}(\Omega)} \le C\, a_0\, \|u^{(2)} - u^{(1)}\|_{L^2(\Omega)} + C\, a_1 \Big(\int\limits_\Omega (\rho^{(1)} + \rho^{(2)})^{\theta_1} |u^{(2)} - u^{(1)}|^2\, dx \Big)^{1/2} \ \Big(\int\limits_\Omega (\rho^{(1)} + \rho^{(2)})^{\frac{\theta_1 r_1}{2-r_1}}\, dx \Big)^{\frac{2-r_1}{2r_1}} .
$$

If $N \ge 2$ or ($N = 1$ and $0 < \theta_1 < 1$) we conclude from (31) and (32) the property $\theta_1 r_1 \le 2 - r_1$. Thus, using also the estimate (18) and the inequality (28) we obtain

$$
(33) \qquad \|\varphi_1\|_{L^{q_1}(\Omega)} \le C \left(1 + \|\rho^{(1)} + \rho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma}\right)^{1/2} .
$$

If $N = 1$ and $1 \le \theta_1 < 2$. Then there holds the representation

$$
\frac{\theta_1 \cdot r_1}{2 - r_1} = 2\gamma \cdot \tau + 1 - \tau\,,\ \tau = \frac{\frac{\theta_1 r_1}{2-r_1} - 1}{2\gamma - 1} \in (0,1)\,.
$$

This gives

$$
\int\limits_\Omega (\rho^{(1)} + \rho^{(2)})^{\frac{\theta_1 r_1}{2-r_1}}\, dx \le C \Big[\int\limits_\Omega (\rho^{(1)} + \rho^{(2)})^{2\gamma}\, dx \Big]^\eta ,
$$

where

$$
\eta = \frac{\theta_1/2 - (2 - r_1)/(2r_1)}{2\gamma - 1}\,.
$$

Hence we conclude via inequality (28) that

$$
(34) \qquad \|\varphi_1\|_{L^{q_1}(\Omega)} \le C \left(1 + \|\rho^{(1)} + \rho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma}\right)^{\frac{1}{2}+\eta} .
$$

It is important to note that

$$
\left(\frac{1}{2} + \eta\right) \cdot \frac{2\gamma - 1}{\gamma - (1/q_1)} = \frac{1}{\gamma - (1/q_1)} \left(\gamma - \frac{1}{2} + \frac{\theta_1}{2} - \frac{1}{r_1} + \frac{1}{2}\right) < 1\,.
$$

In fact, this estimate follows from (31).

4b) In the case $N = 1$ we define (taking into account that $0 < \theta_2 < \frac{1}{2\gamma - 2}$)

$$
q_2 = \frac{4(1 + \theta_2)}{1 - 2\theta_2(\gamma - 1)} > 2\,;\ r_2 = \frac{2\theta_2 + 2}{2\theta_2 + 1} > 1\,.
$$

Then, from (30), we derive the inequality

$$
(35) \qquad \begin{aligned} \|\varphi_2\|_{L^{q_2}(\Omega)} &\le C \Big(1 + \int\limits_\Omega \left(1 + |u^{(2)} - u^{(1)}|^2\right)^{\theta_2} |u^{(2)} - u^{(1)}|^2\, dx\Big)^{(2\theta_2+1)/(2\theta_2+2)} \\ &\le C \left(1 + \|\rho^{(1)} + \rho^{(2)}\|_{L^{2\gamma}(\Omega)}^{2\gamma}\right)^{(2\theta_2+1)/(2\theta_2+2)} . \end{aligned}
$$

It is important to observe that

$$\frac{2\theta_2+1}{2\theta_2+2}\cdot\frac{2\gamma-1}{\gamma-1/q_2}<1\,,\ \text{since}\ q_2=\frac{4(1+\theta_2)}{1-2\theta_2(\gamma-1)}\,.$$

In the case $N=2$ we define (taking into account that $0<\theta_2<\frac{1}{2\gamma-1}$):

$$q_2=\frac{4(1+\theta_2)}{1-\theta_2(2\gamma-1)}>2\,;\ r_2=\frac{2q_2}{2+q_2}>1\,,\ \text{since}\ q_2>2\,.$$

Then we obtain from (30), (28) and Sobolev's imbedding theorem the estimates

$$\begin{aligned}\|\varphi_2\|_{L^{q_2}(\Omega)} &\le C\,\|\sqrt{g}\,|u^{(2)}-u^{(1)}|\,\|_{L^2(\Omega)}\cdot\Big[\int\limits_\Omega\big(1+|u^{(2)}-u^{(1)}|^{\theta_2 q_2}\big)\,dx\Big]^{1/q_2}\\ &\le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{(1/2)+(\theta/2)}\,.\end{aligned}\tag{36}$$

Here it is important to note that

$$\Big(\frac12+\frac{\theta_2}{2}\Big)\cdot\frac{2\gamma-1}{\gamma-1/q_2}<1\,,\ \text{since}\ q_2=\frac{4(1+\theta_2)}{1-\theta_2\cdot(2\gamma-1)}\,.$$

In the case $N\ge3$ we define (here we have $0<\theta_2<\frac{1}{N\gamma-1}$):

$$\begin{aligned}&\delta=\frac78\,,\ r_2=\frac{2\theta_2+2}{2\theta_2+1}\cdot\delta+\frac{2N}{N-2}\cdot\frac{1}{2\theta_2+1}\cdot(1-\delta)\,;\\ &q_2=\frac{Nr_2}{N-r_2}>2\,,\ \text{since the inequalities}\\ &\frac{2N}{N-2}<r_2<N\,,\ 0<\theta_2<\frac{1}{N\gamma-1}\ \text{and}\ N\ge3\ \text{are satisfied}\,.\end{aligned}\tag{37}$$

Then we have from (30), (28) and Sobolev's imbedding theorem:

$$\begin{aligned}\|\varphi_2\|_{L^{q_2}(\Omega)} &\le C\Big(1+\int\limits_\Omega|u^{(2)}-u^{(1)}|^{(2\theta_2+1)r_2}\,dx\Big)^{1/r_2}\\ &\le C\Big(1+\Big[\int\limits_\Omega|u^{(2)}-u^{(1)}|^{2\theta_2+2}\,dx\Big]^{\delta}\cdot\Big[\int\limits_\Omega|u^{(2)}-u^{(1)}|^{(2N)/(N-2)}\,dx\Big]^{1-\delta}\Big)^{1/r_2}\\ &\le C\Big(1+\|\sqrt{g}\,|u^{(2)}-u^{(1)}|\,\|^{\delta/r_2}_{L^2(\Omega)}\cdot\|\nabla(u^{(2)}-u^{(1)})\|^{\frac{(2N)(1-\delta)}{(N-2)r_2}}_{L^2(\Omega)}\Big)\\ &\le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{\delta/r_2+\frac{N}{N-2}(1-\delta)/r_2}\,.\end{aligned}\tag{38}$$

It is important that

$$\Big(\frac{\delta}{r_2}+\frac{1-\delta}{r_2}\cdot\frac{N}{N-2}\Big)\cdot\frac{2\gamma-1}{\gamma-1/q_2}<1\,.$$

5) From the above considerations in all cases there holds the estimate

$$\|\varphi_1\|^{\frac{2\gamma-1}{\gamma-1/q_1}}_{L^{q_1}(\Omega)}+\|\varphi_2\|^{\frac{2\gamma-1}{\gamma-1/q_2}}_{L^{q_2}(\Omega)}\le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{\nu_1}\,,\ \nu_1\in(0,1)\,.$$

Hence, we find from (27) the inequality

$$\frac{d}{dt}y(t)+C_1\,A(t)\le C_2\,,$$

where C_1 , C_2 are positive constants.

¿From this we conclude

$$\begin{cases}\sup\limits_{0<t<T}\Big[\|\rho^{(1)}(t)\|_{L^\gamma(\Omega)}+\|\rho^{(2)}(t)\|_{L^\gamma(\Omega)}\Big] &\le C\,,\\ \int\limits_0^T\|\rho^{(1)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\rho^{(2)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}\,dt &\le C\,.\end{cases}\tag{39}$$

Further we obtain from (39), (28) and boundary condition (7) that

$$\begin{cases} \int_0^T \|u^{(1)}(t)\|^2_{W^{1,2}(\Omega)} + \|u^{(2)}(t)\|^2_{W^{1,2}(\Omega)}\, dt & \le C\,, \\ \int_0^T \|g^{1/2}|u^{(2)} - u^{(1)}|(t)\|^2_{L^2(\Omega)}\, dt & \le C\,. \end{cases} \tag{40}$$

4. Estimates for the Densities of the Mixture from above and below

In this section, we derive L^∞ - bounds for the densities and its inverses from the effective viscous flux equations. The technique of proof reminds to the method of J. Moser for elliptic equations. In our case, the interaction term needs some additional treatment.

First, let us present some estimates for the function $\varphi(x,t)$.

1) If we are in the case $N=1$, then we set (observe $0<\theta_1<2$, $0<\theta_2<\frac{1}{2\gamma-2}$)

$$\varepsilon_1 = \min\left\{ \frac{1}{2\theta_2+1}\,,\, \frac{\frac{2\gamma}{\theta_1}-1}{\frac{2\gamma}{\theta_1}+1} \right\}\,,\ \varepsilon_1\in(0,1)\,.$$

Then we have

$$1+\varepsilon_1 \le \frac{2\theta_2+2}{2\theta_2+1}\,,\ \frac{1+\varepsilon_1}{1-\varepsilon_1}\le 2\gamma\theta_1\,.$$

¿From the imbedding theorem and equation (15) we find

$$\begin{aligned} \|\varphi\|_{L^\infty(\Omega)} &\le C\,\|\nabla\varphi\|_{L^{1+\varepsilon_1}(\Omega)} \le C\,\|g|u^{(2)}-u^{(1)}|\,\|_{L^{1+\varepsilon_1}(\Omega)} \\ &\le C\left(1 + a_0\Big[\int_\Omega |u^{(2)}-u^{(1)}|^{1+\varepsilon_1}\,dx\Big]^{1/(1+\varepsilon_1)}\right) + \\ &+ C\left(a_2\Big[\int_\Omega |u^{(2)}-u^{(1)}|^{(2\theta_2+1)(1+\varepsilon_1)}\,dx\Big]^{1/1+\varepsilon_1}\right. \\ &\left.\quad + a_1\Big[\int_\Omega (\rho^{(1)}+\rho^{(2)})^{\theta_1(1+\varepsilon_1)}\cdot|u^{(2)}-u^{(1)}|^{1/(1+\varepsilon_1)}\,dx\Big]^{1/1+\varepsilon_1}\right)\,. \end{aligned}$$

¿From the choice of ε_1 and inequality (28) we have

$$\begin{aligned} &a_0\,\|u^{(2)}-u^{(1)}\|_{L^{1+\varepsilon_1}(\Omega)} \le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{1/2}\,, \\ &a_2\,\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{(2\cdot\theta_2+1)(1+\varepsilon_1)}} \le C\left(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{(2\theta_2+1)/(2\theta_2+2)}\,, \\ &a_1\|(\rho^{(1)}+\rho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|\,\|_{L^{1+\varepsilon_1}(\Omega)} \le C\left(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{1/2+\theta_1/4\gamma}\,. \end{aligned}$$

Therefore

$$\|\varphi\|_{L^\infty(\Omega)} \le C\left(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{\beta_1}\,,$$

where $\beta_1=\max\{\frac{2\theta_2+1}{2\theta_2+2}, \frac{1}{2}+\frac{\theta_1}{4\gamma}, \frac{1}{2}\}<1$.

¿From this and (39) we have

$$\|\varphi(t)\|_{L^\infty(\Omega)} \in L^1(0,T)\,. \tag{41}$$

2) In the case $N=2$ we set (observe $0<\theta_1<1$, $0<\theta_2<\frac{1}{2\gamma-1}$):

$$\begin{aligned} &\varepsilon_2 = \frac{1}{2}\cdot\min\{1,\gamma-1\}\,, \\ &r_1 = \frac{2\gamma}{2\theta_1(1+\varepsilon_2)} > 1\,,\ r_2 = \frac{r_1}{r_1-1}\,,\ \frac{1}{2}+\frac{\theta_1}{2\gamma}<1\,. \end{aligned}$$

By Sobolev's imbedding theorem and equation (15) we find the following estimates:

$$\|\varphi\|_{L^\infty(\Omega)} \le C\,\|\nabla\varphi\|_{2(1+\varepsilon_2)} \le C\left(1 + a_0\Big[\int\limits_\Omega |u^{(2)}-u^{(1)}|^{2(1+\varepsilon_2)}\,dx\Big]^{1/(2+2\varepsilon_2)}\right.$$

$$\left. + \, a_1\Big[\int\limits_\Omega \big((\rho^{(1)}+\rho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|\big)^{2(1+\varepsilon_2)}\,dx\Big]^{1/(2+2\varepsilon_2)} + a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{q_2}(\Omega)}\right),$$

where $q_2 = 2(2\theta_2+1)(1+\varepsilon_2)$.

By the choice of ε_2 , r_1 , and by inequality (28) we obtain

$$a_0\|u^{(2)}-u^{(1)}\|_{L^{2+2\varepsilon_2}(\Omega)} \le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{1/2},$$

$$a_1\|(\rho^{(1)}+\rho^{(2)})^{\theta_1}|u^{(2)}-u^{(1)}|\,\|_{L^{2+2\varepsilon_2}(\Omega)} \le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{\theta_1/(2\gamma)+1/2},$$

$$a_2\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{(2\theta_2+1)(1+\varepsilon_2)\cdot 2}(\Omega)} \le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{\nu_2},$$

where $\nu_2 = \frac{1-\varepsilon_2}{2(1+\varepsilon_2)} + \frac{\theta_2(1+3\varepsilon_2)+2\varepsilon_2}{2(1+\varepsilon_2)} < 1$ since $0<\theta_2<\frac{1}{2\gamma-1}$ and $\varepsilon_2 = \frac12\min\{1,\gamma-1\}$. Thus, in this case, we have the estimate

$$\|\varphi\|_{L^\infty(\Omega)} \le C\Big(1+\|\rho^{(1)}+\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\Big)^{\beta_2},$$

where $\beta_2 = \text{const.} > 0$, $\beta_2 \in (0,1)$. ¿From this and (39) there follows:

$$\|\varphi(t)\|_{L^\infty(\Omega)} \in L^1(0,T)\,. \tag{42}$$

3) In the case $N \ge 3$ we set (observe that $0<\theta_1<\frac{2}{N}$, $0<\theta_2<\frac{1}{N\gamma-1}$):

$$\begin{cases} \delta &= \min\big\{\frac12\,,\ \frac{\gamma-1}{2}\,,\ \frac{N-1}{N+1}\cdot(1+\theta_2-N\theta_2)\big\}\,; \\ r_1 &= (1+\delta)N\,; \\ r_2 &= \max\big\{1+N(1-\theta_1/\gamma)\cdot(1-N\theta_1/2\gamma)^{-1}\,, \\ & \quad 1+N(1+\delta)\big(1-(N-1)\theta_2\big)^{-1}\big\}\,. \end{cases}$$

Then the following estimates hold:

3a) $0<2\delta<1\,,\ 0<2\delta<\gamma-1\,; r_1, r_2 \in (N,+\infty)\,;$

3b) $r_1\theta_1 \le 2\gamma$ since $r_1\cdot\theta_1 = N(1+\delta)\cdot\theta_1 \le 2+2\delta \le 2\gamma\,;$

3c) $N(1+\delta)/r_2 < 2+2\theta_2-2\theta_2 N + \delta\cdot(1-2N\theta_2)$ since

$$\begin{aligned} 2+2\theta_2-2\theta_2N+\delta\cdot(1-2N\theta_2) &= 2+2\theta_2-2\theta_2N-\delta\cdot\frac{N+1}{N-1} \\ +\,\delta\Big(\frac{2N}{N-1}-2N\cdot\theta_2\Big) &> 2+2\theta_2-2\theta_2\cdot N-\delta\cdot\frac{N+1}{N-1} \\ > 1+\theta_2-N\cdot\theta_2 &> (1+\delta)\cdot N/r_2\,. \end{aligned}$$

Now we conclude from the imbedding theorem and equation (15) the following estimates:

$$\|\varphi\|_{L^\infty(\Omega)} \le C\,\|\nabla\varphi\|_{L^{r_1}(\Omega)} \le C\,\Big(1 + a_2\,\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{(2\theta_2+1)r_1}(\Omega)} +$$

$$+ a_0\,\|u^{(2)}-u^{(1)}\|_{L^{r_1}(\Omega)} + a_1\,\|(\rho^{(1)}+\rho^{(2)})^{\theta_1}\,|u^{(2)}-u^{(1)}|\,\|_{L^{r_1}(\Omega)}\Big)\,.$$

Furthermore, from (1) and the boundary condition (7) and on account of (3) there follows the estimate

$$\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)}$$

$$\le C\,\Big(1+\|\rho^{(1)}\|^{\gamma}_{L^{\gamma r_2}(\Omega)} + \|\rho^{(2)}\|^{\gamma}_{L^{\gamma r_2}(\Omega)} + a_0\,\|u^{(2)}-u^{(1)}\|_{L^{Nr_2/(N+r_2)}(\Omega)}$$

$$+ \, a_1\,\|(\rho^{(1)}+\rho^{(2)})^{\theta_1}\,|u^{(2)}-u^{(1)}|\,\|_{L^{Nr_2/(N+r_2)}(\Omega)} + a_2\,\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{q_4}(\Omega)}\Big),$$

where $q_4 = \frac{(2\theta_2+1)Nr_2}{N+r_2}$. From the inequality $r_1 > Nr_2/(N+r_2)$ we obtain

$$
\begin{aligned}
&\|\varphi\|_{L^\infty(\Omega)} + \|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)} \\
&\le C\left(1 + \|\rho^{(1)}\|^{\gamma}_{L^{\gamma r_2}(\Omega)} + \|\rho^{(2)}\|^{\gamma}_{L^{\gamma r_2}(\Omega)} + a_0\,\|u^{(2)} - u^{(1)}\|_{L^{r_1}(\Omega)}\right) \qquad (43)\\
&\quad + a_1\,\|(\rho^{(1)} + \rho^{(2)})^{\theta_1}\,|u^{(2)} - u^{(1)}|\,\|_{L^{r_1}(\Omega)} + a_2 \cdot \|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{(2\theta_2+1)r_1}(\Omega)}\,.
\end{aligned}
$$

Finally, we look at the last three expressions which we estimate in the following way:

$$
\begin{aligned}
&\text{A) } a_0\,\|u^{(2)} - u^{(1)}\|_{L^{r_1}(\Omega)} \le \varepsilon\left(\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)}\right)\\
&\quad + C\,a_0\,\|u^{(2)} - u^{(1)}\|_{L^2(\Omega)} \le \varepsilon\left(\|\nabla u^{(2)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(1)}\|_{L^{r_2}(\Omega)}\right)\\
&\quad + C\left(1 + \|\rho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{1/2}.
\end{aligned}
$$

Here

a) the number $\varepsilon \in (0,1)$ will be determined later;
b) the estimate (28) has been used;

$$
\begin{aligned}
&\text{B) } a_1\,\|(\rho^{(1)} + \rho^{(2)})^{\theta_1} \cdot |u^{(2)} - u^{(1)}|\,\|_{L^{r_1}(\Omega)}\\
&\le C\,\|u^{(2)} - u^{(1)}\|_{L^\infty(\Omega)}\left(\int_\Omega (\rho^{(1)} + \rho^{(2)})^{r_1\,\theta_1}\,dx\right)^{1/r_1}\\
&\le C\left(1 + \|\rho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{\theta_1/(2\gamma)} \|\nabla(u^{(2)} - u^{(1)})\|^{\alpha}_{L^{r_2}(\Omega)}\,\|u^{(2)} - u^{(1)}\|^{1-\alpha}_{L^{\frac{2N}{N-2}}(\Omega)}\,.
\end{aligned}
$$

Here

a) $r_1 \cdot \theta_1 \le 2\gamma$, since 3b) is satisfied;
b) $\alpha = \frac{N-2}{2N} \cdot \left(\frac{N-2}{2N} - (N - r_2)/(Nr_2)\right)^{-1} = \left(\frac{1}{2} - \frac{1}{N}\right) \cdot \left(\frac{1}{2} - \frac{1}{r_2}\right)^{-1} \in (0,1)$ comes from Lemma 2.1;

Using (28) and the imbedding theorem we find

$$
\begin{aligned}
&a_1\,\|(\rho^{(1)} + \rho^{(2)})^{\theta_1}\,|u^{(2)} - u^{(1)}|\,\|_{L^{r_1}(\Omega)} \le \varepsilon\,(\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)})\\
&\quad + C\left(1 + \|\rho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{1/2 + \theta_1/(2\gamma(1-\alpha))}.
\end{aligned}
$$

It is important that we have $\frac{1}{2} + \frac{\theta_1}{2\gamma(1-\alpha)} < 1$.

$$
\text{C) } a_2\,\|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{(2\theta_2+1)r_1(\Omega)} \le C\,a_2\,\|\nabla(u^{(2)} - u^{(1)})\|^{\beta\,(2\theta_2+1)}_{L^{r_2}(\Omega)}\,\|u^{(2)} - u^{(1)}\|^{(1-\beta)(2\theta_2+1)}_{L^{2\theta_2+2}(\Omega)}\,,
$$

where

a) the inequality follows from Lemma 2.1,
b) $\beta = \left[1/(2\theta_2 + 2) - 1/(2r_1\theta_2 + r_1)\right]\left(1/(2\theta_2 + 2) - (N - r_2)/(Nr_2)\right)^{-1} \in (0,1)$.

Using (28) we find $\nu_3 = \frac{(1-\beta)(2\theta_2+1)}{1-\beta(2\theta_2+1)}\,\frac{1}{2\theta_2+2}$ such that

$$
\begin{aligned}
&a_2\,\|u^{(2)} - u^{(1)}\|^{2\theta_2+1}_{L^{r_1(2\theta_2+1)}(\Omega)} \le \varepsilon\left(\|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)}\right)\\
&\quad + C\left(1 + \|\rho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{\nu_3}.
\end{aligned}
$$

Here it is important, that $\beta\,(2\theta_2 + 1)^2 < 1$. This inequality gives us the estimates

$$
\beta(2\theta_2 + 1) < 1\,,\ (1 - \beta)(2\theta_2 + 1) < \left(1 - \beta(2\theta_2 + 1)\right)(2\theta_2 + 2)\,.
$$

Therefore, choosing $\varepsilon \in (0,1)$ appropriately we achieve the inequality

(44)
$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla u^{(1)}\|_{L^{r_2}(\Omega)} + \|\nabla u^{(2)}\|_{L^{r_2}(\Omega)}$$
$$\le C\left(1 + \|\rho^{(1)}\|^{\gamma r_2}_{L^{\gamma r_2}(\Omega)} + \|\rho^{(2)}\|^{\gamma r_2}_{L^{\gamma r_2}(\Omega)}\right)^{1/r_2} + C\left(1 + \|\rho^{(1)}\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\rho^{(2)}\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{\beta_2},$$

where

a) $\beta_2 = \max\{\frac{1}{2}\ ,\ \frac{1}{2} + \frac{\theta_1}{2\gamma(1-\alpha)}\ ,\ \frac{(1-\beta)(2\theta_2+1)}{1-\beta(2\theta_2+1)} \cdot \frac{1}{2\theta_2+2}\} \in (0,1)$.

4) Here we look at the terms $I_1\ ,\ I_2\ ,\ I_3$ from (21) in the case $m > \gamma > 1\ ,\ N \ge 1$. On account of (21) and (39) we find, for all $t \in [0,T]$,

$$I_1 \le C\, y(t) \int_\Omega (\rho^{(1)} + \rho^{(2)})^\gamma\, dx \le C\, y(t)\,,$$

where C is a positive constant not depending on m. Furthermore, there holds the inequality

$$I_2 \le C\, y(t)\, \|\varphi(t)\|_{L^\infty(\Omega)}\,, \qquad\qquad t \in [0,T]\,,$$

and, again, the positive constant C does not depend on m.

On account of Lemma 2.3 the term I_3 can be estimated from below in the following way:

$$I_3 \ge \frac{M}{D^{(0)}} \int_\Omega \left(D^{(1)}\, k^{(1)}\, \nu_{22}\, \rho^{(1)})^{m+1} + D^{(2)}\, k^{(2)}\, \nu_{11}\, (\rho^{(2)})^{m+1}\right)(\rho^{(1)}/\rho^{(1)}_{ref} + \rho^{(2)}/\rho^{(2)}_{ref})^{\gamma-1}\, dx$$
$$\ge C \int_\Omega \left(D^{(1)}\, (\rho^{(1)})^{m+1} + D^{(2)} \cdot (\rho^{(2)})^{m+1}\right)\left(\rho^{(1)} + \rho^{(2)}\right)^{\gamma-1} dx = C\, A(t)\,,$$

where $M = \text{const} > 0$ comes from Lemma 2.3 and C is a positive constant not depending on m. Hence we conclude from (20) the following inequality for all $t \in (0,T)$:

$$\frac{1}{m-1} \cdot \frac{d}{dt} y(t) + C_1\, A(t) \le C_2\, \left(y(t) + y(t) \cdot \|\varphi(t)\|_{L^\infty(\Omega)}\right)\,, \tag{45}$$

where C_1, C_2 are positive constants independent of m.

4a) In the cases $N = 1$, $N = 2$ we obtain from (45) and (41), (42), for all $m > \gamma > 1$, $t \in (0,T)$, the inequality

$$\frac{1}{m-1} \cdot \frac{d}{dt} y(t) \le G_1(t) \cdot y(t)\,, \tag{46}$$

where $G_1(t) = C_2\left(1 + \|\varphi(t)\|_{L^\infty(\Omega)}\right) \in L^1(0,T)$ and where the function $G_1(t)$ does not depend on m.

4b) In the case $N = 3$ we obtain from (45) and (44) that, for all $m > \max\left(\gamma, \gamma\,(r_2 - 1)\right) > 1$, $t \in (0,T)$, the following inequality holds:

(47)
$$\frac{1}{m-1} \cdot \frac{d}{dt} y(t) + C_3\, A(t) \le C_4\, y(t) \cdot$$
$$\left(1 + \left(\|\rho^{(1)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)} + \|\rho^{(2)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{\beta_2} + \left(\|\rho^{(1)}(t)\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)} + \|\rho^{(2)}(t)\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)}\right)^{1/r_2}\right).$$

Due to the estimates for the densities (39) one easily checks the inequality

$$\left(\int_\Omega D^{(1)}\, (\rho^{(1)})^m + D^{(2)}\, (\rho^{(2)})^m\, dx\right) \cdot \left(1 + \|\rho^{(1)}\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)} + \|\rho^{(2)}\|^{r_2\gamma}_{L^{r_2\gamma}(\Omega)}\right)^{1/r_2}$$
$$\le C \cdot \left(\left(A(t)\right)^{1-\frac{\gamma}{m r_2}} + y(t)\right)\,,$$

where C is a positive constant not depending on m.
Hence we obtain from (47) that

$$\frac{1}{m-1}\cdot\frac{d}{dt}y(t)\le C^m+G_2(t)\cdot y(t)\,,\qquad t\in(0,T)\,,\tag{48}$$

with C being a positive constant not depending on m, and so does the function defined by

$$G_2(t)=C\left(1+\|\rho^{(1)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}+\|\rho^{(2)}(t)\|^{2\gamma}_{L^{2\gamma}(\Omega)}\right)^{\beta_2}.$$

We have $G_2(t)\in L^1(0,T)$.
Now we find from (46) and (48), for all $N\ge 1$, $m>m_0$, $t\in[0,T]$, the estimate

$$\begin{aligned}y(t)&=\int\limits_\Omega D^{(1)}\cdot(\rho^{(1)})^m+D^{(2)}\cdot(\rho^{(2)})^m\,dx\\&\le\big(C(T)\big)^m\cdot\Big(1+\|\rho_0^{(1)}\|_{L^m(\Omega)}+\|\rho_0^{(2)}\|_{L^m(\Omega)}\Big)^m,\end{aligned}$$

where $C(T)$ is a positive constant not depending on m. From this, there follows:

$$\sup_{0<t<T}\Big(\|\rho^{(1)}(t)\|_{L^\infty(\Omega)}+\|\rho^{(2)}(t)\|_{L^\infty(\Omega)}\Big)\le C\left(1+\|\rho_0^{(1)}\|_{L^\infty(\Omega)}+\|\rho_0^{(2)}\|_{L^\infty(\Omega)}\right).\tag{49}$$

Now, we present an estimate *from below* for the densities of the mixture:
5) Let $n\in(0,+\infty)$ be a positive number. Then,analogously to equation (20), we find

$$\begin{aligned}&-\frac{1}{n+1}\cdot\frac{d}{dt}\int\limits_\Omega(\rho^{(1)})^{-n}+(\rho^{(2)})^{-n}\,dx\\&+\frac{1}{D^{(0)}}\int\limits_\Omega(\rho^{(1)})^{-n}\Big[\nu_{22}(p^{(1)}-p_1^{(1)})-\nu_{12}(p^{(2)}-p_1^{(2)})-(\nu_{22}+\nu_{12})\varphi\Big]\,dx\\&+\frac{1}{D^{(0)}}\int\limits_\Omega(\rho^{(2)})^{-n}\Big[\nu_{11}(p^{(2)}-p_1^{(2)})-\nu_{21}(p^{(1)}-p_1^{(1)})+(\nu_{11}+\nu_{21})\varphi\Big]\,dx=0\,.\end{aligned}$$

We define for all $t\in[0,T]$ the function $Z(t)=\int\limits_\Omega(p^{(1)})^{-n}+(p^{(2)})^{-n}\,dx$.

Then the estimate (49) implies the inequality

$$\frac{1}{n+1}\cdot\frac{d}{dt}Z(t)\le C\,Z(t)\,\big(1+\|\varphi(t)\|_{L^\infty(\Omega)}\big)\,,$$

where C again is a positive constant not depending on n.

On account of (41), (42), (43) and (49) we derive from the differential inequality the estimate for all $n>0$:

$$Z(t)=\int\limits_\Omega(\rho^{(1)})^{-n}+(\rho^{(2)})^{-n}\,dx\le\big(C(T)\big)^{n+1}\left(1+\|1/\rho_0^{(1)}\|_{L^n(\Omega)}+\|1/\rho_0^{(2)}\|_{L^n(\Omega)}\right)^n,$$

where $C(T)$ is a positive constant not depending on n. ¿From this there follows

$$\sup_{0<t<T}\left(\|\frac{1}{\rho^{(1)}(t)}\|_{L^\infty(\Omega)}+\|\frac{1}{\rho^{(2)}(t)}\|_{L^\infty(\Omega)}\right)\le C\left(1+\|\frac{1}{\rho_0^{(1)}}\|_{L^\infty(\Omega)}+\|\frac{1}{\rho_0^{(2)}}\|_{L^\infty(\Omega)}\right).\tag{50}$$

5. Estimates for Gradients of the Velocities and the Densities

In this section w show that it is possible to estimate the first derivatives of the functions $u^{(1)}(x,t)$, $u^{(2)}(x,t)$, $\rho^{(1)}(x,t)$, $\rho^{(2)}(x,t)$. Let $s\in(N,+\infty)$ be any number.

1) First we have by equation (1) and the boundary condition (7) the following estimate:

$$\|u^{(2)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \\ \leq C\left(\|g\cdot(u^{(2)}-u^{(1)})\|_{L^s(\Omega)} + \|\nabla p^{(1)}\|_{L^s(\Omega)} + \|\nabla p^{(2)}\|_{L^s(\Omega)}\right).$$

Having completed (49) one proceeds with the inequality

$$\|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \\ \leq C\left(1 + \|(u^{(2)}-u^{(1)})\|^{2\theta_2+1}_{L^{s(2\theta_2+1)}(\Omega)} + \|\nabla\rho^{(1)}\|_{L^s(\Omega)} + \|\nabla\rho^{(2)}\|_{L^s(\Omega)}\right). \tag{51}$$

We take into account, that from (28) follows that

$$\|\nabla u^{(1)}\|_{L^2(\Omega)} + \|\nabla u^{(2)}\|_{L^2(\Omega)} \leq C. \tag{52}$$

1a) If we have the case $N=1$, $N=2$, then there holds the estimate

$$\|(u^{(2)}-u^{(1)})\|_{L^{s(2\theta_2+1)}(\Omega)} \leq C\,\|\nabla(u^{(2)}-u^{(1)})\|_{L^2(\Omega)}\,,$$

and thus we find from (51) and (52) the estimate

$$\|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \leq C\left(1 + \|\nabla\rho^{(1)}\|_{L^s(\Omega)} + \|\nabla\rho^{(2)}\|_{L^s(\Omega)}\right). \tag{53}$$

1b) In the case $N\geq 3$ then there holds the inequality (Lemma 2.1):

$$\|u^{(2)}-u^{(1)}\|_{L^{s(2\theta_2+1)}(\Omega)} \leq C\,\|\nabla(u^{(2)}-u^{(1)})\|^{\alpha}_{L^\infty(\Omega)}\cdot\|u^{(2)}-u^{(1)}\|^{1-\alpha}_{L^{2N/(N-2)}(\Omega)}\,,$$

where $\alpha\in(0,1)$, $\alpha = 1-\frac{2}{N}-2/(2s\theta_2+s)$, if $s(2\theta_2+1) > \frac{2N}{N-2}$, and $\alpha=0$ if $s(2\theta_2+1)\leq\frac{2N}{N-2}$. Here, we take into account that $\alpha\cdot(2\theta_2+1)<1$, since $0<\theta_2<1/(N\gamma-1)$.

Therefore, by (52) and the estimate

$$\|u^{(2)}-u^{(1)}\|_{L^{2N/(N-2)}(\Omega)} \leq c\,\|\nabla(u^{(2)}-u^{(1)})\|_{L^2(\Omega)}\,,$$

we conclude the following inequality, for all $\varepsilon>0$,

$$\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{s(2\theta_2+1)}(\Omega)} \leq \varepsilon\,\|\nabla(u^{(2)}-u^{(1)})\|_{L^\infty(\Omega)} + C(\varepsilon). \tag{54}$$

¿From this, we obtain further:

$$\|u^{(2)}-u^{(1)}\|^{2\theta_2+1}_{L^{s(2\theta_2+1)}(\Omega)} \leq \varepsilon\left[\|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)}\right] + C(\varepsilon). \tag{55}$$

Now, we find from (51) and (55) the estimate

$$\|u^{(1)}\|_{W^{2,s}(\Omega)} + \|u^{(2)}\|_{W^{2,s}(\Omega)} \leq C\left(1 + \|\nabla\rho^{(1)}\|_{L^s(\Omega)} + \|\nabla\rho^{(2)}\|_{L^s(\Omega)}\right). \tag{56}$$

Hence we have proven the estimate (53), (56) in all cases $N\geq 1$.

2) In our considerations we use an important estimate for the velocities from [17], [18], [16], [10]. On account of the estimates for the densities (49), (50) we have $\rho^{(i)}\in L^\infty(\Omega\times(0,T))$ for $i=1,2$ and in view of the inequality from [17], [18], [16], [10] we conclude from (1), for all $s>N$, the estimate

$$\|\nabla u^{(1)}\|_{L^\infty(\Omega)} + \|\nabla u^{(2)}\|_{L^\infty(\Omega)} \\ \leq C\left(1 + \ln\left(2 + \|\nabla\rho^{(1)}\|_{L^s(\Omega)} + \|\nabla\rho^{(2)}\|_{L^s(\Omega)}\right) + \|g\cdot(u^{(2)}-u^{(1)})\|_{L^s(\Omega)}\right).$$

Because of (49), (52) and (54) we have

$$\|\nabla u^{(1)}\|_{L^\infty(\Omega)} + \|\nabla u^{(2)}\|_{L^\infty(\Omega)} \leq C\left(1 + \ln(2 + \|\nabla\rho^{(1)}\|_{L^s(\Omega)} + \|\nabla\rho^{(2)}\|_{L^s(\Omega)})\right). \tag{57}$$

3) The estimates for the derivatives $\frac{\partial\rho^{(i)}}{\partial x_j}(x,t)\,;\ i=1,2\,;\ j=1\,\ldots,N$ are derived from the equation

$$\frac{\partial}{\partial t}(\nabla\rho^{(i)}) + \nabla((u^{(i)}\cdot\nabla)\rho^{(i)}) + \nabla(\rho^{(i)}\cdot\operatorname{div}u^{(i)}) = 0\,,$$

which, in turn, follows from (2). Therefore, we obtain from (49) for $s \in (N, \infty)$ the estimate

$$\frac{d}{dt}\Big(\int\limits_\Omega |\nabla \rho^{(1)}|^s + |\nabla \rho^{(2)}|^s \, dx \Big) \le C \left(\int\limits_\Omega \left(|\nabla \rho^{(1)}|^s + |\nabla \rho^{(2)}|^s \right) \left(|\nabla u^{(1)}| + |\nabla u^{(2)}| \right) dx \right)$$
$$+ C \left(\int\limits_\Omega \left(|\nabla \rho^{(1)}|^s + |\nabla \rho^{(2)}|^s \right)^{(s-1)/s} \left(|\nabla \operatorname{div} u^{(1)}| + |\nabla \operatorname{div} u^{(2)}| \right) dx \right).$$

Using (56) and (57) we proceed and get

$$\frac{d}{dt} L(t) \le C\big(1 + L(t) + L(t)\, \ln(2 + L(t))\big)$$

with L defined by $L(t) = \int\limits_\Omega |\nabla \rho^{(1)}(t)|^s + |\nabla \rho^{(2)}(t)|^s \, dx \,, \qquad t \in [0, T].$

¿From the last differential inequality we receive, for $s \in (N, +\infty)$, the estimate

$$\sup_{0<t<T} \left(\|\nabla \rho^{(1)}(t)\|_{L^s(\Omega)} + \|\nabla \rho^{(2)}(t)\|_{L^s(\Omega)} \right) \le C \,. \tag{58}$$

Furthermore, from (2) and (56):

(59)
$$\sup_{0<t<T} \left(\|\frac{\partial \rho^{(1)}}{\partial t}(t)\|_{L^s(\Omega)} + \|\frac{\partial \rho^{(2)}}{\partial t}(t)\|_{L^s(\Omega)} + \|u^{(1)}(t)\|_{W^{2,s}(\Omega)} + \|u^{(2)}(t)\|_{W^{2,s}(\Omega)} \right) \le C \,.$$

4) The estimates for the derivatives $\frac{\partial}{\partial t}\Big(\frac{\partial u^{(i)}}{\partial x_j}(x,t) \Big)$; $i = 1, 2$; $j = 1, \ldots, N$; come from the following system which is, in turn, derived from (1):

$$\begin{aligned} &\frac{\partial}{\partial t}\Big(\sum_{j=1}^{2} \mu_{ij} \Delta u^{(j)} + (\mu_{ij} + \lambda_{ij}) \nabla \operatorname{div} u^{(j)} \Big) \\ &\quad + (-1)^{i+1} \Big(g \cdot \frac{\partial}{\partial t}(u^{(2)} - u^{(1)}) + \frac{\partial g}{\partial t} \cdot (u^{(2)} - u^{(1)}) \Big) - \nabla \Big(\frac{\partial p^{(i)}}{\partial t} \Big) = 0 \,. \end{aligned} \tag{60}$$

Firstly, from (60), in view of the estimates (49), (50) and (59), there follows the inequality

$$\|\nabla \Big(\frac{\partial u^{(1)}}{\partial t} \Big)\|^2_{L^2(\Omega)} + \|\nabla \Big(\frac{\partial u^{(2)}}{\partial t} \Big)\|^2_{L^2(\Omega)} \le C \,, \tag{61}$$

since $g = g(x,t) \ge 0$.

Finally, by the properties of the system (60) and the imbedding theorem, we have for $s \in (N, +\infty)$ the estimate

$$\|\frac{\partial u^{(1)}}{\partial t}\|_{W^{1,s}(\Omega)} + \|\frac{\partial u^{(2)}}{\partial t}\|_{W^{1,s}(\Omega)} \le C \left(\|\frac{\partial p^{(1)}}{\partial t}\|_{L^s(\Omega)} + \|\frac{\partial p^{(2)}}{\partial t}\|_{L^s(\Omega)} \right.$$
$$\left. + \|g \Big(\frac{\partial u^{(2)}}{\partial t} - \frac{\partial u^{(1)}}{\partial t} \Big)\|_{L^s(\Omega)} + \|\frac{\partial g}{\partial t}(u^{(2)} - u^{(1)})\|_{L^s(\Omega)} \right).$$

¿From this, applying (49), (50) and (59) we find the estimate

$$\|\frac{\partial u^{(1)}}{\partial t}\|_{W^{1,s}(\Omega)} + \|\frac{\partial u^{(2)}}{\partial t}\|_{W^{1,s}(\Omega)} \le C \left(1 + \|\frac{\partial u^{(2)}}{\partial t} - \frac{\partial u^{(1)}}{\partial t}\|_{L^s(\Omega)} \right).$$

Therefore, we conclude with Lemma 2.1 and (61):

$$\sup_{0<t<T} \left(\|\frac{\partial u^{(1)}}{\partial t}(t)\|_{W^{1,s}(\Omega)} + \|\frac{\partial u^{(2)}}{\partial t}(t)\|_{W^{1,s}(\Omega)} \right) \le C \,. \tag{62}$$

5) In the case $\rho_0^{(1)}$, $\rho^{(2)} \in W^{l,r}(\Omega)$, $r > 1$, $l > 1$, $r \cdot (l-1) > N$ it is easy to see, using (49), (50), (58), (59) and (62), that for all $k = 1, 2, \ldots, l$ and $i = 1, 2$ the following inclusions

hold:

$$\frac{\partial^k u^{(i)}}{\partial t^k} \in L^\infty\big(0,T\,;\; W^{l+1-k,r}(\Omega)\big)\,, \quad \frac{\partial^k \rho^{(i)}}{\partial t^k} \in L^\infty\big(0,T\,;\; W^{l-k,r}(\Omega)\big)\,. \tag{63}$$

Thus we have proven all a priori estimates stated in the theorem.

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