ON QUASICONFORMAL SELF-MAPPINGS OF THE UNIT DISK SATISFYING POISSON'S EQUATION

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Abstract. Let $\mathcal{QC}(K,g)$ be a family of K quasiconformal mappings of the open unit disk onto itself satisfying the PDE $\Delta w = g$, $g \in C(\overline{\mathbb{U}})$, $w(0) = 0$. It is proved that $\mathcal{QC}(K,g)$ is a uniformly Lipschitz family. Moreover, if $|g|_\infty$ is small enough, then the family is uniformly bi-Lipschitz. The estimations are asymptotically sharp as $K \to 1$ and $|g|_\infty \to 0$, so $w \in \mathcal{QC}(K,g)$ behaves almost like a rotation for sufficiently small K and $|g|_\infty$.

1. Introduction and statement of the main result

In this paper $\mathbb{U}$ denotes the open unit disk in $\mathbb{C}$, and S^1 denotes the unit circle. Also, by D and Ω we denote open regions in $\mathbb{C}$. For a complex number $z = x + iy$, its norm is given by $|z| = \sqrt{x^2 + y^2}$. For a real 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

we will consider the matrix norm $|A| = \sup\{|Az| : |z| = 1\}$ and the matrix function $l(A) = \inf\{|Az| : |z| = 1\}$.

A real-valued function u, defined in an open subset D of the complex plane $\mathbb{C}$, is harmonic if it satisfies Laplace's equation:

$$\Delta u(z) := \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0 \qquad (z = x + iy).$$

A complex-valued function $w = u + iv$ is harmonic if both u and v are real harmonic.

We say that a function $u : D \to \mathbb{R}$ is ACL (absolutely continuous on lines) in the region D, if for every closed rectangle $R \subset D$ with sides parallel to the x and y-axes, u is absolutely continuous on a.e. horizontal and a.e. vertical line in R. Such a function has of course, partial derivatives u_x, u_x a.e. in D.

The definition carries over to complex valued functions.

Definition 1.1. A homeomorphism $w : D \to \Omega$, between open regions $D, G \subset \mathbb{C}$, is $K-$quasiconformal ($K \geq 1$) (abbreviated $K - q.c.$) if

(1) w is ACL in D,
(2) $|w_{\bar z}| \leq k|w_z|$ a.e. ($k = \frac{K-1}{K+1}$).

Here

$$w_z := \frac{1}{2}\left(w_x - iw_y\right) \text{ and } w_{\bar z} := \frac{1}{2}\left(w_x + iw_y\right)$$

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are complex partial derivatives (cf. [1], pp. 3, 23–24).

If by $\nabla w(z)$ we denote the formal derivative of $w = u + iv$ at z:

$$\nabla w = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then the condition (2) of Definition 1.1 can be written as

$$K^{-1}(|\nabla w|)^2 \le J_w \le K(l(\nabla w))^2 \quad \text{a.e. on } D, \tag{1.1}$$

where $J_w = \det(\nabla u)$ is the Jacobian of w. The above fact follows from the following well-known formulae

$$J_w(z) = |w_z|^2 - |w_{\bar z}|^2, \quad |\nabla w| = |w_z| + |w_{\bar z}|, \quad l(\nabla w) = ||w_z| - |w_{\bar z}||.$$

Notice that if w is $K-$quasiconformal, then w^{-1} is $K-$quasiconformal as well (this follows from (1.1)).

Let P be the Poisson kernel, i.e. the function

$$P(z, e^{i\theta}) = \frac{1-|z|^2}{|z-e^{i\theta}|^2},$$

and let G be the Green function of the unit disk, i.e. the function

$$G(z,\omega) = \frac{1}{2\pi} \log \left| \frac{1 - z\overline{\omega}}{z - \omega} \right| \quad z, \omega \in \mathbb{U},\ z \ne \omega. \tag{1.2}$$

The functions $z \mapsto P(z, e^{i\theta})$, $z \in \mathbb{U}$, and $z \mapsto G(z,\omega)$, $z \in \mathbb{U} \setminus \{\omega\}$ are harmonic.

Let $f : S^1 \to \mathbb{C}$ be a bounded integrable function on the unit circle S^1 and let $g : U \to \mathbb{C}$ be continuous. The solution of the equation $\Delta w = g$ in the unit disk satisfying the boundary condition $w|_{S^1} = f \in L^1(S^1)$ is given by

$$\begin{aligned} w(z) &= P[f](z) - G[g](z) \\ &:= \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\varphi}) f(e^{i\varphi}) d\varphi - \int_{\mathbb{U}} G(z,\omega) g(\omega)\, dm(\omega), \end{aligned} \tag{1.3}$$

$|z| < 1$, where $dm(\omega)$ denotes the Lebesgue measure in the plane. It is well known that if f and g are continuous in S^1 and in $\overline{\mathbb{U}}$ respectively, then the mapping $w = P[f] - G[g]$ has a continuous extension $\tilde w$ to the boundary, and $\tilde w = f$ on S^1. See [9, pp. 118–120].

We will consider those solutions of the PDE $\Delta w = g$ that are quasiconformal as well and will investigate their Lipschitz character.

Recall that a mapping $w : D \to \Omega$ is said to be $C-$Lipschitz ($C > 0$) ($c-$co-Lipschitz ($c > 0$)) if

$$|w(z_1) - w(z_2)| \le C|z_1 - z_2|, \quad z_1, z_2 \in D, \tag{1.4}$$

$$(c|z_1 - z_2| \le |w(z_1) - w(z_2)|, \quad z_1, z_2 \in D). \tag{1.5}$$

A mapping w is bi-Lipschitz if it is Lipschitz and co-Lipschitz.

O. Martio [17] was the first who considered harmonic quasiconformal mappings on the complex plane. Recent papers [10], [12], [14], [21] and [13] bring much light on the topic of quasiconformal harmonic mappings on the plane. See also [11] for the extension of the problem on the space. In [16] it was established the Lipschitz character of q.c. harmonic self-mappings of the unit disk with respect to hyperbolic metric and it was generalized to the arbitrary domain in [18]. See [27],

[28], [26], [30] for additional results concerning the Lipschitz character of harmonic quasiconformal mappings w.r.t the hyperbolic metric.

The following theorem is a generalization of an analogous theorem for the unit disk due to Pavlović [21] and of an asymptotically sharp version of Pavlović theorem due to Partyka and Sakan [20] in the case of harmonic quasiconformal mappings.

The following fact is the main result of the paper.

Theorem 1.2. *Let $K \geq 1$ be arbitrary and let $g \in C(\overline{\mathbb{U}})$ and $|g|_\infty := \sup_{w\in\mathbb{U}} |g(w)|$. Then there exist constants $N(K)$ and $M(K)$ with $\lim_{K\to 1} M(K) = 1$ such that : If w is a K-quasiconformal self-mapping of the unit disk $\mathbb{U}$ satisfying the PDE $\Delta w = g$, with $w(0) = 0$, then for $z_1, z_2 \in \mathbb{U}$:*

$$\left(\frac{1}{M(K)} - \frac{7|g|_\infty}{6}\right)|z_1 - z_2| \leq |w(z_1) - w(z_2)| \leq (M(K) + N(K)|g|_\infty)\,|z_1 - z_2|.$$

The proof of Theorem 1.2, given in Section 3, depends on the following two propositions:

Proposition 1.3. [13] *Let w be a quasiconformal C^2 diffeomorphism from a bounded plane domain D with $C^{1,\alpha}$ boundary onto a bounded plane domain Ω with $C^{2,\alpha}$ boundary. If there exist constants a and b such that*

$$|\Delta w| \leq a|\nabla w|^2 + b\,, \quad z \in D, \tag{1.6}$$

then w has bounded partial derivatives in D. In particular it is a Lipschitz mapping in D.

Proposition 1.4 (Mori's Theorem)**.** [5, 22, 31] *If w is a K-quasiconformal self-mapping of the unit disk $\mathbb{U}$ with $w(0) = 0$, then there exists a constant $M_1(K)$, satisfying the condition $M_1(K) \to 1$ as $K \to 1$, such that*

$$|w(z_1) - w(z_2)| \leq M_1(K)|z_1 - z_2|^{K^{-1}}. \tag{1.7}$$

See also [2] and [19] for some constants that are not asymptotically sharp.

The mapping $|z|^{-1+K^{-1}}z$ shows that the exponent K^{-1} is optimal in the class of arbitrary K-quasiconformal homeomorphisms.

2. Auxiliary results

In this section, we establish some lemmas needed in the proof of the main results.

Lemma 2.1. *Let w be a harmonic function defined on the unit disk and assume that its derivative $v = \nabla w$ is bounded on the unit disk (or equivalently, according to Rademacher's theorem [7], let w be Lipschitz continuous). Then there exists a mapping $A \in L^\infty(S^1)$ defined on the unit circle S^1 such that $\nabla w(z) = P[A](z)$ and for almost every $e^{i\theta} \in S^1$ the relation*

$$\lim_{r\to 1-} \nabla w(re^{i\theta}) = A(e^{i\theta}) \tag{2.1}$$

holds. Moreover the function $f(e^{i\theta}) := w(e^{i\theta})$ is differentiable almost everywhere in $[0, 2\pi]$ and the formula

$$A(e^{i\theta}) \cdot (ie^{i\theta}) = \frac{\partial}{\partial\theta} f(e^{i\theta})$$

holds.

Proof. For the proof of the first statement of the lemma, see, for example, [3, Theorem 6.13 and Theorem 6.39].

Next, since

$$|\frac{\partial}{\partial\theta}w(re^{i\theta})| = |r\nabla w(re^{i\theta})\frac{\partial}{\partial\theta}e^{i\theta}| \leq |r\nabla w(re^{i\theta})|\cdot|\frac{\partial}{\partial\theta}e^{i\theta}|$$

$$\leq \operatorname{ess\,sup}_{\theta}|A(e^{i\theta})|\cdot|\frac{\partial}{\partial\theta}e^{i\theta}|,$$

the Lebesgue Dominated Convergence Theorem yields

$$\begin{aligned} f(e^{i\theta}) &= \lim_{r\to 1} w(re^{i\theta}) \\ &= \lim_{r\to 1-}\int_{\theta^0}^{\theta}\frac{\partial}{\partial\varphi}w(re^{i\varphi})d\varphi + f(e^{i\theta_0}) \\ &= \int_{\theta^0}^{\theta}\lim_{r\to 1}\frac{\partial}{\partial\varphi}w(re^{i\varphi})d\varphi + f(e^{i\theta_0}) \\ &= \int_{\theta^0}^{\theta}\lim_{r\to 1} r\nabla w(re^{i\varphi})\frac{\partial}{\partial\varphi}e^{i\varphi}d\varphi + f(e^{i\theta_0}) \\ &= \int_{\theta^0}^{\theta}A(e^{i\varphi})\cdot\frac{\partial}{\partial\varphi}e^{i\varphi}d\varphi + f(e^{i\theta_0}). \end{aligned}$$

Differentiating in θ we get

$$\frac{\partial}{\partial\theta}f(e^{i\theta}) = A(e^{i\theta})\cdot\frac{\partial}{\partial\theta}e^{i\theta} = A(e^{i\theta})(ie^{i\theta})$$

almost everywhere in S^1. □

Lemma 2.2. *If $f(e^{it}) = e^{i\psi(t)}$, $\psi(2\pi) = \psi(0)+2\pi$, is a diffeomorphism of the unit circle onto itself, then*

$$|f(e^{it}) - f(e^{is})| \leq |\psi'|_\infty|e^{it} - e^{is}|, \tag{2.2}$$

where $|\psi'|_\infty = \max\{\psi'(\tau) : 0 \leq \tau \leq 2\pi\} = \max\{|\partial_\tau f(e^{i\tau})| : 0 \leq \tau \leq 2\pi\}$.

Proof. Take the function

$$h(t) = \frac{|f(e^{it}) - f(e^{is})|}{|e^{it} - e^{is}|}.$$

Then we have

$$h(t) = \frac{\sin\frac{\psi(t)-\psi(s)}{2}}{\sin\frac{t-s}{2}}. \tag{2.3}$$

In order to estimate the maximum of the function h, we found out that the stationary points of it satisfy the equation

$$\tan\frac{\psi(t)-\psi(s)}{2} = \tan\frac{t-s}{2}\cdot\psi'(t). \tag{2.4}$$

Substituting (2.4) to (2.3) we obtain

$$h^2(t) = \frac{\left(1+\tan^2\frac{\psi(t)-\psi(s)}{2}\right)\psi'^2(t)}{1+\tan^2\frac{\psi(t)-\psi(s)}{2}\psi'^2(t)}. \tag{2.5}$$

Now since

$$2\pi = \psi(2\pi)-\psi(0) = \int_0^{2\pi}\psi'(\tau)d\tau,$$

it follows that $|\psi'|_\infty \geq 1$. If $|\psi'(t)| \leq 1$ then from (2.5) it follows $|h(t)| \leq 1 \leq |\psi'|_\infty$. If $|\psi'(t)| > 1$, then again employing (2.5) we obtain $|h(t)| \leq |\psi'|_\infty$. This implies the lemma. □

Lemma 2.3. *If* $z \in \mathbb{U}$, *and*

$$I(z) = \frac{1}{2\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{|z-\omega|\cdot|1-\bar{z}\omega|}\,dm(\omega)\ ,$$

then

$$\frac{1}{2}\ \leq I(z) \leq\ \frac{2}{3}. \tag{2.6}$$

Both inequalities are sharp. Moreover the function $z \mapsto I(z)$, *is a radial function and decreasing for* $|z| \in [0,1]$.

Proof. For a fixed z, we introduce the change of variables

$$\frac{z-\omega}{1-\bar{z}\omega} = \xi,$$

or, what is the same,

$$\omega = \frac{z-\xi}{1-\bar{z}\xi}.$$

Then

$$\begin{aligned}
I :=& \frac{1}{2\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{|z-\omega|\cdot|1-\bar{z}\omega|}\,dm(\omega)\\
=& \ \frac{1}{2\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{|\xi|\cdot|1-\bar{z}\omega|^2}\,dm(\omega)\\
=& \ \frac{1}{2\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{|\xi|\cdot|1-\bar{z}\omega|^2}\frac{(1-|z|^2)^2}{|1-\bar{z}\xi|^4}\,dm(\xi)\\
=& \ \frac{1}{2\pi}\int_{\mathbb{U}} \frac{(1-|\xi|^2)(1-|z|^2)^3}{|\xi|\cdot|1-\bar{z}\xi|^6\,|1-\bar{z}\omega|^2}\,dm(\xi).
\end{aligned}$$

Since

$$\begin{aligned}
1-\bar{z}\omega &= 1-\bar{z}\frac{z-\xi}{1-\bar{z}\xi}\\
&= \frac{1-|z|^2}{1-\bar{z}\xi},
\end{aligned}$$

we see that

$$I = \frac{1}{2\pi}\int_{\mathbb{U}} \frac{(1-|z|^2)(1-|\xi|^2)}{|\xi|\cdot|1-\bar{z}\xi|^4}\,dm(\xi).$$

In the polar coordinates, we have

$$I = (1-|z|^2)\int_0^1 (1-\rho^2)\,d\rho\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-\bar{z}\rho e^{it}|^4}.$$

By Parseval's formula (see [24, Theorem 10.22]), we get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-\bar{z}\rho e^{it}|^4} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|(1-\bar{z}\rho e^{it})^2|^2}\\ &= \frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{n=0}^{\infty}(n+1)(\bar{z}\rho)^n e^{nit}\Big|^2\,dt\\ &= \sum_{n=0}^{\infty}(n+1)^2|z|^{2n}\rho^{2n},\end{aligned}$$

whence

$$I = (1-|z|^2)\sum_{n=0}^{\infty}\frac{2(n+1)^2}{(2n+1)(2n+3)}|z|^{2n}.$$

Now the desired inequality follows from the simple inequality

$$\frac{1}{2} \le c_n := \frac{2(n+1)^2}{(2n+1)(2n+3)} \le \frac{2}{3} \quad (n = 0, 1, 2, \ldots).$$

Setting $|z|^2 = r$, and $\varphi(r) = I(z)$, we obtain

$$\varphi'(r) = \sum_{n=1}^{\infty} n(c_n - c_{n-1})r^{n-1}.$$

Since $c_n \le c_{n-1}$ it follows that φ is decreasing, as desired. □

We need the following well-known propositions.

Proposition 2.4. [25] *Let X be an open subset of $\mathbb{R}$, and Ω be a measure space. Suppose that a function $F\colon X\times\Omega\to\mathbb{R}$ satisfies the following conditions:*

1. *$F(x,\omega)$ is a measurable function of x and ω jointly, and is integrable over ω, for almost all $x\in X$ held fixed.*
2. *For almost all $\omega\in\Omega$, $F(x,\omega)$ is an absolutely continuous function of x. (This guarantees that $\partial F(x,\omega)/\partial x$ exists almost everywhere).*
3. *$\partial F/\partial x$ is "locally integrable" – that is, for all compact intervals $[a,b]$ contained in X:*

$$\int_a^b\int_\Omega\left|\frac{\partial}{\partial x}F(x,\omega)\right|\,d\omega\,dx < \infty\,. \tag{2.7}$$

Then $\int_\Omega F(x,\omega)\,d\omega$ is an absolutely continuous function of x, and for almost every $x\in X$, its derivative exists and is given by

$$\frac{d}{dx}\int_\Omega F(x,\omega)\,d\omega = \int_\Omega\frac{\partial}{\partial x}F(x,\omega)\,d\omega\,.$$

The following proposition is well-known as well.

Proposition 2.5. [29, p. 24–26] *Let ρ be a bounded (absolutely) integrable function defined on a bounded domain $\Omega \subset \mathbb{C}$. Then the potential type integral*

$$I(z) = \int_{\Omega} \frac{\rho(\omega)\, dm(\omega)}{|z-\omega|}$$

belongs to the space $C(\mathbb{C})$.

Lemma 2.6. *Let ρ be continuous on the closed unit disc $\overline{\mathbb{U}}$. Then the integral*

$$J(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \log\left|\frac{1-z\overline{\omega}}{z-\omega}\right| \rho(\omega)\, dm(\omega)$$

belongs to the space $C^1(\mathbb{U})$. Moreover

$$\nabla J(z) = \frac{1}{2\pi} \int_{\mathbb{U}} \nabla \log\left|\frac{1-z\overline{\omega}}{z-\omega}\right| \rho(\omega)\, dm(\omega).$$

Proof. Straightforward calculations yield

$$\nabla_z \frac{1}{2\pi}\log\left|\frac{1-z\overline{\omega}}{z-\omega}\right| = \frac{1}{2\pi}\left(\frac{1}{\omega-z} - \frac{\bar{\omega}}{1-z\bar{\omega}}\right), \tag{2.8}$$

and consequently

$$|\nabla_z \frac{1}{2\pi}\log\left|\frac{1-z\overline{\omega}}{z-\omega}\right|| = \frac{1}{2\pi}\frac{1-|\omega|^2}{|z-\omega||\bar{z}\omega-1|},\ \ z\neq\omega. \tag{2.9}$$

(Here $\nabla_z\varphi(z,\omega)$ denotes the gradient of the function φ treated as a function of z). Let $\Omega = \mathbb{U}$, and let μ be the Lebesgue measure of $\mathbb{U}$.

According to Lemma 2.3, condition (2.7) of Proposition 2.4 is satisfied. Applying now Proposition 2.4, and relation (2.8) together with Proposition 2.5, we obtain the desired conclusion. □

Lemma 2.7. *If g is continuous on $\overline{\mathbb{U}}$, then the mapping $G[g]$ has a bounded derivative, i.e. it is Lipschitz continuous and the inequalities*

$$|\partial G[g]| \le \frac{1}{3}|g|_\infty, \tag{2.10}$$

and

$$|\bar{\partial} G[g]| \le \frac{1}{3}|g|_\infty \tag{2.11}$$

hold on the unit disk. Moreover $\nabla G[g]$ has a continuous extension to the boundary, and for $e^{i\theta}\in S^1$ there hold

$$\partial G[g](e^{i\theta}) = -\frac{-e^{i\theta}}{4\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{|e^{i\theta}-\omega|^2} g(\omega)\, dm(\omega), \tag{2.12}$$

and

$$\bar{\partial} G[g](e^{i\theta}) = -\frac{e^{i\theta}}{4\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{|e^{i\theta}-\omega|^2} g(\omega)\, dm(\omega). \tag{2.13}$$

Finally, for $e^{i\theta}\in S^1$

$$|\partial G[g]| \leq \frac{1}{4}|g|_\infty, \tag{2.14}$$

and

$$|\bar\partial G[g]| \leq \frac{1}{4}|g|_\infty. \tag{2.15}$$

Proof. First of all for $z \neq \omega$ we have

$$\begin{aligned} G_z(z,\omega) &= \frac{1}{4\pi}\left(\frac{1}{\omega - z} - \frac{\bar\omega}{1 - z\bar\omega}\right) \\ &= \frac{1}{4\pi}\frac{(1-|\omega|^2)}{(z-\omega)(z\bar\omega - 1)}, \end{aligned}$$

and

$$G_{\bar z}(z,\omega) = \frac{1}{4\pi}\frac{(1-|\omega|^2)}{(\bar z - \bar\omega)(\bar z\omega - 1)}.$$

By Lemma 2.6 the potential type integral

$$\partial G[g](z) = \frac{1}{4\pi}\int_{\mathbb{U}} \frac{1-|\omega|^2}{(z-\omega)(z\bar\omega - 1)} g(\omega)\, dm(\omega),$$

exists and belongs to the space $C(\mathbb{U})$.

According to Lemma 2.3 it follows that

$$|\partial G[g]| \leq \frac{1}{4\pi}|g|_\infty \int_{\mathbb{U}} \frac{1-|\omega|^2}{|z-\omega||\bar z\omega - 1|}\, dm(\omega),$$

and

$$|\partial G[g]| \leq \frac{1}{3}|g|_\infty.$$

The inequality (2.10) is proved. Similarly we establish (2.11).

According to Lemma 2.5 it follows

$$\partial G[f](z) = \int_{\mathbb{U}} G_z(z,\omega) g(\omega)\ dm(\omega). \tag{2.16}$$

Next we have

$$\lim_{z\to e^{i\theta}, z\in\mathbb{D}} G_z(z,\omega) = -\frac{1}{4\pi}\frac{e^{-i\theta}(1-|\omega|^2)}{|e^{i\theta}-\omega|^2} \tag{2.17}$$

and

$$\lim_{z\to e^{i\theta}, z\in\mathbb{D}} G_{\bar z}(z,\omega) = -\frac{1}{4\pi}\frac{e^{i\theta}(1-|\omega|^2)}{|e^{i\theta}-\omega|^2}. \tag{2.18}$$

In order to deduce (2.12) from the last two relations, we use the Vitali theorem (see [6, Theorem 26.C]):

Let X be a measure space with finite measure μ, and let $h_n : X \to \mathbb{C}$ be a sequence of functions that is uniformly integrable, i.e. such that for every $\varepsilon > 0$ there exists $\delta > 0$, independent of n, satisfying

$$\mu(E) < \delta \implies \int_E |h_n|\, d\mu < \varepsilon. \tag{†}$$

Now: if $\lim_{n\to\infty} h_n(x) = h(x)$ *a.e., then*

$$\lim_{n\to\infty}\int_X h_n\,d\mu = \int_X h\,d\mu. \tag{‡}$$

In particular, if

$$\sup_n \int_X |h_n|^p\,d\mu < \infty, \quad \textit{for some } p > 1,$$

then (†) *and* (‡) *hold.*

Hence, to prove (2.12), it suffices to prove that

$$\sup_{z\in\mathbb{U}}\int_{\mathbb{U}}\left(\frac{1-|\omega|^2}{|z-\omega|\cdot|1-\bar z\omega|}|g(\omega)|\right)^p dm(\omega) \ < \infty, \quad \text{for } p = 3/2.$$

In order to prove this inequality, we proceed as in the case of Lemma 2.3. We obtain

$$\begin{aligned}
I_{p,g}(z) &= \int_{\mathbb{U}}\left(\frac{1-|\omega|^2}{|z-\omega|\cdot|1-\bar z\omega|}|g(\omega)|\right)^p dm(\omega)\\
&\le |g|_\infty^p\int_{\mathbb{U}}\left(\frac{1-|\omega|^2}{|z-\omega|\cdot|1-\bar z\omega|}\right)^p dm(\omega)\\
&= |g|_\infty^p\int_{\mathbb{U}}\frac{(1-|z|^2)^{2-p}(1-|\omega|^2)^p}{|\xi|^p\,|1-\bar z\xi|^4}\,dm(\xi)\\
&\le |g|_\infty^{3/2}(1-|z|^2)^{1/2}\int_0^1 \rho^{-1/2}(1-\rho^2)^{3/2}\,d\rho\int_0^{2\pi}|1-\bar z\rho e^{i\varphi}|^{-4}\,d\varphi\\
&\le |g|_\infty^{3/2}(1-|z|^2)^{1/2}\int_0^1 \rho^{-1/2}(1-\rho^2)^{3/2}(1-|z|\rho)^{-3}\,d\rho.
\end{aligned}$$

Now the desired result follows from the elementary inequality

$$\int_0^1 \rho^{-1/2}(1-\rho^2)^{3/2}(1-|z|\rho)^{-3}\,d\rho \ \le\ C(1-|z|^2)^{-1/2}.$$

This proves (2.12). Similarly we prove (2.13). The inequalities (2.14) and (2.15) follow from (2.12) and (2.13) and Lemma 2.3. □

A mapping $w : D \to \Omega$ is *proper* if the preimage of every compact set in Ω is compact in D. In the case where $D = \Omega = \mathbb{U}$, the mapping w is proper if and only if $|w(z)| \to 1$ as $|z| \to 1$.

Lemma 2.8 (The main lemma)**.** *Let* w *be a solution of the PDE* $\Delta w = g$ *that maps the unit disk onto itself properly. Let in addition* w *be Lipschitz continuous. Then there exist for a.e.* $t = e^{i\theta} \in S^1$*:*

$$\nabla w(t) := \lim_{r\to 1-}\nabla w(rt) \tag{2.19}$$

and

$$J_w(t) := \lim_{r\to 1-} J_w(re^{i\theta}), \tag{2.20}$$

and the following relation

$$\begin{aligned} J_w(t) = \psi'(\theta)\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(t)-f(e^{i\varphi})|^2}{|t-e^{i\varphi}|^2}d\varphi \\ + \psi'(\theta)\int_0^1 r\Big(\frac{1}{2\pi}\int_0^{2\pi}P(re^{i\varphi},t)\,\langle g(rt),f(t)\rangle\,d\varphi\Big)dr, \end{aligned} \tag{2.21}$$

holds. Here ψ *is defined by*

$$e^{i\psi(\theta)} := f(e^{i\theta}) = w|_{S^1}(e^{i\theta}).$$

If w *is biharmonic (*$\Delta\Delta w = 0$*), then we have:*

$$\begin{aligned} J_w(t) = \psi'(\theta)\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(t)-f(e^{i\varphi})|^2}{|t-e^{i\varphi}|^2}d\varphi \\ + \frac{\psi'(\theta)}{2}\int_0^1 \langle g(rt),f(t)\rangle\,dr,\ \ t\in S^1. \end{aligned} \tag{2.22}$$

For an arbitrary continuous g *and* $|g|_\infty = \max_{|z|\le 1}|g(z)|$ *the inequality*

$$|J_w(t)-\psi'(\theta)\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(t)-f(e^{i\theta})|^2}{|t-e^{i\theta}|^2}d\theta| \le \frac{\psi'(\theta)|g|_\infty}{2},\ \ t\in S^1 \tag{2.23}$$

holds.

Proof. First of all, according to Lemma 2.7, $G[g]$ has a bounded derivative, and there exists the function $\nabla G[g](e^{i\theta})$, $e^{i\theta}\in S^1$, which is continuous and satisfies the limit relation

$$\lim_{z\to e^{i\theta}, z\in\mathbb{D}}\nabla G[g](z) = \nabla G[g](e^{i\theta}).$$

Since $w = P[f]-G[g]$ has bounded derivative, from Lemma 2.1, it follows that there exists

$$\lim_{r\to 1-}\nabla P[f](re^{i\theta}) = \nabla P[f](e^{i\theta}).$$

Thus $\lim_{r\to 1-}\nabla w(re^{i\theta}) = \nabla w(e^{i\theta})$.

It follows that the mapping χ: $\chi(\theta) = f(e^{i\theta}) := f(t)$, $t\in S^1$, defines the outer normal vector field $\mathbf{n}_\chi$ almost everywhere in S^1 at the point $\chi(\theta) = f(e^{i\theta}) = e^{i\psi(\theta)} = (\chi_1,\chi_2)$ by the formula:

$$\mathbf{n}_\chi(\chi(\theta)) = \psi'(\theta)\cdot f(e^{i\theta}). \tag{2.24}$$

Let $\varpi(r,\theta) := w(re^{i\theta})$. According to Lemma 2.1, we obtain:

$$\lim_{r\to 1-}\varpi_\theta(r,\theta) = \chi'(\theta). \tag{2.25}$$

On the other hand, for almost every $\theta\in S^1$ we have

$$\frac{\chi_j(\theta)-\varpi_j(r,\theta)}{1-r} = \varpi_r(\rho_{j,r,\theta},\theta)$$

where $r<\rho_{j,r,\theta}<1$, $j=1,2$. Thus we have:

$$\lim_{r\to 1-} \varpi_{j_r}(r,\theta) = \lim_{r\to 1-} \frac{\chi_j(\theta)-\varpi_j(r,\theta)}{1-r},\ j\in\{1,2\}. \tag{2.26}$$

Denote by p polar coordinates, i.e. $p(r,\theta) = re^{i\theta}$.

We derive

$$\begin{aligned}
\lim_{r\to 1-} J_{w\circ p}(r,\theta) &= \lim_{r\to 1-} \left\langle \frac{\chi - P[f]}{1-r}, \psi'(\theta)\cdot f(e^{i\theta})\right\rangle + \Lambda \\
&= \lim_{r\to 1-} \frac{1}{2\pi}\int_0^{2\pi} \frac{1+r}{|e^{i\theta}-re^{i\varphi}|^2}\left\langle f(e^{i\theta})-f(e^{i\varphi}), \psi'(\theta)\cdot f(e^{i\theta})\right\rangle d\varphi + \Lambda \\
&= \lim_{r\to 1-} \psi'(\theta)\frac{1}{2\pi}\int_0^{2\pi} \frac{1+r}{|e^{i\theta}-re^{i\varphi}|^2}\left\langle f(e^{i\theta})-f(e^{i\varphi}), f(e^{i\theta})\right\rangle d\varphi + \Lambda \\
&= \psi'(\theta)\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta})-f(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^2} d\varphi + \Lambda,
\end{aligned} \tag{2.27}$$

where

$$\Lambda = \lim_{r\to 1-}\left\langle \frac{G[g]}{1-r}, -i\chi_\theta\right\rangle.$$

In order to estimate Λ, observe first that

$$\begin{aligned}
\lim_{z\to e^{i\theta}, z\in\mathbb{D}} \frac{G(z,\omega)}{1-|z|} &= \lim_{z\to e^{i\theta}, z\in\mathbb{D}} \frac{G(z,\omega)-G(e^{i\theta},\omega)}{1-|z|} \\
&= \lim_{r\to 1-} \frac{G(re^{i\theta},\omega)-G(e^{i\theta},\omega)}{1-r} = -\left.\frac{\partial G(re^{i\theta},\omega)}{\partial r}\right|_{r=1}.
\end{aligned} \tag{2.28}$$

Since

$$\frac{\partial G(re^{i\theta},\omega)}{\partial r} = z_r G_z(re^{i\theta},\omega) + \bar{z}_r G_{\bar z}(re^{i\theta},\omega),\ z_r = e^{i\theta}, \bar{z}_r = e^{-i\theta},$$

using (2.17) and (2.18) we obtain

$$\lim_{z\to e^{i\theta}, z\in\mathbb{D}} \frac{G(z,\omega)}{1-|z|} = \frac{1}{2\pi}P(e^{i\theta},\omega). \tag{2.29}$$

On the other hand we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{U}} P(\omega, e^{i\theta})\left\langle g(\omega), f(e^{i\theta})\right\rangle\, dm(\omega) \\
&\quad= \int_0^1 r\Big(\frac{1}{2\pi}\int_0^{2\pi} P(re^{i\varphi}, e^{i\theta})\left\langle g(re^{i\varphi}), f(e^{i\theta})\right\rangle d\varphi\Big)dr.
\end{aligned} \tag{2.30}$$

Next, we have

$$J_{w\circ p}(r,\theta) = rJ_w(re^{i\theta}). \tag{2.31}$$

Combining (2.27), (2.29), (2.30) and (2.31) we obtain (2.21). Relations (2.22) and (2.23) follow form (2.21) and (1.3). If w is biharmonic, then g is harmonic and thus

$$\frac{1}{2\pi}\int_0^{2\pi} P(re^{i\varphi}, e^{i\theta})\left\langle g(re^{i\varphi}), f(e^{i\theta})\right\rangle d\varphi = \left\langle g(r^2e^{i\theta}), f(e^{i\theta})\right\rangle.$$

This yields relation (2.22). □

Lemma 2.9. *If $x \geq 0$ is a solution of the inequality $x \leq ax^\alpha + b$, where $a \geq 1$ and $0 \leq a\alpha < 1$, then*

$$x \leq \frac{a + b - \alpha a}{1 - \alpha a}. \tag{2.32}$$

Observe that for $\alpha = 0$, (2.32) coincides with $x \leq a + b$, i.e. $x \leq ax^\alpha + b$.

Proof. We will use the Bernoulli's inequality. $x \leq ax^\alpha + b = a(1 + x - 1)^\alpha + b \leq a(1 + \alpha(x - 1)) + b$. Relation (2.32) now easily follows. □

3. The main results

Theorem 3.1. *Let $g \in C(\overline{\mathbb{U}})$. The family $\mathcal{QC}(K, g)$ of K-quasiconformal ($K \geq 1$) self-mappings of the unit disk $\mathbb{U}$ satisfying the PDE $\Delta w = g$, $w(0) = 0$, is uniformly Lipschitz, i.e. there is a constant $M' = M'(K, g)$ satisfying:*

$$|w(z_1) - w(z_2)| \leq M'|z_1 - z_2|, \; z_1, z_2 \in \mathbb{U}, \; w \in \mathcal{QC}(K, g). \tag{3.1}$$

Moreover $M'(K, g) \to 1$ as $K \to 1$ and $|g|_\infty \to 0$.

In Remark 3.7 bellow is given a quantitative bound of $M'(K, g)$.

Proof. Combining Proposition 1.3 and Lemma 2.8, in the special case where the range of a function is the unit disk, we obtain that there exist ∇w and J_w almost everywhere in S^1, and the following inequality

$$J_w(t) \leq \psi'(\theta) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi} - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right) \tag{3.2}$$

holds.

Now from

$$|\nabla w(re^{i\theta})|^2 \leq K J_w(re^{i\theta}),$$

we obtain

$$\lim_{r \to 1-} |\nabla w(re^{i\theta})|^2 \leq \lim_{r \to 1-} K J_w(re^{i\theta}), \tag{3.3}$$

almost everywhere in $[0, 2\pi]$. From Lemma 2.1, we deduce that

$$\lim_{r \to 1-} \frac{\partial(w(re^{i\theta}))}{\partial\theta} = \frac{\partial f(e^{i\theta})}{\partial\theta} = \psi'(\theta) e^{i\psi(\theta)} \tag{3.4}$$

almost everywhere in $[0, 2\pi]$. Since

$$\frac{\partial w \circ S}{\partial\theta}(r, \theta) = ru'(re^{i\theta})(ie^{i\theta}),$$

using (3.4) we obtain that

$$\psi'(\theta) \leq \lim_{r \to 1} |\nabla w(re^{i\theta})|. \tag{3.5}$$

From (3.2)-(3.5) we infer that

$$|\nabla w(e^{i\theta})|^2 \leq K|\nabla w(e^{i\theta})| \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right)$$

i.e.

$$|\nabla w(e^{i\theta})| \leq K \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{i\varphi}) - f(e^{i\theta})|^2}{|e^{i\varphi} - e^{i\theta}|^2} d\varphi + \frac{|g|_\infty}{2} \right). \tag{3.6}$$

Let

$$M = \operatorname{ess}\sup_{0\le\tau\le 2\pi} |\nabla w(e^{i\tau})|.$$

According to Lemma 2.2 and to relation (3.5) we obtain

$$|f(e^{i\varphi}) - f(e^{i\theta})| \le M|e^{i\varphi} - e^{i\theta}|. \tag{3.7}$$

Let $\mu = K^{-1}$, $\gamma = -1 + K^{-2}$, and let $\nu = 1 - 1/K$. Let $\varepsilon > 0$. Then there exists θ_ε such that

$$M(1-\varepsilon) \le |\nabla w(e^{i\theta_\varepsilon})|.$$

Applying now relation (3.6) and using (1.7), we obtain

$$\begin{aligned}(1-\varepsilon)M &\le K\left(M^\nu \frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma \frac{|f(e^{i\theta_\varepsilon}) - f(e^{i\varphi})|^{2-\nu}}{|e^{i\theta_\varepsilon} - e^{i\varphi}|^{\mu^2+\mu}} d\varphi + \frac{|g|_\infty}{2}\right)\\ &\le KM^\nu M_1(K)^{1+\mu}\frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma d\varphi + K\frac{|g|_\infty}{2}\\ &\le M_2(K)M^\nu + \frac{K|g|_\infty}{2},\end{aligned}$$

where

$$M_2(K) = KM_1(K)^{1+\mu}\frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma d\varphi.$$

Letting $\varepsilon \to 0$ we obtain

$$M \le M_2(K)M^\nu + \frac{K|g|_\infty}{2}. \tag{3.8}$$

From (3.8) we obtain

$$M \le C_0 := \left(M_2(K) + \frac{K|g|_\infty}{2}\right)^{1/(1-\nu)} = \left(M_2(K) + \frac{K|g|_\infty}{2}\right)^K. \tag{3.9}$$

From Lemma 2.9, if

$$M_1(K)^{1+\mu}\frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta_\varepsilon} - e^{i\varphi}|^\gamma d\varphi < \frac{1}{K-1}$$

and $g \ne 0$, we obtain

$$M \le C_1 := \frac{M_2(K) + K|g|_\infty/2 - \nu M_2(K)}{1 - \nu M_2(K)}. \tag{3.10}$$

Let $C_2 := \min\{C_0, C_1\}$.

If $g \equiv 0$ then by (3.9) we get

$$M \le C_2 := (M_2(K))^{1/(1-\mu)}. \tag{3.11}$$

To continue observe that $w - G[g]$ is harmonic. Thus

$$|\nabla w(z) - \nabla G(z)| \le \operatorname{ess}\sup_{0\le\tau\le 2\pi} |\nabla w(e^{i\tau}) - \nabla G[g](e^{i\tau})|.$$

According to Lemma 2.3 and Lemma 2.7 it follows that:

$$|\nabla w(z)| \le \operatorname{ess}\sup_{0\le\tau\le 2\pi} |\nabla w(e^{i\tau})| + \frac{2}{3}|g|_\infty + \frac{1}{2}|g|_\infty.$$

Therefore the inequality (3.1) does hold for

$$M' = C_2 + \frac{7}{6}|g|_\infty. \tag{3.12}$$

Using (1.7), it follows that

$$\lim_{|g|_\infty \to 0, K\to 1} M'(K) = 1.$$

□

Lemma 3.2. *If w is a K-q.c. self-mapping of the unit disk satisfying the PDE $\Delta w = g$ and $w(0) = 0$, $w|_{S^1}(e^{i\theta}) = f(e^{i\theta}) = e^{i\psi(\theta)}$, $g \in C(\overline{\mathbb{U}})$, then for almost every $\theta \in [0, 2\pi]$ the relation*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi \le K\psi'(\theta) + \frac{|g|_\infty}{2} \tag{3.13}$$

holds.

Proof. From (2.23) it follows that

$$\frac{J_w(e^{i\theta})}{\psi'(\theta)} \ge \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{i\theta}) - f(e^{i\varphi})|^2}{|e^{i\theta} - e^{i\varphi}|^2} d\varphi - \frac{|g|_\infty}{2}. \tag{3.14}$$

Using Lemma 2.1 we obtain

$$\psi'(\theta) = |\frac{\partial f(e^{i\theta})}{\partial\theta}| = |\lim_{r\to 1-} \frac{\partial w(re^{i\theta})}{\partial \theta}|. \tag{3.15}$$

On the other hand

$$\frac{\partial w(re^{i\theta})}{\partial\theta} = izw_z(re^{i\theta}) - i\bar{z}w_{\bar{z}}(re^{i\theta}) \ (z = re^{i\theta}). \tag{3.16}$$

Therefore

$$|\lim_{r\to 1-} \frac{\partial w(re^{i\theta})}{\partial\theta}| \ge ||w_z(t)| - |w_{\bar{z}}(t)|| = l(\nabla w(t)) \ (t = e^{i\theta}). \tag{3.17}$$

As w is K-q.c., according to (1.1) it follows that

$$\frac{J_w(t)}{(l(\nabla w(t)))^2} \le K. \tag{3.18}$$

Combining (3.14) - (3.18) we obtain (3.13). □

Lemma 3.3. *Under the conditions and notations of Lemma 3.2, there exists a function $m_1(K)$ such that $\lim_{K\to 1} m_1(K) = 1$ and*

$$m(K) := \max\left\{m_1(K) - \frac{|g|_\infty}{4}, \frac{4 - 5|g|_\infty}{8}\right\} \le K\psi'(\theta), \ \textit{for a.e.}\, \theta \in [0, 2\pi]. \tag{3.19}$$

Proof. Applying (1.7) to the mapping w^{-1}, we obtain

$$|f(z) - f(w)| \ge M_1(K)^{-K}|z_1 - z_2|^K.$$

Using now relation (3.13) we obtain

$$\begin{aligned} K\psi'(\theta) &\geq \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{i\theta})-f(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^2}d\varphi - \frac{|g|_\infty}{2} \\ &\geq M_1(K)^{-2K}\frac{1}{2\pi}\int_0^{2\pi}|e^{i\theta}-e^{i\varphi}|^{2K-2}d\varphi - \frac{|g|_\infty}{2}. \\ &= m_1(K) - \frac{|g|_\infty}{2}, \end{aligned} \tag{3.20}$$

where

$$m_1(K) = M_1(K)^{-2K}\frac{1}{2\pi}\int_0^{2\pi}|e^{i\theta}-e^{i\varphi}|^{2K-2}d\varphi.$$

Let us prove the second part of the inequality (3.19). Since $w(0) = 0$ we infer that $P[f](0) = -G[g](0)$. Thus

$$P[f](0) = \int_{\mathbb{U}} G(0,\omega)g(\omega)\,dm(\omega),$$

i.e. in polar coordinates

$$P[f](0) = \frac{1}{2\pi}\int_0^1\int_0^{2\pi} r\log\frac{1}{r}g(\omega)\,dm(\omega).$$

Hence

$$|P[f](0)| \leq |g|_\infty\int_0^1 r\log\frac{1}{r}dr = \frac{|g|_\infty}{4}.$$

Next we have

$$\begin{aligned} K\psi'(\theta) &\geq \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{i\theta})-f(e^{i\varphi})|^2}{|e^{i\theta}-e^{i\varphi}|^2}d\varphi - \frac{|g|_\infty}{2} \\ &\geq \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}\left(1-\operatorname{Re}\overline{f(e^{i\theta})}f(e^{i\varphi})\right)d\varphi - \frac{|g|_\infty}{2} \\ &\geq \frac{1-|P[f](0)|}{2} - \frac{|g|_\infty}{2} \\ &\geq \frac{4-5|g|_\infty}{8}. \end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21) we obtain (3.19). □

Theorem 3.4. *If w is a K-q.c. orientation preserving self-mapping of the unit disk satisfying the PDE $\Delta w = g$, $w(0) = 0$, $g \in C(\overline{\mathbb{U}})$, then for*

$$m(K) = \max\left\{M_1(K)^{-2K}\frac{1}{2\pi}\int_0^{2\pi}|e^{i\theta}-e^{i\varphi}|^{2K-2}d\varphi - \frac{|g|_\infty}{4}, \frac{4-5|g|_\infty}{8}\right\},$$

the inequality

$$l(\nabla w) \geq \frac{m(K)}{K^2} - \frac{7|g|_\infty}{6} \tag{3.22}$$

where $l(\nabla w(z)) = \min\{|\nabla w(z)t| : |t| = 1\}$, holds.

Proof. Assume, as we may, that

$$\frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \geq \left(\frac{m(K)}{K^2} - \frac{7|g|_\infty}{6}\right) \geq 0. \tag{3.23}$$

From (3.19) and the definition of quasiconformality we deduce that:

$$\frac{m(K)}{K^2} \leq \frac{\psi'(\theta)}{K} \leq \frac{|\nabla w(e^{i\theta})|}{K} \leq l(\nabla w),$$

i.e

$$\frac{m(K)}{K^2} \leq |w_z| - |w_{\bar z}|$$

almost everywhere on the unit circle.

According to relations (2.14) and (2.15) we obtain

$$\frac{m(K)}{K^2} - \frac{|g|_\infty}{2} \leq |P[f]_z| - |P[f]_{\bar z}| \tag{3.24}$$

almost everywhere on the unit circle.

To continue observe that, as w is q.c., it follows that f is a homeomorphism. Hence by Choquet-Radó-Kneser theorem $P[f]$ is a diffeomorphism (see [15], [4] or [23]).

Thus $h := P[f]$ is a harmonic diffeomorphism. According to the Heinz theorem ([8])

$$|h_z|^2 + |h_z|^2 \geq \frac{1}{\pi^2},$$

which, in view of the fact that $|h_z| \geq |h_{\bar z}|$, implies that

$$|h_z| \geq \frac{\sqrt{2}}{2\pi}.$$

Thus the functions

$$a(z) := \frac{\overline{h_{\bar z}}}{h_z}, \quad b(z) := \frac{1}{h_z}\left(\frac{m(K)}{K^2} - \frac{|g|_\infty}{2}\right)$$

are holomorphic and bounded on the unit disk. As $|a| + |b|$ is bounded on the unit circle by 1 (see (3.23) and (3.24)), it follows that it is bounded on the whole unit disk by 1 because

$$|a(z)| + |b(z)| \leq P[|a|_{S^1}](z) + P[|b|_{S^1}](z), \quad z \in \mathbb{U}.$$

This in turn implies that for every $z \in \mathbb{U}$

$$l(\nabla h) \geq \frac{m(K)}{K^2} - \frac{|g|_\infty}{2}. \tag{3.25}$$

By (2.10) and (2.11) we obtain

$$l(\nabla w) \geq \frac{m(K)}{K^2} - \frac{1}{2}|g|_\infty - \frac{2}{3}|g|_\infty. \tag{3.26}$$

□

Having in mind the fact $l(\nabla w(z)) = |\nabla w^{-1}(w(z))|^{-1}$, and putting Theorem 3.1 and Theorem 3.4 together we obtain:

Theorem 3.5. *Let $\mathcal{QC}(K,g)$ be the family of orientation preserving K-q.c. self-mappings of the unit disk satisfying the equation $\Delta w = g$ and $w(0) = 0$. Then for $|g|_\infty$ small enough (for example if $|g|_\infty \le \frac{12}{15+28K^2}$) the family $\mathcal{QC}(K,g)$ is uniformly bi-Lipschitz, i.e. there exists $M_0(K,g) \ge 1$ such that*

$$M_0(K,g)^{-1} \le \frac{|w(z_1)-w(z_2)|}{|z_1-z_2|} \le M_0(K,g),\ w \in \mathcal{QC}(K,g),\ \text{for } z_1,z_2 \in \mathbb{U},\ z_1 \ne z_2.$$

Moreover

$$\lim_{|g|_\infty \to 0, K\to 1} M_0(K,g) = 1.$$

Example 3.6. Let $w(z) = |z|^\alpha z$, with $\alpha > 1$. Then w is twice differentiable $(1+\alpha)-$quasiconformal self-mapping of the unit disk. Moreover

$$\Delta w = \alpha(2+\alpha)\frac{|z|^\alpha}{\bar z} = g.$$

Thus $g = \Delta w$ is continuous and bounded by $\alpha(2+\alpha)$. However w is not co-Lipschitz (i.e. it does not satisfy (1.5)), because $l(\nabla w)(0) = |w_z(0)| - |w_{\bar z}(0)| = 0$. This means that the condition "$|g|_\infty$ is small enough" in Theorem 3.5 cannot be replaced by the condition "g is arbitrary".

Remark 3.7. Let $\mathcal{QC}_K(\mathbb{U})$ be the family of $K-$quasiconformal self-mappings of the unit disk. Let $M_1(K)$ be the Mori's constant:

$$M_1(K) = \inf\{M : |f(z_1)-f(z_2)| \le M|z_1-z_2|^{1/K}, z_1,z_2 \in \mathbb{U}, f \in \mathcal{QC}_K(\mathbb{U}), f(0)=0\}.$$

In [22] is proved that

$$M_1(K) \le 16^{1-1/K}\min\left\{\left(\frac{23}{8}\right)^{1-1/K}, (1+2^{3-2K})^{1/K}\right\}.$$

Since for $\alpha > -1$

$$\frac{1}{2\pi}\int_0^{2\pi} |e^{i\theta}-e^{i\varphi}|^\alpha d\varphi = \frac{2^{\alpha+1}}{\pi}\frac{\sqrt{\pi}\Gamma[\frac{1+\alpha}{2}]}{\alpha\Gamma[\frac{\alpha}{2}]},$$

our proofs, in the case of harmonic mappings ($g \equiv 0$), yield the following estimates for co-Lipschitz constant

$$m_2 := \frac{2^{2K-2}\Gamma[K-1/2]}{\sqrt{\pi}(K^3-K^2)\Gamma[K-1]M_1(K)^{2K}} \tag{3.27}$$

which is

$$\ge \frac{1}{K^2M_1(K)^{2K}} \ge \frac{46^2}{K^2 46^{2K}},$$

and therefore is better than the corresponding constant

$$m_1 := \frac{2^{K(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2+K-1)^{3K}} \tag{3.28}$$

obtained in the paper [20] for every K (see the appendix below).

Similarly we obtain the following estimate for the Lipschitz constant (see (3.9) and (3.12)).

$$M' = \left(KM_1(K)^{1+1/K}\left(\frac{2^{K^{-2}}\Gamma[K^{-2}/2]}{\sqrt{\pi}(K^{-2}-1)\Gamma[(K^{-2}-1)/2]}\right) + \frac{K|g|_\infty}{2}\right)^K + \frac{7|g|_\infty}{6}.$$

The last constant (if $g \equiv 0$) is not comparable with the corresponding constant

$$K^{3K+1}2^{5(K-1/K)/2}$$

obtained in the same paper [20] (it is better if K is large enough but it is not for K close to 1). It seems that in the proof of Theorem 3.1 there is some small place for improvement of M' (taking $\nu \neq 1 - K^{-1}$).

3.1. **Appendix.** Let us prove that $m_2 \geq m_1$, where m_1 and m_2 are defined in (3.27) and (3.28). Since $(3 \cdot (3^2 + 3 - 1))^{3/2} > 46$, the inequality follows directly if $K \geq 3$.
Assume now that $1 \leq K \leq 3$. First of all we have

$$\frac{46^2}{K^2 46^{2K}} - \frac{2^{K(1-K^2)(3+1/K)/2}}{K^{3K+1}(K^2+K-1)^{3K}} \geq \frac{1}{K^2}\left(46^{2-2K}\left(1 - \frac{46^{2K-2} \cdot 2^{2(1-K^2)}}{K^8}\right)\right).$$

Therefore, the inequality

$$46^{2K-2} \cdot 2^{2(1-K^2)} \leq K^8 \tag{3.29}$$

implies $m_2 \geq m_1$.
Let $K \leq 2$. Then $\frac{46}{2^{1+K}} < 16 = 2^4$. By Bernoulli's inequality $2^{K-1} = (1+1)^{K-1} \leq 1 + K - 1 = K$ for $K \leq 2$. This yields (3.29).
Assume now that $2 \leq K \leq 3$. Then

$$\frac{46}{2^{1+K}} < e^2.$$

Thus

$$\left(\frac{46}{2^{1+K}}\right)^{K-1} \leq e^{2(K-1)}.$$

Therefore, if we prove

$$e^{K-1} \leq K^2 \text{ for } 2 \leq K \leq 3$$

we will prove the inequality $m_2 \geq m_1$ completely.
Let $x = K - 1$. Then

$$\begin{aligned} K^2 - e^{K-1} &= 1 + 2x + x^2 - 1 - x - x^2/2 - x^3/3! - x^4/4! - ... \\ &= x(1 + x/2 - x^2/3! - x^3/4! - ...) \\ &\geq x(1 - x^3/4! - ...) \\ &\geq x(1 - 0.5(e^2 - 1 - 2 - 2^2/2 - 2^3/6)) > x/2, \end{aligned}$$

as desired.

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