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ON R. VON MISES' CONDITION FOR THE DOMAIN OF ATTRACTION OF $\exp(-e^{-x})$[1]

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There exist well-known necessary and sufficient conditions for a distribution function to belong to the domain of attraction of the double exponential distribution Λ. For practical purposes a simple sufficient condition due to von Mises is very useful. It is shown that each distribution function F in the domain of attraction of Λ is tail equivalent to some distribution function satisfying von Mises' condition.

Suppose $X_1, X_2, X_3, \cdots$ are independent real-valued random variables with common distribution function F. We say that F is in the domain of attraction of the double exponential distribution (notation $F \in D(\Lambda)$; $\Lambda(x) = \exp(-e^{-x})$) if there exist two sequences of real constants $\{b_n\}$ and $\{a_n\}$ (with $a_n > 0$ for $n = 1, 2, \cdots$) such that for all real x

$$(1) \qquad \lim_{n\to\infty} P\left\{\frac{\max(X_1, X_2, \cdots, X_n) - b_n}{a_n} \leqq x\right\} = \exp(-e^{-x}) .$$

It is convenient to use the symbol x_0 for the upper bound of X_1 defined by

$$x_0(F) = \sup\{x \mid F(x) < 1\} .$$

Necessary and sufficient conditions for $F \in D(\Lambda)$ are well-known (Gnedenko (1943), de Haan (1970)) but rather intricate. The following relatively simple criterion is due to R. von Mises ((1936) page 285):

Suppose $F(x)$ is a distribution function with a density $f(x)$ which is positive and differentiable on a left neighborhood of x_0. If

$$(2) \qquad \lim_{x\uparrow x_0} \frac{d}{dx}\left(\frac{1-F(x)}{f(x)}\right) = 0 ,$$

then $F \in D(\Lambda)$.

A distribution function F satisfying (2) will be called a *von Mises function.*

We shall prove

THEOREM. *A distribution function F lies in the domain of attraction of* Λ *if and only if there exists a von Mises function* F_* *such that* $x_0(F_*) = x_0(F) = x_0$ *and*

$$(3) \qquad \lim_{x\uparrow x_0} \frac{1-F(x)}{1-F_*(x)} = 1 .$$

REMARK. Relation (3) implies (see Resnick (1971) Lemma 2.5) that for the

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convergence of the distribution functions F^n and $F_*{}^n$ the same norming constants $a_n > 0$ and b_n may be used.

PROOF. The if statement is an immediate consequence of Gnedenko (1943), Théorème 7.

Now suppose $F \in D(\Lambda)$ with endpoint x_0. We define the sequence $U_0, U_1, \cdots$ by

$$U_0(x) = 1 - F(x)$$
$$U_{n+1}(x) = \int_x^{x_0} U_n(t)\, dt \qquad n = 0, 1, 2, \cdots .$$

By de Haan ((1970) Lemma 2.5.1 or (1971) Lemma 6 and Theorem 8) the distribution function F_n defined by $F_n(x) = \max(0, 1 - U_n(x))$ belongs to $D(\Lambda)$ if F_{n-1} does. In particular, the integral above converges. Then $F_n \in D(\Lambda)$ for $n = 0, 1, 2, \cdots$ and by de Haan ((1970) Theorem 2.5.2 or (1971) Theorem 10) we have

$$\lim_{x \uparrow x_0} \frac{U_{n-1}(x) U_{n+1}(x)}{U_n{}^2(x)} = 1 \qquad n = 1, 2, \cdots . \tag{4}$$

We now define the function U_* on $(-\infty, x_0)$ by

$$U_*(x) = \{U_3(x)\}^4 \{U_4(x)\}^{-3} .$$

Then $U_*(x)$ is twice differentiable on a left neighborhood of x_0 and

$$\frac{d}{dx} \log U_* = -4 \frac{U_2}{U_3} + 3 \frac{U_3}{U_4} = \frac{3 - 4U_2 U_3{}^{-2} U_4}{U_4 U_3{}^{-1}} . \tag{5}$$

Consider

$$\frac{U_4 U_3{}^{-1}}{3 - 4U_2 U_3{}^{-2} U_4} = \frac{U_*}{\frac{d}{dx} U_*} .$$

By (4) the denominator is asymptotic to -1 as $x \uparrow x_0$ and both $(d/dx) U_4 U_3{}^{-1}$ and $U_4 U_3{}^{-1} (d/dx)(3 - 4U_2 U_3{}^{-2} U_4)$ vanish as $x \uparrow x_0$. Hence

$$\lim_{x \uparrow x_0} \frac{d}{dx} \left(\frac{U_*(x)}{U_*'(x)} \right) = 0 . \tag{6}$$

Observe that

$$U_0 = \frac{U_0 U_2}{U_1^2} \cdot \left(\frac{U_1 U_3}{U_2^2} \right)^2 \cdot \left(\frac{U_2 U_4}{U_3^2} \right)^3 \cdot U_* .$$

Hence by (4) we obtain

$$\lim_{x \uparrow x_0} \frac{U_0(x)}{U_*(x)} = 1 . \tag{7}$$

Then $\lim_{x \uparrow x_0} U_*(x) = 0$, and since by (5) U_* is decreasing on a left neighborhood of x_0, there exists a twice differentiable distribution function $F_*(x)$ which coincides with $1 - U_*(x)$ on a left neighborhood of x_0. F_* is a von Mises function by (6) and satisfies (2) by (7). □

COROLLARY. *A distribution function F belongs to $D(\Lambda)$ if and only if there exist* *a positive function c satisfying* $\lim_{x \uparrow x_0} c(x) = 1$ *and a positive differentiable function* ϕ *satisfying* $\lim_{x \uparrow x_0} \phi'(x) = 0$ *such that*

$$1 - F(x) = c(x) \cdot \exp\left\{-\int_{-\infty}^{x} \frac{dt}{\phi(t)}\right\} \qquad \text{for } x < x_0 .$$

This improves the representation theorem (Theorem 2.5.3) in de Haan (1970).

PROOF. Set

$$\phi(x) = \frac{1 - F_*(x)}{F_*'(x)}$$

in a left neighborhood of x_0.

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