References

1. H. W. Becker & J. Riordan. "The Arithmetic of Bell and Stirling Numbers." *Amer. J. Math.* *70* (1948):385-394.
2. L. Carlitz. "Congruences for Generalized Bell and Stirling Numbers." *Duke Math. J.* *22* (1955):193-205.
3. A. Nijenhuis & H. S. Wilf. "Periodicities of Partition Functions and Stirling Numbers Modulo p." *J. Number Theory* *25* (1987):308-312.

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ON r-GENERALIZED FIBONACCI NUMBERS

François Dubeau
Collège militaire royal de Saint-Jean
Saint-Jean-sur-Richelieu, Québec, Canada, J0J 1R0
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Introduction

Miles [5] defined the r-generalized Fibonacci numbers ($r \geq 2$) as follows:

$$u_{r,n} = 0 \quad (n = -1, -2, -3, \ldots), \tag{1a}$$

$$u_{r,0} = 1, \tag{1b}$$

$$u_{r,n} = \sum_{i=1}^{r} u_{r,n-i} \quad (n = 1, 2, 3, \ldots). \tag{1c}$$

In such a way, for $r = 2$, we get the ordinary Fibonacci numbers. The object of this paper is to present, in the first section, an elementary proof of the convergence of the sequences of ratios

$$\left\{t_{r,n} = \frac{u_{r,n}}{u_{r,n-1}}\right\}_{n=1}^{\infty}$$

using neither the theory of difference equations nor the theory of continued fractions. In the second section, we consider a geometric interpretation of the r-generalized Fibonacci numbers that is a natural generalization of the golden rectangle. Finally, in the third section, we consider electrical schemes generating these numbers.

1. Convergence Results

For each $r \geq 2$, we consider the sequence of ratios

$$t_{r,n} = u_{r,n}/u_{r,n-1} \quad (n = 1, 2, 3, \ldots).$$

Rather than using the theory of difference equations to obtain a formula for $u_{r,n}$ and use it to prove the convergence of the sequence to the unique positive root of the polynomial

$$p_r(x) = x^r - \sum_{i=1}^{r} x^{r-i} \quad \text{(see [5])},$$

we present here a proof based on a fixed point argument using the way the $u_{r,n}$ are generated.

Observe that $u_{r,n} > 0$ for $n \geq 0$. Hence, dividing (1c) by $u_{r,n-1}$, we get

$$t_{r,n} = 1 + \sum_{i=2}^{r} \frac{u_{r,n-i}}{u_{r,n-1}} \quad (n \geq 1)$$

and, using the definition of $t_{r,i}$, we obtain

$$t_{r,n} = 1 + \sum_{i=2}^{r} \frac{1}{\prod_{j=1}^{i-1} t_{r,n-j}} \quad (n \geq r). \tag{2}$$

From (1), we also have

$$u_{r,n} = 2u_{r,n-1} - u_{r,n-r-1} \text{ for } n \geq 2;$$

hence, dividing by $u_{r,n-1}$, we obtain

$$t_{r,n} = 2 - \frac{1}{\prod_{i=1}^{r} t_{r,n-i}} \quad (n \geq r+1). \tag{3}$$

Now, since $t_{r,n} \geq 1$ for $n = 1, \ldots, r$, using (2) we have $t_{r,n} \geq 1$ for all $n \geq 1$ and, using (3), we also have $t_{r,n} \leq 2$ for all $n \geq 1$.

Using (2) and (3) we can generate a sequence of upper bounds $\{B_{r,\ell}\}_{\ell=0}^{\infty}$ and a sequence of lower bounds $\{b_{r,\ell}\}_{\ell=0}^{\infty}$ for $t_{r,n}$ as follows. We have

$$1 = b_{r,0} \leq t_{r,n} \leq B_{r,0} = 2 \quad (n \geq 1)$$

and, assuming that $b_{r,\ell-1}$ and $B_{r,\ell-1}$ are known and such that

$$b_{r,\ell-1} \leq t_{r,n} \leq B_{r,\ell-1} \text{ for all } n \geq r(\ell - 1) + 1,$$

we generate $b_{r,\ell}$ and $B_{r,\ell}$ using (2) and (3) in such a way that

$$b_{r,\ell} = 1 + \sum_{i=2}^{r} \frac{1}{B_{r,\ell-1}^{i-1}} \leq t_{r,n} \leq 2 - \frac{1}{B_{r,\ell-1}^{r}} = B_{r,\ell} \tag{4}$$

for all $n \geq r\ell + 1$.

The problem is now related to the convergence of the sequences

$$\{b_{r,\ell}\}_{\ell=0}^{\infty} \quad \text{and} \quad \{B_{r,\ell}\}_{\ell=0}^{\infty}.$$

We consider the two functions

$$f_r(x) = 1 + \sum_{i=2}^{r} \frac{1}{x^{i-1}} \quad \text{and} \quad F_r(x) = 2 - \frac{1}{x^r}$$

From (4), $B_{r,\ell} = F_r(B_{r,\ell-1})$ and $b_{r,\ell} = f_r(B_{r,\ell-1})$; hence, the result we look for will be obtained from the study of the two functions $f_r(\cdot)$ and $F_r(\cdot)$.

Lemma 1: Let $r \geq 2$ and $F_r(x) = 2 - \frac{1}{x^r}$.

(a) The equation $x = F_r(x)$ has two solutions in the interval $(0, \infty)$. One solution is 1 and the other, noted α_r, is in the interval $(1, 2)$.

(b) Let $\{x_i\}_{i=0}^{\infty}$ be a sequence defined by $x_{i+1} = F_r(x_i)$ for $i = 0, 1, 2, \ldots$.

  (i) If $x_0 \in (1, \alpha_r)$, the sequence $\{x_i\}_{i=0}^{\infty}$ is strictly increasing and converges to α_r.

(ii) If $x_0 \in (\alpha_r, \infty)$, the sequence $\{x_i\}_{i=0}^{\infty}$ is strictly decreasing and converges to α_r.

Proof: If $x \in (0, \infty)$, then

$$F'(x) = \frac{r}{x^{r+1}} > 0 \quad \text{and} \quad F''(x) = \frac{r(r+1)}{x^{r+2}} < 0;$$

hence, $F_r(\cdot)$ is a strictly increasing continuous concave function on $(0, \infty)$. Also

$$\lim_{x \to 0^+} F_r(x) = -\infty, \quad \lim_{x \to +\infty} F_r(x) = 2,$$

$F_r(1) = 1$ and $F'(1) = r > 1$, then $F_r(x) < x$ on $(0, 1)$ and there exists a real number α_r such that $F_r(x) > x$ on $(1, \alpha_r)$ and $F_r(x) < x$ on (α_r, ∞) (see Figure 1). The results follow from these observations. □

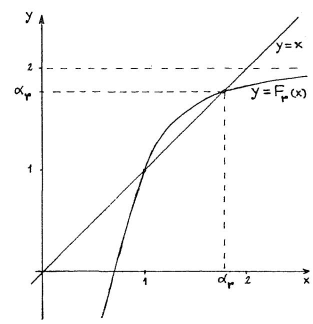

FIGURE 1. Graph of $y = F_r(x)$

Lemma 2: Let $r \geq 2$ and let

$$f_r(x) = 1 + \sum_{i=2}^{r} \frac{1}{x^{i-1}}.$$

The equation $x = f_r(x)$ has a unique solution β_r in the interval $(0, \infty)$. Also β_r is the unique positive root of the polynomial

$$p_r(x) = x^r - \sum_{i=1}^{r} x^{r-i}.$$

Proof: If $x \in (0, \infty)$, we have

$$f_r'(x) = -\sum_{i=2}^{r} \frac{(i-1)}{x^i} < 0 \quad \text{and} \quad f_r''(x) = \sum_{i=2}^{r} \frac{i(i-1)}{x^{i+1}} > 0;$$

therefore, $f_r(\cdot)$ is a strictly decreasing continuous convex function on $(0, \infty)$. Also

$$\lim_{x \to 0^+} f_r(x) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} f_r(x) = 1 \text{ (see Figure 2).}$$

It follows that there exists a unique positive x such that $x = f_r(x)$. Also, for $x > 0$, $x = f_r(x)$ is equivalent to

$$x^r = \sum_{i=1}^{r} x^{r-i}$$

and the result follows. □

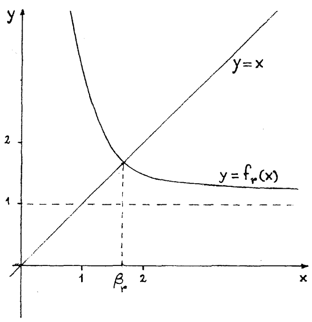

FIGURE 2. Graph of $y = f_r(x)$

Lemma 3: Let $r \geq 2$. For $x \neq 1$, $x = f_r(x)$ is equivalent to $x = F_r(x)$, and it follows that

$$\beta_r = \alpha_r \in \left(2\left(1 - \frac{1}{2^r}\right),\ 2\right).$$

Proof: $x = F_r(x)$ is equivalent to $x^r(x - 1) = x^r - 1$. For $x \neq 1$, $x = F_r(x)$ is equivalent to

$$x^r = \sum_{i=0}^{r-1} x^i$$

which is also equivalent to $x = f_r(x)$. Hence,

$$\alpha_r = f_r(\alpha_r) \geq f_r(2) = 2\left(1 - \frac{1}{2^r}\right). \quad \square$$

From Lemmas 1-3 we can conclude that i) the sequence $\{B_{r,\ell}\}_{\ell=0}^{\infty}$ is strictly decreasing and converges to α_r, ii) the sequence $\{b_{r,\ell}\}_{\ell=0}^{\infty}$ is strictly increasing and converges to α_r. Then, using (4), we have the following result.

Theorem 1: Let $r \geq 2$, $u_{r,n}$ given by (1), and

$$t_{r,n} = \frac{u_{r,n}}{u_{r,n-1}} \qquad \text{for } n \geq 1.$$

The sequence $\{t_{r,n}\}_{n=1}^{\infty}$ converges to the unique positive root α_r of the polynomial

$$p_r(x) = x^r - \sum_{i=1}^{r} x^{r-i}. \quad \square$$

We could call α_r the r-generalized golden number; hence, we have the following result.

Theorem 2: The sequence of r-generalized golden numbers $\{\alpha_r\}_{r=2}^{\infty}$ is a strictly increasing sequence converging to 2.

Proof: Let $2 \le r_1 \le r_2$. We have $F_{r_1}(x) < F_{r_2}(x)$ for all $x \in (1, \infty)$; Hence,

$$\alpha_{r_1} = F_{r_1}(\alpha_{r_1}) < F_{r_2}(\alpha_{r_1}).$$

It follows that $\alpha_{r_1} \in (1, \alpha_{r_2})$. Then the sequence $\{\alpha_r\}_{r=2}^{\infty}$ is strictly increasing and upper bounded by 2. It converges and we have

$$\lim_{r\to\infty} \alpha_r = \lim_{r\to\infty}\left(2 - \frac{1}{\alpha_r^r}\right) = 2. \quad \square$$

Remark 1: Somer [8] considered the proof of Theorem 2 based on continued fractions.

Remark 2: We have shown that α_r is the unique positive root of the polynomial

$$p_r(x) = x^r - \sum_{i=1}^{r} x^{r-i}.$$

We can also easily observe that $p_r(x)$ has

(i) only one negative real root if r is even,
(ii) no negative real root if r is odd,

because $p_r(x) = 0$ is equivalent to

$$x^r = \frac{x^r - 1}{x - 1} \quad \text{for } x < 0$$

(see Miles [5] for a complete study of the polynomial $p_r(x)$).

Remark 3: We could consider that $u_{r,i}$ are given positive real numbers for $i = 0, \ldots, r-1$ and that $u_{r,n}$ are generated using (1c) for $n \ge r$. In this way, we could show that $t_{r,n} \ge 1$ for $n \ge r$ and $t_{r,n} \le 2$ for $n \ge 2r$. More generally, it follows that we could start with any given real numbers $u_{r,i}$ ($i = 0, \ldots, r-1$) and use the method described here to show

$$\lim_{n\to\infty} t_{r,n} = \alpha_r,$$

which is the positive root of $p_r(x)$, as soon as r successive values $u_{r,i}$ of the same sign appear.

2. A Geometric Interpretation

Let us consider the sequence of r-tuples $\{\vec{v}_{r,n}\}_{n=0}^{\infty}$ generated by induction. Let $\vec{v}_{r,0} = (u_{r,0}, u_{r,1}, \ldots, u_{r,r-1})$. Assuming that $\vec{v}_{r,j}$ is already generated for $j = 0, \ldots, n-1$, we generate $\vec{v}_{r,n}$ as follows:

(i) determine the unique integers i and k such that $n = i + kr$, $0 < i \le r$ and $k \ge 0$ [in other words, $i = 1 + (n-1) \bmod r$],

(ii) the coordinates of $\vec{v}_{r,n}$ are those of $\vec{v}_{r,n-1}$ except for the i^{th} coordinate of $\vec{v}_{r,n}$ which is the sum of the r coordinates of $\vec{v}_{r,n-1}$.

From this construction, we can show that the coordinates of $\vec{v}_{r,n}$ are successively $u_{r,n}, u_{r,n+1}, \ldots, u_{r,n+r-1}$ where $u_{r,n+r-1}$ is the i^{th} coordinate, and the sum of the coordinates of $\vec{v}_{r,n}$ is $u_{r,n+r}$.

To each $\vec{v}_{r,n}$ we can associate the parallelepiped rectangle in $\mathbb{R}^r$ having this point as the vertex that is not on the axis. This construction for $r > 2$ is a natural generalization of what happens in the case $r = 2$. Figures 3 and 4 illustrate the cases $r = 2$ and $r = 3$, respectively.

$$\vec{v}_{2,0} = (1,\ 1)$$
$$\vec{v}_{2,1} = (2,\ 1)$$
$$\vec{v}_{2,2} = (2,\ 3)$$
$$\vec{v}_{2,3} = (5,\ 3)$$
$$\vec{v}_{2,4} = (5,\ 8)$$
$$\vec{v}_{2,5} = (13,\ 8)$$
$$\vdots$$

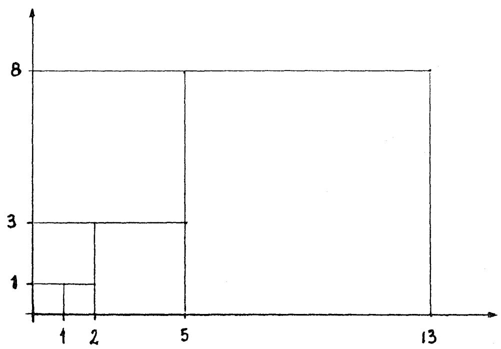

FIGURE 3. Case $r = 2$

$$\vec{v}_{3,0} = (1,\ 1,\ 2)$$
$$\vec{v}_{3,1} = (4,\ 1,\ 2)$$
$$\vec{v}_{3,2} = (4,\ 7,\ 2)$$
$$\vec{v}_{3,3} = (4,\ 7,\ 13)$$
$$\vec{v}_{3,4} = (24,\ 7,\ 13)$$
$$\vec{v}_{3,5} = (24,\ 44,\ 13)$$
$$\vdots$$

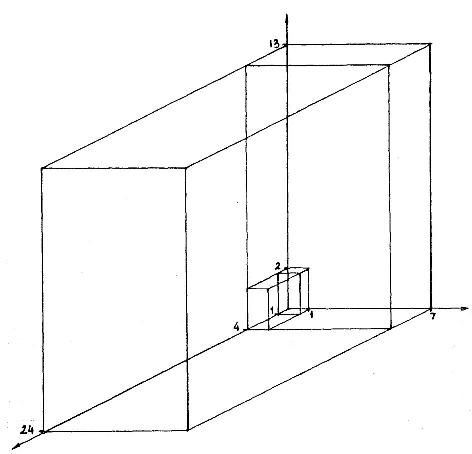

FIGURE 4. Case $r = 3$

Normalizing the vectors $\vec{v}_{r,n}$ with respect to the uniform norm $\|\cdot\|_\infty$, we observe that

$$\lim_{k\to\infty} \frac{\vec{v}_{r,\,i+kr}}{\|\vec{v}_{r,\,i+kr}\|_\infty} = \vec{d}_{r,\,i} \qquad (i = 1,\ \ldots,\ r)$$

where $\vec{d}_{r,\,i}$ is a unit vector, with respect to the uniform norm, having the coordinates $1/\alpha_r^{r-1},\ 1/\alpha_r^{r-2},\ \ldots,\ 1/\alpha_r^2,\ 1/\alpha_r,\ 1$, and such that 1 is the i^{th} coordinate. Figures 5 and 6 illustrate the vectors $\vec{d}_{r,\,i}$ $(i = 1,\ \ldots,\ r)$ for $r = 2$ and $r = 3$, respectively.

$$\alpha_2 = 1.618034\ldots$$
$$\vec{d}_{2,1} = (1,\ 1/\alpha_2)$$
$$\vec{d}_{2,2} = (1/\alpha_2,\ 1)$$

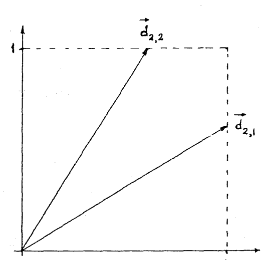

FIGURE 5. Case $r = 2$

$\alpha_3 = 1.8392868\ldots$

$\vec{d}_{3,1} = (1,\ 1/\alpha_3^2,\ 1/\alpha_3)$

$\vec{d}_{3,2} = (1/\alpha_3,\ 1,\ 1/\alpha_3^2)$

$\vec{d}_{3,3} = (1/\alpha_3^2,\ 1/\alpha_3,\ 1)$

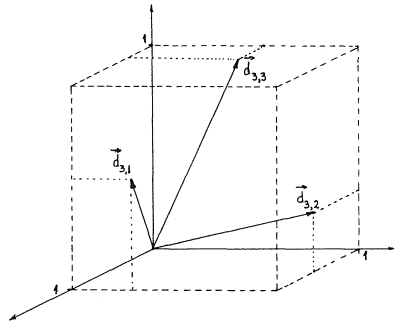

FIGURE 6. Case $r = 3$

Moreover, the volume V of the parallelepiped generated by the vectors $\vec{d}_{r,1}$, $\vec{d}_{r,2}$, ..., $\vec{d}_{r,r}$ is

$$V_r = \det(\vec{d}_{r,1}, \ldots, \vec{d}_{r,r}) = \left(1 - \frac{1}{\alpha_r^r}\right)^{r-1}.$$

Since $\lim\limits_{r\to+\infty} \alpha_r = 2$, it follows that $\lim\limits_{r\to\infty} V_r = 1$.

We can present an informal interpretation of the last result. If we consider coordinatewise convergence, we can define for the sequence $\{\vec{d}_{r,i}\}_{r=i}^{\infty}$ the limit

$$\vec{d}_{\infty,i} = \lim_{r\to\infty} \vec{d}_{r,i} = (2^{1-i},\ 2^{2-i},\ \ldots,\ 2^{-2},\ 2^{-1},\ 1,\ 0,\ 0,\ \ldots)$$

which is a vector in the infinite-dimensional euclidean space $\mathbb{R}^{\infty}$ (or the set of infinite sequences). Hence, the semi-infinite determinant

$$V_{\infty} = \det(\vec{d}_{\infty,1},\ \vec{d}_{\infty,2},\ \ldots) = \lim_{r\to\infty} V_r$$

is triangular and has 1's along the diagonal, so $V_{\infty} = 1$.

3. Electrical Schemes

It is well known that we can generate the sequence

$$\left\{\frac{u_{2,n+1}}{u_{2,n}}\right\}_{n=0}^{\infty}$$

using electrical circuits (see [1], [2], [3], [4], [6], [7]). Recently, Beran [2] wondered if it was also possible for the sequence

$$\left\{\frac{u_{3,n+1}}{u_{3,n}}\right\}_{n=0}^{\infty}.$$

We present here one method to generate the sequence

$$\left\{\frac{u_{r,n+1}}{u_{r,n}}\right\}_{n=0}^{\infty}$$

using electrical circuits.

Let us define the resistances

$$\Omega_{j,i}^{r} = \frac{u_{r,j+i}}{u_{r,j}}$$

for $j \geq 0$ and $i \geq -j$. Hence defined, connecting in series r successive resistances

$$\Omega_{j,i+k}^{r} \quad (k = 0, \ldots, r-1)$$

we obtain the resistance next to the last one $\Omega_{j,i+r}^{r}$ because

$$\Omega_{j,i+r}^{r} = \sum_{k=0}^{r-1} \Omega_{j,i+k}^{r}.$$

Also, connecting in parallel r successive resistances

$$\Omega_{j+k,i}^{r} \quad (k = 0, \ldots, r-1)$$

we obtain again the resistance next to the last one $\Omega_{j+r,i}^{r}$ because

$$\Omega_{j+r,i}^{r} = \frac{1}{\sum_{k=0}^{r-1} 1/\Omega_{j+k,i}^{r}}.$$

Using these observations, we can generate a sequence of sets $\{S_n^r\}_{n=0}^{\infty}$, where S_n^r is the set of resistances having values $\Omega_{n,i}^{r}$ for $i = -r, -r+1, \ldots, -1, 0, 1, \ldots, r-1, r$. The process is by induction.

For $n = 0$, we have:

(i) $\Omega_{0,i}^{r} = 0$ for $i = -r, \ldots, -1$;

(ii) $\Omega_{0,0}^{r} = 1$;

(iii) $\Omega_{0,i}^{r} = \sum_{j=1}^{i} \Omega_{0,i-j}^{r}$ for $i = 1, \ldots, r$.

Assuming that the resistances in the sets $S_0^r, S_1^r, S_2^r, \ldots, S_{n-1}^r$ are available, we can generate the resistances in the set S_n^r as follows:

(i) for $i = -r, \ldots, -1$, we have $\Omega_{n,i}^{r} = \dfrac{1}{\sum_{j=1}^{r} 1/\Omega_{n-j,j+i}^{r}}$

and $\Omega_{n-j,j+i}^{r} \in S_{n-j}^{r}$ for $j = 1, \ldots, r$

(in these expressions we do not consider a term for which the index j is such that $n - j < 0$). Then the resistance $\Omega_{n,i}^{r}$ can be constructed if we use the already constructed resistances and connect them in parallel.

(ii) $\Omega_{n,0}^{r} = 1$.

(iii) for $i = 1, \ldots, r$, we have $\Omega_{n,i}^{r} = \sum_{j=1}^{r} \Omega_{n,i-j}^{r}$,

where $\Omega_{n,i-j}^{r} \in S_n^r$ for $j = 1, \ldots, r$.

These resistances are already known and can be connected in series to obtain the desired resistance.

If we consider the rational resistances hence built, in each set S_n^r their smallest common denominator is $u_{r,n}$ if we start with $u_{r,0}, \ldots, u_{r,r-1}$ having no common factor, i.e., $(u_{r,0}, u_{r,1}, \ldots, u_{r,r-1}) = 1$. Then, if we write these rational numbers using their common denominator $u_{r,n}$, the numerators form the sequence $\{u_{r,n+i}\}_{i=-r}^{r}$.

References

1. S. L. Basin. "The Appearance of Fibonacci Numbers and the Q Matrix in Electrical Network Theory." *Math. Mag.* *36* (1963):84-97.
2. L. Beran. "Schemes Generating the Fibonacci Sequence." *Math. Gazette 70* (1986):38-40.
3. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.
4. V. E. Hoggatt, Jr., & M. Bicknell. "A Primer for the Fibonacci Numbers: Part XIV." *Fibonacci Quarterly* *12* (1974):147-156.
5. E. P. Miles. "Generalized Fibonacci Numbers and Associated Matrices." *Amer. Math. Monthly* *67* (1960):745-752.
6. A. M. Morgan-Voyce. "Ladder Network Analysis Using Fibonacci Numbers." *IRE Transactions on Circuit Theory*, Vol. CT-6, September 1959, pp. 321-322.
7. W. P. Risk. "Thevenin Equivalents of Ladder Networks." *Fibonacci Quarterly* *20* (1982):245-248.
8. L. Somer. Problem H-197 (and solution). *Fibonacci Quarterly* *12* (1974) 110-111.

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NOTE ON A FAMILY OF FIBONACCI-LIKE SEQUENCES

John C. Turner

University of Waikato, Hamilton, New Zealand

(Submitted May 1987)

In [2] P. Asveld gave a solution to the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j \text{ with } G_0 = G_1 = 1. \tag{1}$$

In [2] we showed that the solution to the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + S_n,\ G_1 = S_1,\ G_2 = S_1 + S_2, \tag{2}$$

where S_n is the n^{th} term of *any* sequence $\{S_n\} \equiv S$, is given by the n^{th} term of the convolution of the Fibonacci sequence F with the sequence S. That is, the solution of (2) can be expressed as

$$G_n = (F * S)_n,$$

using $*$ to mean *convolution*.

This note shows how Asveld's family can be dealt with by the convolution technique, using generating functions. Although we do not work through the details in the note, it is clear that a comparison of the two final solutions would yield interesting identities relating Asveld's tabulated polynomials and coefficients, and the coefficients from our solution.