On Radiative Corrections due to Soft Photons.

by

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SUMMARY

It is shown how the infra-red divergent part, due to soft virtual photons, of any scattering amplitude can be factorized in a Lorentz invariant manner.

It is further shown that if the energy resolution for a process is negligible compared to the electron rest mass then the whole soft photon contribution to the transition probability density can be factorized leaving a Lorentz-invariant remainder.

This factorization is then extended to the more realistic case of an energy resolution which is of the same order of magnitude as the electron rest mass, or larger.

Finally, the magnitudes of radiative corrections are discussed.

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I. Introduction

It has been known for a long time that the so-called infra-red divergences encountered in quantum electrodynamics are connected with the fact that the number of soft photons is not an observable. It is also well-known that the infra-red divergences cancel exactly when soft photons are treated collectively. A proof of this cancellation to all orders of perturbation theory was given by Nakanishi [1].

In their work on infra-red divergences Jauch and Rohrlich [2] introduced a low energy cut-off equal to the energy resolution ΔE and then succeeded in expressing the soft photon contributions in closed form as a factor in the scattering cross-section. Since the appearance of their paper the factorization of the soft photon contributions has been accomplished in different ways by several authors [3] with the use of an infra-red cut-off.

In a recent paper Yennie, Frautschi and Suura [4] find that the soft photon contributions to a scattering cross-section can be factorized in such a way that the remainder is cut-off independent and can be calculated with the usual methods of perturbation theory.

The same result has been obtained independently by the present author, and a proof of the cut-off independent factorization in perturbation theory is given in this paper. The difficulties that occur in the paper of Yennie, Frautschi and Suura in connection with what they call "overlapping infra-red divergences" will not appear in the following treatment. They are avoided by a splitting of the electromagnetic field into two "components", one for soft photons and one for hard photons, similar to the method of Murota [5].

In high energy electromagnetic processes only part of the low energy radiation can be factorized; the rest is left to be handled by perturbation theory. A discussion of this is contained in the last part of the paper.

With regard to metric, normalization factors etc. we shall follow the conventions used in the text-book of Jauch and Rohrlich [6].

We shall consider a process

$$|i\rangle \rightarrow |f\rangle \tag{1}$$

in which r particles with momenta $p_1, \ldots, p_r$ and charges $Q_1, \ldots, Q_r$ take part. With this process there is connected a (Lorentz-) invariant amplitude, M, defined in terms of the S-matrix element by

$$\langle f | S - 1 | i \rangle = \delta (P_f - P_i) M \tag{2}$$

Here P_i and P_f are the total momenta of the states $|i\rangle$ and $|f\rangle$ respectively.

In general M is infra-red divergent. However, as is well-known - and as is shown below - the infra-red divergences deriving from M cancel with the infra-red divergences deriving from soft photon emission accompanying the process (1). Since "soft" is not an invariant concept, the discussion of soft photons must be restricted to a specific Lorentz-frame L. We assume that in L the energy loss due to soft photon emission does not exceed ΔE, with

$$\Delta E \ll m \quad (m = \text{electron rest mass}) \tag{3}$$

In Section II an investigation is made of the origin of the infra-red divergences, and a preliminary separation of the infra-red contributions is performed. In Section III the infra-red divergent part of M is factorized in a Lorentz-invariant manner, and a "non-infra-red invariant amplitude", $\bar{M}$, is defined. In Section IV the transition probability density for process (1) with energy resolution ΔE, is derived in the form $F(\Delta E) |\bar{M}|^2$ and the infra-red factor $F(\Delta E)$ is studied in detail. In Section V, finally, some questions are discussed that are of interest for practical calculations on electrodymanic processes at high energies. Then we omit restriction (3).

II. Preliminary discussion

Throughout the following we shall make all considerations with reference to the Lorentz-frame L, mentioned above.

We define a region R_ε of the 4-dimensional k-space by

$$k \in R_\varepsilon \quad \text{if and only if} \quad \begin{cases} |\underline{k}| \leq \varepsilon \\ k_0 \leq \varepsilon \end{cases} \quad \varepsilon \ll m \tag{4}$$

To the division of k-space into R_ε and its complementary region there corresponds a separation of the free electromagnetic field into two parts (like in ref. [5])

$$a_\mu(x) = a_{s\mu}(x) + a_{h\mu}(x) \tag{5}$$

where

$$\begin{aligned} a_{s\mu}(x) &= \frac{1}{(2\pi)^{3/2}} \int\limits_{k_0=|\underline{k}|\leq\varepsilon} \frac{d^3k}{\sqrt{2k_0}} \left[a_\mu(\underline{k})e^{ikx} + a^*_\mu(\underline{k})e^{-ikx} \right] \\ a_{h\mu}(x) &= \frac{1}{(2\pi)^{3/2}} \int\limits_{k_0=|\underline{k}|>\varepsilon} \frac{d^3k}{\sqrt{2k_0}} \left[a_\mu(\underline{k})e^{ikx} + a^*_\mu(\underline{k})e^{-ikx} \right] \end{aligned} \tag{6}$$

As a consequence of the commutation relations

$$\begin{aligned} \left[a_\mu(\underline{k}), a_\nu(\underline{k}')\right] &= \left[a^*_\mu(\underline{k}), a^*_\nu(\underline{k}')\right] = 0; \\ \left[a_\mu(\underline{k}), a^*_\nu(\underline{k}')\right] &= g_{\mu\nu}\,\delta(\underline{k}-\underline{k}') \end{aligned} \tag{7}$$

we then have the new relations

$$\left[a_{s\mu}(x), a_{h\nu}(x')\right] = \left[a_{s\mu}(x), a^*_{h\nu}(x')\right] = \left[a^*_{s\mu}(x), a_{h\nu}(x')\right] =$$

$$\left[a^*_{s\mu}(x), a^*_{h\nu}(x')\right] = 0 \tag{8}$$

showing that soft photons (those described by the field $a_{s\mu}(x)$) and hard photons (those described by the field $a_{h\mu}(x)$) can be treated as different particles with different graphical representations in perturbation theory (Fig. 1).

To see this more explicitly let

$$S = \sum_{n=0}^{\infty} S^{(n)} \tag{9}$$

be an expansion of the S-operator in powers of the coupling constant e. Then the n^{th} term of the expansion is

$$S^{(n)} = \frac{e^n}{n!} \int \dots \int dx_n \dots dx_1 \; O^{\mu_1 \dots \mu_n}(x_1, \dots, x_n) T(a_{\mu_1}(x_1) \dots a_{\mu_n}(x_n)) \tag{10}$$

where $O^{\mu_1 \dots \mu_n}(x_1, \dots x_n)$ is an operator which is symmetric in $x_1, \dots, x_n$ and is composed of all fields except the electromagnetic field, and T means time-ordered product. According to (8) the expression (10) can be split into

$$S^{(n)} = \frac{e^n}{n!} \sum_{n'=0}^{n} \binom{n}{n'} \int \dots \int dx_n \dots dx_1 \, O^{\mu_1 \dots \mu_n}(x_1, \dots, x_n) \, T\left(a_{h\mu_1}(x_1) \dots a_{h\mu_{n'}}(x_{n'})\right) \cdot$$

$$\cdot T\left(a_{s\mu_{n'+1}}(x_{n'+1}) \dots a_{s\mu_n}(x_n)\right) = \sum_{n'=0}^{n} \frac{e^{n'}}{n'!} \frac{e^{n-n'}}{(n-n')!} \int \dots \int dx_n \dots dx_1 \cdot$$

$$\cdot O^{\mu_1 \dots \mu_n}(x_1, \dots, x_n) \, T\left(a_{h\mu_1}(x_1) \dots a_{h\mu_{n'}}(x_{n'})\right) \, T\left(a_{s\mu_{n'+1}}(x_{n'+1}) \dots a_{s\mu_n}(x_n)\right) \tag{11}$$

Let the "black box", $\hat{M}$, in Fig. 2 represent the sum over all diagrams for the process (1) that do not include any soft photons. The corresponding transition amplitude will also be denoted by "$\hat{M}$". Then $\hat{M}$ must be corrected for virtual and real soft photon interactions. Only soft photons giving rise to infra-red divergences, i.e. to logarithmic momentum dependences need to be considered, since terms which are linear in soft photon momenta, are smaller by a factor $\frac{\Delta E}{m}$ and can be neglected because of (3). It appears that a soft photon gives rise to an infra-red divergence only if it is attached at a "single photon corner" to an external line of $\hat{M}$.*) Thus the only corrections to $\hat{M}$ that one needs to consider are those due to interchange of soft photons between external lines and emission of soft photons (with a total energy $\leq \Delta E$) from the external lines (see Fig. 4).

Consider first one of the external lines of $\hat{M}$, say the i^{th} line. Assume that it represents an outgoing particle. Then we can write

$$\hat{M} = \beta^* (p_i) \hat{M}_i \quad (\beta^*(p_i) = \text{boson emission factor}) \tag{12b}$$

if that line represents a boson, and

$$\hat{M} = \bar{u}(p_i) \hat{M}_i \quad (\bar{u} (p_i) \text{ is a spinor}) \tag{12f}$$

if it represents a fermion. If $Q_i = 0$, no soft photons can be attached to the i^{th} line. Assume $Q_i \neq 0$. Let us attach n soft photons with nomenta $k_1, \ldots, k_n$ (counted as outgoing) to that line. If we suppress photon factors and neglect terms in the numerators that are linear in soft photon

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*) This is a well-known fact. It can easily be proved that a soft photon which is attached to

a) the interior of $\hat{M}$ (Fig. 3a) or

b) a closed loop (Fig. 3b) or

c) a boson line at a "two-photon corner" (Fig. 3c)

gives a contribution which is not infra-red divergent but vanishes when the photon momentum goes to zero.

momenta, the added soft photons change $\hat{M}$ into

$$\frac{(eQ_i)^n}{n!}\sum_{(j_1,\dots,j_n)}\beta^*(p_i)\,\frac{2ip_i^{\mu_{j_1}}\,2ip_i^{\mu_{j_2}}\dots 2ip_i^{\mu_{j_n}}}{2k_{j_1}\cdot p_i\,[2(k_{j_1}+k_{j_2})\cdot p_i]\dots[2(k_{j_1}+\dots+k_{j_n})\cdot p_i]}\,\hat{M}; \quad \text{(boson case)} \qquad (13b)$$

$$\frac{(eQ_i)^n}{n!}\sum_{(j_1,\dots,j_n)}\bar{u}(p_i)\,\frac{\gamma^{\mu_{j_1}}(i\not{p}_i-m)\gamma^{\mu_{j_2}}(i\not{p}_i-m)\dots\gamma^{\mu_{j_n}}(i\not{p}_i-m)}{2k_{j_1}\cdot p_i\,[2(k_{j_1}+k_{j_2})\cdot p_i]\dots[2(k_{j_1}+\dots+k_{j_n})\cdot p_i]}\,\hat{M}; \quad \text{(fermion case)} \qquad (13f)$$

The summations are over all permutations of $j_1, \dots, j_n$ among $1, \dots, n$. Rearranging the factors in (13b,f) (with the use of the Dirac equation for (13f)) and inserting (12b,f) we get for bosons and fermions the identical result

$$\frac{1}{n!}\prod_{j=1}^{n}(ieQ_ip_i^{\mu_j})\sum_{(j_1,\dots,j_n)}\frac{1}{(k_{j_1}\cdot p_i)[(k_{j_1}+k_{j_2})\cdot p_i]\dots[(k_{j_1}+\dots+k_{j_n})\cdot p_i]}\,\hat{M} = \frac{1}{n!}\prod_{j=1}^{n}\left(\frac{ieQ_ip_i^{\mu_j}}{k_j\cdot p_i}\right)\hat{M} \qquad (14)$$

If the i^{th} particle were incoming rather than outgoing, every factor in the denominator of (14) should include a minus sign. Thus the contribution from the diagrams with n soft photons attached to the i^{th} external line (representing an in- or outgoing, charged or uncharged particle) of $\hat{M}$ is, apart from photon factors, given by

$$\frac{1}{n!}\prod_{j=1}^{n}\left(\frac{i\,e\,Q_i\,p_i^{\mu_j}}{k_j\cdot\varepsilon_i\,p_i}\right)\quad\hat{M} \qquad (15)$$

where

$$\varepsilon_i = \begin{cases} +1 \\ -1 \end{cases} \quad \text{if the } i^{th} \text{ particle is } \begin{cases}\text{incoming} \\ \text{outgoing}\end{cases} \qquad (16)$$

Consider next a diagram with m emitted and l internal soft photons. Assume that the m'^{th} emitted photon has a momentum $\underline{k}_{m'}$, and polarization $e_{m'}$, and is emitted from the $i_{m'}{}^{th}$ external line of $\hat{M}$.

The l'^{th} internal photon is assumed to have a momentum $k'_{l'}$ and to join the $i_{l'1}^{th}$ and the $i_{l'2}^{th}$ external lines of $\hat{M}$. The generalized quantity corresponding to (15) - including photon factors $(i(2\pi)^{-3/2} \cdot (2k_{m'o})^{\frac{1}{2}} e_{m'\mu_{m'}}$ for emitted and $i(2\pi)^{-4}(k'^2_{l'})^{-1}$ for internal soft photons), and also including integrations over internal momenta - is now

$$\frac{1}{(m+2\ell)!}\prod_{m'=1}^{m}\left(\frac{i}{(2\pi)^{3/2}}\frac{ieQ_{i_{m'}}p_{i_{m'}}\cdot e_{m'}}{\sqrt{2k_{m'o}}\,(k_{m'}\cdot\varepsilon_{i_{m'}}p_{i_{m'}})}\right)\cdot$$

$$\cdot\prod_{\ell'=1}^{\ell}\left(\frac{-i}{(2\pi)^4}\int_{(R_\varepsilon)}\frac{d^4k'_{\ell'}\;e^2Q_{i_{\ell'1}}Q_{i_{\ell'2}}\;p_{i_{\ell'1}}\cdot p_{i_{\ell'2}}}{(k'^2_{\ell'}+\lambda^2)(k'_{\ell'}\cdot\varepsilon_{i_{\ell'1}}p_{i_{\ell'1}})(-k'_{\ell'}\cdot\varepsilon_{i_{\ell'2}}p_{i_{\ell'2}})}\right)\hat{M};\;(k_{m'o}=\sqrt{\underline{k}^2_{m'}+\lambda^2})\qquad(17)$$

(λ = fictious photon mass; $\lambda \ll \varepsilon, \Delta E$)

We introduce now the following definitions which will prove useful.

$$s^{\mu}(k) = \frac{i\,e}{(2\pi)^{3/2}}\sum_{i=1}^{r}\frac{2\,Q_i\,p_i^{\mu}}{k^2+2k\cdot\varepsilon_i\,p_i}\qquad(18)$$

and

$$\hat{A} = \frac{1}{2\pi i}\int_{(R_\varepsilon)}\frac{d^4k}{k^2+\lambda^2}\,s_{\mu}(k)\,s^{\mu}(-k)\qquad(19)$$

The expression (17) must be summed over all diagrams in which soft photons are emitted. This means

i) a summation over the

$$\frac{(m+2l)\,!}{m\,!\,l\,!\,2^l}\qquad(20)$$

different ways in which the l pairs of endpoints of internal soft photons may be chosen among the $m + 2l$ soft photon corners. This summation is denoted by $\sum'$.

ii) a summation over each of the $m + 2l$ indices $i_1, \ldots, i_m$, $i_{11}, \ldots, i_{l2}$ from 1 to r.

iii) a summation over l from 0 to ∞.

So using the definitions (18) and (19) we get (since $k^2 \cong 0$ for soft photons)

$$\sum_{l=0}^{\infty} \sum_{\substack{i_1,\ldots,i_m,\\ i_{11},\ldots,i_{l2}=1}}^{r} \sum{}' \frac{1}{(m+2l)!} \prod_{m'=1}^{m} \left(\frac{i}{(2\pi)^{3/2}} \frac{ieQ_{i_{m'}} P_{i_{m'}} \cdot e_{m'}}{\sqrt{2k_{m'0}}\,(k_{m'} \cdot \varepsilon_{i_{m'}} P_{i_{m'}})} \right) \cdot$$

$$\cdot \prod_{l'=1}^{l} \left(\frac{-i}{(2\pi)^4} \int_{(R_\varepsilon)} \frac{d^4k'_{l'}\, e^2 Q_{i_{l'1}} Q_{i_{l'2}} P_{i_{l'1}} \cdot P_{i_{l'2}}}{(k'^2_{l'}+\lambda^2)(k'_{l'} \cdot \varepsilon_{i_{l'1}} P_{i_{l'1}})(-k'_{l'} \cdot \varepsilon_{i_{l'2}} P_{i_{l'2}})} \right) \hat{M} =$$

$$= \sum_{l=0}^{\infty} \sum{}' \frac{1}{(m+2l)!} \prod_{j=1}^{m} \left(\frac{is(k_j) \cdot e_j}{\sqrt{2k_{j0}}} \right) (-\hat{A})^l \hat{M} = \sum_{l=0}^{\infty} \frac{(-\hat{A})^l}{m!\, l!\, 2^l} \prod_{j=1}^{m} \left(\frac{is(k_j) \cdot e_j}{\sqrt{2k_{j0}}} \right) \hat{M} =$$

$$= \frac{1}{m!} \prod_{j=1}^{m} \left(\frac{is(k_j) \cdot e_j}{\sqrt{2k_{j0}}} \right) e^{-\frac{1}{2}\hat{A}}\, \hat{M} \qquad (21)$$

Thus the result we have obtained in this Section is that the transition amplitude $M(\underline{k}_1, e_1; \ldots; \underline{k}_m, e_m)$ for emission of m soft photons with momenta $\underline{k}_1, \ldots, \underline{k}_m$ and polarizations $e_1, \ldots, e_m$ accompanying the process (1) is given by

$$M(\underline{k}_1, e_1; \ldots; \underline{k}_m, e_m) = \frac{1}{m!} \prod_{j=1}^{m} \left(\frac{is(k_j) \cdot e_j}{\sqrt{2k_{j0}}} \right) e^{-\frac{1}{2}\hat{A}}\, \hat{M}; \quad \left(k_{j0} = \sqrt{\underline{k}_j^2 + \lambda^2}\right) \qquad (22)$$

with the natural convention that if $m = 0$, the product reduces to unity.

III. The infra-red contribution from internal photons

When no soft photons are emitted, i.e. when $m = 0$ in (22) we have simply the invariant amplitude M, that is

$$M = e^{-\frac{1}{2}\hat{A}}\hat{M} \tag{23}$$

We define

$$A = \frac{1}{2\pi i}\int \frac{d^4k}{k^2+\lambda^2} s_\mu(k) s^\mu(-k) \tag{24}$$

and

$$\bar{M} = e^{\frac{1}{2}(A-\hat{A})}\hat{M} \tag{25}$$

By definition $\hat{M}$ is non-infra-red. From (19) and (24) it follows that $A - \hat{A}$ is non-infra-red. Hence $\bar{M}$ is non-infra-red. From (23) and (25) we conclude that

$$M = e^{-\frac{1}{2}A}\bar{M} \tag{26}$$

Since M and A are Lorentz-invariant, so is $\bar{M}$. With (26) we have accomplished a Lorentz-invariant and cut-off independent separation of M into an infra-red part, $e^{-\frac{1}{2}A}$, and a non-infra-red part $\bar{M}$. We shall call $\bar{M}$ the "non-infra-red invariant amplitude", and from now on consider (26) as the (exact) definition of $\bar{M}$.

If M and $\bar{M}$ are expressed by perturbation expansions in α,

$$M = \sum_{n=0}^{\infty} M^{(n)} \tag{27}$$

$$\bar{M} = \sum_{n=0}^{\infty} \bar{M}^{(n)} \tag{28}$$

then $\bar{M}^{(n)}$ can be obtained from $M^{(o)}, M^{(1)},\ldots,M^{(n)}$ by

$$\bar{M}^{(n)} = \sum_{n'=o}^{n} \frac{1}{(n-n')!} \left(\tfrac{1}{2}A\right)^{n-n'} M^{(n')} \tag{29}$$

This follows from (26). The terms $M^{(o)}, M^{(1)},\ldots,M^{(n)}$ are to be calculated according to the ususal recipe *) (with the same photon mass, λ , as that occurring in A).

IV. The total infra-red contribution

By means of (25), the amplitude for photon emission given by (22) can be expressed in terms of the non-infra-red invariant amplitude $\bar{M}$, as follows

$$M(\underline{k}_1,e_1;\ldots;\underline{k}_m,e_m) = \frac{1}{m!}\prod_{j=1}^{m}\left(\frac{is(k_j)\cdot e_j}{\sqrt{2k_{jo}}}\right) e^{-\frac{1}{2}A}\,\bar{M}; \quad \left(k_{jo}=\sqrt{\underline{k}_j^2+\lambda^2}\right) \tag{30}$$

Using the rule for summation over polarizations

$$\sum_{pol} e^{\mu}e^{\nu} = g^{\mu\nu} \tag{31}$$

we then obtain the following probability density

*) See for instance ref. [6], table 8-2.

$$P(\underline{k}_1,\ldots,\underline{k}_m)=\sum_{pol}\sum_{perm} M^*(\underline{k}_{j_1},e_{j_1};\ldots;\underline{k}_{j_m},e_{j_m})\, M(\underline{k}_1,e_1;\ldots;\underline{k}_m e_m)=$$

$$=\sum_{pol} m!\,|M(\underline{k}_1,e_1;\ldots\underline{k}_m,e_m)|^2=\frac{1}{m!}\prod_{j=1}^{m}\left[\frac{1}{2k_{j0}}s^{\mu}(k_j)\,s_{\mu}^{*}(k_j)\right]e^{-\mathrm{Re}\{A\}}|\bar{M}|^2 \tag{32}$$

The restriction on the total energy of the emitted soft photons

$$\sum_{j=1}^{m} k_{jo} \leq \Delta E \tag{33}$$

is now introduced through the function

$$I(k_{1o},\ldots,k_{mo})=\begin{cases}1, & \text{for } \sum_{j=1}^{m} k_{jo} \leq \Delta E\\ 0, & \text{for } \sum_{j=1}^{m} k_{jo} > \Delta E\end{cases} \tag{34}$$

which will be used in the form

$$I(k_{10},\ldots,k_{mo})=\int_0^{\Delta E} dx\,\delta\Big(\sum_{j=1}^{m}k_{j0}-x\Big)\,dx=$$
$$=\frac{1}{2\pi}\int_0^{\Delta E}dx\int_{-\infty}^{\infty}dy\; e^{i(\sum_{j=1}^{m}k_{j0}-x)\,y} \tag{35}$$

We are now prepared to write down an expression for the probability density *), $P(\Delta E)$, for the process (1) with an energy loss due to soft photon emission not exceeding ΔE. The natural value for the limit between soft and hard photon energies ε (see Eq. (6)) is then just $\varepsilon = \Delta E$.

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*) Here and in the following we omit a factor $(2\pi)^{-4}\,\delta(P_f-P_i)$. (See ref.[6] eq. (8-34))

Using (32) and (35) we get

$$P(\Delta E)=\sum_{m=0}^{\infty}\int_{\substack{|\underline{k}_j|\leq \Delta E\\ k_{j0}=\sqrt{\underline{k}_j^2+\lambda^2}}} d^3k_1\ldots d^3k_m\, I(k_{10},\ldots,k_{m0})\, P(\underline{k}_1,\ldots,\underline{k}_m)=$$

$$=\frac{1}{2\pi}\int_0^{\Delta E}dx\int_{-\infty}^{\infty}dy\, e^{-ixy}\sum_{m=0}^{\infty}\frac{1}{m!}\Big[\int_{\substack{|\underline{k}|\leq \Delta E\\ k_0=\sqrt{\underline{k}^2+\lambda^2}}}\frac{d^3k}{2k_0}\, s^{\mu}(k)s_{\mu}^{*}(k)\, e^{ik_0y}\Big]^m e^{-\mathrm{Re}\{A\}}\,|\overline{M}|^2=$$

$$=\frac{1}{2\pi}\int_0^{\Delta E}dx\int_{-\infty}^{\infty}dy\, e^{-ixy}\exp\Big[\int_{\substack{|\underline{k}|\leq \Delta E\\ k_0=\sqrt{\underline{k}^2+\lambda^2}}}\frac{d^3k}{2k_0}\, s^{\mu}(k)s_{\mu}^{*}(k)\, e^{ik_0y}\Big]^m e^{-\mathrm{Re}\{A\}}|\overline{M}|^2 \qquad (36)$$

We can write $P(\Delta E)$ on the form

$$P(\Delta E) = b\, e^{G(\Delta E)}\, |\overline{M}|^2 \qquad (37)$$

with

$$b=\frac{1}{2\pi}\int_0^{\Delta E}dx\int_{-\infty}^{\infty}dy\, e^{-ixy}\exp\Big[\int_{\substack{|\underline{k}|\leq \Delta E\\ k_0=\sqrt{\underline{k}^2+\lambda^2}}}\frac{d^3k}{2k_0}\, s^{\mu}(k)s_{\mu}^{*}(k)\,(e^{ik_0y}-1)\Big] \qquad (38)$$

$$G(\Delta E)=-\mathrm{Re}\{A\}+\int_{\substack{|\underline{k}|\leq \Delta E\\ k_0=\sqrt{\underline{k}^2+\lambda^2}}}\frac{d^3k}{2k_0}\, s^{\mu}(k)\, s_{\mu}^{*}(k) \qquad (39)$$

The functions b and $G(\Delta E)$ are to be transformed into more useful forms. Introducing into (38) and (39) the definitions of A and $s^{\mu}(k)$, equations (24) and (18), we get *)

$$b=\frac{1}{2\pi}\int_0^{\Delta E}dx\int_{-\infty}^{\infty}dy\, e^{-ixy}\exp\Big[C\int_0^{\Delta E}\frac{d\kappa}{\kappa}\,(e^{i\kappa y}-1)\Big] \qquad (40)$$

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*) Because of the factor $(e^{ik_0y}-1)$ in the k-integral of (38) b is not infra-red divergent so we can at once put $\lambda = 0$, i.e. $|\underline{k}| = k_0$ in the expression for b.

with

$$C = \frac{\alpha}{\pi}\sum_{i,j=1}^{r} Q_i Q_j \, (-p_i \cdot p_j) \, \frac{1}{4\pi} \int\limits_{\ell_0 = |\underline{\ell}| = 1} \frac{d\Omega_\ell}{(\ell \cdot \varepsilon_i p_i)(-\ell \varepsilon_j p_j)} \tag{41}$$

and

$$G(\Delta E) = \frac{\alpha}{\pi}\sum_{i,j=1}^{r} Q_i Q_j (-p_i \cdot p_j) \, \mathrm{Re}\left\{ \frac{1}{4\pi} \int\limits_{\substack{|\underline{k}| \leq \Delta E \\ k_0 = \sqrt{\underline{k}^2 + \lambda^2}}} \frac{d^3k}{k_0 \, (k \cdot \varepsilon_i p_i)(-k \cdot \varepsilon_j p_j)} - \right.$$
$$\left. - \frac{1}{i\pi^2} \int \frac{d^4k}{(k^2+\lambda^2)(k^2 + 2k \cdot \varepsilon_i p_i)(k^2 - 2k \cdot \varepsilon_j p_j)} \right. \tag{42}$$

By a change of the names of the integration variables ($x \rightarrow x\,\Delta E$, $y \rightarrow \frac{y}{\Delta E}$, $\kappa \rightarrow \kappa\,\Delta E$), (40) transforms into

$$b = \frac{1}{2\pi} \int_0^1 dx \int_{-\infty}^{\infty} dy \; e^{-ixy} \, \exp\left[C \int_0^1 \frac{d\kappa}{\kappa} \left(e^{i\kappa y} - 1\right) \right] \tag{43}$$

which shows that b is independent of ΔE. To simplify (43) we start with the integral occurring in the exponential *)

$$\int_0^1 \frac{d\kappa}{\kappa}\left(e^{i\kappa y} - 1\right) = \int_0^{1/y} \frac{d\kappa}{\kappa}\left(e^{i\kappa y} - 1\right) + \int_{1/y}^{\infty} \frac{d\kappa}{\kappa} e^{i\kappa y} - \int_{1/y}^{1} \frac{d\kappa}{\kappa} - \int_1^{\infty} \frac{d\kappa}{\kappa} e^{i\kappa y} =$$

$$= \int_0^1 \frac{d\kappa}{\kappa}\left(e^{i\kappa} - 1\right) + \int_1^{\infty} \frac{d\kappa}{\kappa} e^{i\kappa} - \ln y - \int_1^{\infty} \frac{d\kappa}{\kappa} e^{i\kappa y} = \tag{44}$$

*) We have adopted the method of ref. [4] for the calculation of b.

$$= \int_0^1 \frac{d\kappa}{\kappa}(\cos\kappa - 1) + \int_1^\infty \frac{d\kappa}{\kappa}\cos\kappa + i\int_0^\infty \frac{d\kappa}{\kappa}\sin\kappa - \ln\kappa - \int_1^\infty \frac{d\kappa}{\kappa} e^{i\kappa y} =$$

$$= -\gamma + \tfrac{1}{2}\pi i - \ln y - \int_1^\infty \frac{d\kappa}{\kappa} e^{i\kappa y}$$

(γ=Euler's constant)

From this we get the y-integral of (43)

$$e^{-C(\gamma - \frac{1}{2}\pi i)} \int_{-\infty}^{\infty} \frac{dy}{y^C} e^{-ixy} \exp\left[-C\int_1^\infty \frac{d\kappa}{\kappa} e^{i\kappa y}\right] = e^{-C(\gamma-\frac{1}{2}\pi i)} \int_{-\infty}^{\infty} \frac{dy}{y^C} e^{-ixy} +$$

$$+ e^{-C(\gamma - \frac{1}{2}\pi i)} \int_{-\infty}^{\infty} \frac{dy}{y^C} e^{-ixy} \left\{ \exp\left[-C\int_1^\infty \frac{d\kappa}{\kappa} e^{i\kappa y}\right] - 1 \right\} \qquad (45)$$

The integrand in the last term of (45) is analytic for $\mathrm{Im}\{y\} > 0$. When $x \leq 1$, its norm decreases sufficiently rapidly for $\mathrm{Im}\{y\} > 0$, $y \to \infty$ that the contour can be closed by a semicircle in the upper half-plane. Thus the last term of (45) vanishes and b reduces to

$$b = \frac{1}{2\pi} e^{-C(\gamma - \frac{1}{2}\pi i)} \int_0^1 dx \int_{-\infty}^{\infty} \frac{dy}{y^C} e^{-ixy} = \frac{1}{2\pi i} e^{-C(\gamma - \pi i)} \int_0^1 dx\, x^{C-1} \int_{-i\infty}^{i\infty} \frac{dy}{y^C} e^{-y} =$$

$$= \frac{e^{-C\gamma}}{C} \cdot \frac{e^{C\pi i}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dy}{y^C} e^{-y} \qquad (46)$$

By cutting the y-plane along the positive real axis, and using formulae for the Γ-function, we can solve the integral in the last membrum of (46) and obtain

$$b = \frac{e^{-C\gamma}}{C} \cdot \frac{e^{C\pi i}}{2\pi i} \int_0^\infty \frac{dy}{y^C} e^{-y}\left(1 - e^{-2C\pi i}\right) = \frac{e^{-C\gamma}}{C} \cdot \frac{\sin(C\pi)}{\pi} \Gamma(1-C) =$$

$$= \frac{e^{-C\gamma}}{C} \cdot \frac{1}{\Gamma(C)} = \frac{e^{-\gamma C}}{\Gamma(1+C)} \qquad (47)$$

This is our final expression for b *).

We now turn to C and $G(\Delta E)$. By the use of Feynman's method of integration, expressed by the formula

$$\frac{1}{ab} = \int_0^1 \frac{dx}{\left[ax+b(1-x)\right]^2} \tag{48}$$

with the prescription that an eventual singularity is avoided by a detour into the complex x-plane, (41) and (42) are transformed into

$$C = \frac{\alpha}{\pi} \sum_{i,j=1}^{r} Q_i\, Q_j\, (-p_i p_j) \int_0^1 dx\; f\left(\varepsilon_i p_i x - \varepsilon_j p_j (1-x)\right) \tag{49}$$

$$f(p) = \frac{1}{4\pi} \int_{l_0 = |\underline{l}| = 1} \frac{d\Omega_l}{(l \cdot p)^2} \tag{50}$$

and

$$G(\Delta E) = \frac{\alpha}{\pi} \sum_{i,j=1}^{r} Q_i\, Q_j\, (-p_i \cdot p_j)\, \mathrm{Re}\left\{ \int_0^1 dx\; g\left(\varepsilon_i p_i x - \varepsilon_j p_j (1-x); \Delta E\right)\right\} \tag{51}$$

$$g(p;\Delta E) = \frac{1}{4\pi} \int_{\substack{|\underline{k}| \leq \Delta E \\ k_0 = \sqrt{\underline{k}^2+\lambda^2}}} \frac{d^3k}{k_0\, (k\cdot p)^2} - \frac{1}{i\pi^2} \int \frac{d^4k}{(k^2+\lambda^2)(k^2+2k\cdot p)^2} \tag{52}$$

From (50) we get simply

$$f(p) = \frac{1}{2} \int_{-1}^{1} \frac{dz}{(|\underline{p}|\, z - p_0)^2} = \frac{1}{-p^2} \tag{53}$$

*) A series expansion in C of $b = \dfrac{e^{-\gamma C}}{\Gamma(1+C)}$ starts with $b = 1 - \dfrac{\pi^2}{12} C^2 + \ldots$

Inserting (53) into (49) and performing the x-integration gives the final expression for C

$$C = \frac{\alpha}{\pi}\left[-\sum_{i=1}^{r} Q_i^2 - \sum_{1\leq i<j\leq r} \frac{\varepsilon_i Q_i \varepsilon_j Q_j}{C_{ij}} \ln\left(\frac{1+C_{ij}}{1-C_{ij}}\right)\right]$$

$$C_{ij} = \left[1 - \frac{p_i^2 p_j^2}{(p_i \cdot p_j)^2}\right]^{\frac{1}{2}} \tag{54}$$

Clearly, C is Lorentz-invariant.

The evaluation of the integrals in (52) is not very complicated and yields

$$\frac{1}{i\pi^2}\int \frac{d^4k}{(k^2+\lambda^2)(k^2+2k\cdot p)^2} = \frac{1}{-2p^2} \ln\left(\frac{-p^2}{\lambda^2}\right) \tag{55}$$

$$\frac{1}{4\pi}\int_{\substack{|\underline{k}|\leq \Delta E \\ k_0=\sqrt{\underline{k}^2+\lambda^2}}} \frac{d^3k}{k_0 (k\cdot p)^2} = \frac{1}{-2p^2} \ln\left(\frac{-p^2}{\lambda^2}\right) + \frac{1}{-p^2} \ln\left(\frac{\Delta E}{E}\right) + \frac{1}{-p^2}\left[\frac{1}{2}\ln\left(\frac{4E^2}{-p^2}\right) - \int_0^1 dy \frac{p_0^2}{p_0^2 - \underline{p}^2 y^2}\right] \tag{56}$$

Here the total energy $E = P_{io} = P_{fo}$ has been introduced. Clearly the λ-dependence of (52) vanishes and we get

$$g(p;\Delta E) = \frac{1}{-p^2} \ln\left(\frac{\Delta E}{E}\right) + \frac{1}{-p^2}\left[\frac{1}{2}\ln\left(\frac{4E^2}{-p^2}\right) - \int_0^1 dy \frac{p_0^2}{p_0^2 - \underline{p}^2 y^2}\right] \tag{57}$$

Insertion of (57) into (51) and use of (49) and (53) then yields

$$G(\Delta E) = C \ln \left(\frac{\Delta E}{E}\right) + B \tag{58}$$

where

$$B = \frac{\alpha}{\pi} \left\{ \sum_{i=1}^{r} Q_i^2 \left[1 + \frac{p_i^2}{E^2} h\left(\frac{p_i}{E}\right) \right] - \right.$$

$$\left. - 2 \sum_{1 \leq i < j \leq r} Q_i Q_j \frac{p_i \cdot p_j}{E^2} \operatorname{Re} \left\{ \int_0^1 dx \, h\left(\frac{\varepsilon_i p_i x - \varepsilon_j p_j (1-x)}{E} \right) \right\} \right\} \tag{59}$$

$$h(p) = \frac{1}{p^2} \left[\frac{1}{2} \ln\left(\frac{-p^2}{4}\right) + \int_0^1 dy \frac{p_0^2}{p_0^2 - p^2 y^2} \right]$$

From (37), (47) and (58) we now collect the following result which contains no divergences

$$P(\Delta E) = \left(\frac{\Delta E}{E}\right)^C e^B \frac{e^{-\gamma C}}{\Gamma(1+C)} |\bar{M}|^2 \tag{60}$$

where C is given by (54), B by (59) and $\bar{M}$ by (26), (24) and (18). In perturbation theory $\bar{M}$ is given by (29).

The whole ΔE-dependence in (60) is contained in the first factor and B is a quantity that depends on the components of the momenta $p_1, \ldots, p_r$ in the Lorentz-frame L. The quantity C depends also on these momenta but in an invariant manner. Finally, the non-infra-red invariant amplitude $\bar{M}$ contains all non-infra-red interactions between the r particles.

V. High energy processes

Though the preceding sections the mechanism of collective soft photon effects was treated in a simple but general way. Two results were obtained, (i) the invariant factorization of the infra-red part of the invariant amplitude M, achieved in Section III, and (ii) the closed expression (60) for $P(\Delta E)$ which is correct when the energy resolution is restricted by (3). The result (ii) is, however, only of academic interest so far, because the restriction (3) can be satisfied only in experiments performed at very low energies, and we know beforehand that radiative corrections are unimportant at low energies.

In this Section we shall try to extend the discussion to high energy processes. Then we abandon restriction (3).

Take L to be the CM-system and assume an isotropic energy resolution, which satisfies

$$\Delta E \ll p_{io}, \text{ for each } i \text{ such that } Q_i \neq 0 \qquad (61)$$

This restriction allows us to neglect the "recoil" terms as we did above in (13b,f), because those terms are small of the order $\frac{\Delta E}{p_{io}}$. But we can no longer neglect those diagrams for photon emission that do not give rise to infra-red divergences. In Sections II and IV these diagrams could be neglected because they gave contributions that were linear in soft photon momenta and thus bounded by quantities of magnitude $\gtrsim \frac{\Delta E}{m}$, which were negligibly small because of (3). In high energy processes, however, it usually happens that a ΔE satisfying (61) still is considerably larger than m.

We can generalize (36) so as to include also emission of soft photons through diagrams not giving rise to infra-red divergences. Let the index n refer to the number of emitted photons of this kind and m, as before, to the number of emitted photons giving infra-red divergences. We do not put $\varepsilon = \Delta E$ here but keep $\varepsilon << m$. Then the generalization of (36) is

$$P(\Delta E) = \frac{1}{2\pi}\int_0^{\Delta E} dx \int_{-\infty}^{\infty} dy\, e^{-ixy} \sum_{m=0}^{\infty} \frac{1}{m!}\Big[\int_{\substack{|\underline{k}|\leq\varepsilon \\ k_0=\sqrt{\underline{k}^2+\lambda^2}}} \frac{d^3k}{2k_0}\, s^{\mu}(k)\, s_{\mu}^{*}(k)\, e^{ik_0 y}\Big]^m \cdot$$

$$\cdot\, e^{-\mathrm{Re}\{A\}} \sum_{n=0}^{\infty} \hat{P}_n(y) \qquad (62)$$

with

$$\hat{P}_n(y) = \int_{\varepsilon \leq k_{jo} = |\underline{k}|_j \leq \Delta E} \cdots \int d^3k_1 \ldots d^3k_n\, e^{iy\sum_{j=1}^{n} k_{jo}}\, \hat{R}(\underline{k}_1,\ldots,\underline{k}_n) \qquad (63)$$

$$\hat{R}(\underline{k}_1,\ldots,\underline{k}_n) = \sum_{\substack{\text{diagrams} \\ i_1,\, i_2}} \bar{M}_{i_1}(\underline{k}_1,\ldots,\underline{k}_n)\, \bar{M}^{*}_{i_2}(\underline{k}_1,\ldots,\underline{k}_n) \qquad (64)$$

where $\bar{M}_i(\underline{k}_1,\ldots,\underline{k}_m)$ is the term in the non-infra-red amplitude (for emission of m photons with momenta $\underline{k}_1,\ldots\underline{k}_m$ along the process (1)) that corresponds to the i^{th} diagram. In (63) a summation over all polarizations is understood.

Now if a collection of terms $\bar{M}_{i_1}\bar{M}^{*}_{i_2}$ in the sum (64) can be written on the form (apart from numerical factors)

$$N(\underline{k}_1,\ldots,\underline{k}_{n'})M_{i_1'}(\underline{k}_{n'+1},\ldots,\underline{k}_n)\bar{M}^{*}_{i_2'}(\underline{k}_{n'+1},\ldots,\underline{k}_n) \qquad (65)$$

with

$$N(\underline{k}_1,\ldots,\underline{k}_{n'}) = \prod_{j=1}^{n'} \frac{s^{\mu}(k_j)s^{*}_{\mu}(k_j)}{2k_{jo}} \qquad (66)$$

then the contribution from the factor $N(\underline{k}_1,\dots,\underline{k}_n)$ can be removed to the soft photon ($|\underline{k}| \leq \varepsilon$) integral in (62). If this is done for each term $\bar{M}_{i_1} \bar{M}^*_{i_2}$ in (64) and for each $\hat{P}_n(y)$, then (62), (63) and (64) transform into

$$P(\Delta E) = \frac{1}{2\pi} \int_0^{\Delta E} dx \int_{-\infty}^{\infty} dy\, e^{-ixy} \sum_{m=0}^{\infty} \frac{1}{m!} \Big[\int_{\substack{|\underline{k}| \leq \Delta E \\ k_o = \sqrt{\underline{k}^2 + \lambda^2}}} \frac{d^3k}{2k_o} s^\mu(k) s_\mu^*(k) e^{ik_o y} \Big]^m \cdot e^{-\mathrm{Re}\{A\}} \sum_{n=0}^{\infty} P_n(y) \tag{67}$$

with

$$P_n(y) = \int_{k_{jo} = |\underline{k}_j| \leq \Delta E} \!\!\!\cdots\!\int d^3k_1 \dots d^3k_n\, e^{iy \sum_{j=1}^{n} k_{jo}}\, R(\underline{k}_1,\dots,\underline{k}_n) \tag{68}$$

$$R(\underline{k}_1,\dots,\underline{k}_n) = \sum_{\substack{\text{diagrams} \\ i_1,\, i_2}}{}' \bar{M}_{i_1}(\underline{k}_1,\dots,\underline{k}_n) \bar{M}^*_{i_2}(\underline{k}_1,\dots,\underline{k}_n) \tag{69}$$

In (69) "$\sum'$" means that the summation goes only over such pairs of diagrams that do not give contributions containing factors of the type (66) with logarithmic singularities at $k_{jo} = |\underline{k}_j| = 0$. Summation over m in (67) leads to an exponential. Using the definitions (39) and (41) and the result (58) we can rewrite (67) as

$$P(\Delta E) = \left(\frac{\Delta E}{E}\right)^C e^B \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{k_{jo} = |\underline{k}_j| \leq \Delta E} \!\!\!\cdots\!\int d^3k_1 \dots d^3k_n \int_{-\sum_{j=1}^{n} k_{jo}}^{\Delta E - \sum_{j=1}^{n} k_{jo}} dx \int_{-\infty}^{\infty} dy\, e^{-ixy} \cdot$$

$$\cdot \exp\Big[C \int_0^{\Delta E} \frac{d\kappa}{\kappa} (e^{i\kappa y} - 1) \Big] R(\underline{k}_1,\dots,\underline{k}_n) \tag{70}$$

After a calculation similar to that for b in Section IV, one is led to the following result for the transition probability density

$$P(\Delta E) = e^{B}\,\frac{e^{-\gamma C}}{\Gamma(1+C)}\int_0^{\Delta E} dx\,\left(\frac{x}{E}\right)^{C}\sum_{n=0}^{\infty}\int\cdots\int_{k_{j0}=|\underline{k}_j|\leq\Delta E} d^3k_1\cdots d^3k_n\cdot$$

$$\cdot R(\underline{k}_1,\dots,\underline{k}_n)\,\delta\Big(\Delta E - x - \sum_{j=1}^{n} k_{j0}\Big) \qquad (71)$$

Actually if one compares the formulae (60) and (71) one sees that the latter is a natural generalization of the former to the case when part of the energy loss goes through "non-infra-red photons".

Emission of electron pairs, which may occur if $\Delta E > 2m$ can easily be included in (71) simply by defining $R(\underline{k}_1,\dots,\underline{k}_n)$ (eq.(69)) to include also pair emission.

The quantities B and C occurring in (60) and (71) are expressed in terms of the external momenta by (59) and (54) respectively. In electron scattering processes at high energies and high momentum transfers B and C will both be of the order of magnitude

$$\frac{2\alpha}{\pi}\ln\left(\frac{q^2}{m^2}\right) \; ; \qquad \left(q^2 = \mathrm{Max}\left[(p_i-p_j)^2\right]\right) \qquad (72)$$

So if (60) or (71) is expressed as a perturbation series in α , the effective expansion parameter as far as the soft photon factors are concerned is at most

$$\frac{2\alpha}{\pi}\ \ln\left(\frac{q^2}{m^2}\right)\mathrm{Max}\left[1,\ln\left(\frac{E}{\Delta E}\right)\right. \qquad (73)$$

In realistic experiments, though (61) holds with reasonable accuracy one has a $\ln\left(\frac{E}{\Delta E}\right)$ which is only of the order of a few units. Thus the effective expansion parameter for the soft photon part is of the order of magnitude

$$\frac{2\alpha}{\pi} \ln \left(\frac{q^2}{m^2}\right) \tag{74}$$

The "structure" of the probability density (71) can be visualized in a very schematic way by the formula

$$P(\Delta E) = \int \dots \int F \sum{}' \bar{M} \bar{M}^* \tag{75}$$

Here F is the soft photon part. It can be expanded in powers of the effective parameter (74). The $\bar{M}$ s are terms of non-infra-red invariant amplitudes, and $\sum{}'$ has the same meaning as in (69).

In actual high energy experiments one generally has - beside the isotropic energy loss ΔE - a certain amount of photons and pairs with an energy $> \Delta E$ emitted, mainly along the incoming and outgoing particle momenta. When this emission is taken into account the recoil effects can no longer be neglected. The integrals over the momenta of the emitted particles are limited by complicated surfaces. It would be meaningless to try to write down a formula like (71) for a general energy resolution. However, the "structure" indicated by (75) (with the general effective expansion parameter (74) for F) is still obtained in the general case.

The invariance of $\bar{M}$ means that $\bar{M}$ depends on its momentum variables only through the scalar products that can be formed by them. That $\bar{M}$ is non-infra-red means that inside the integral over internal photon momenta k_1', k_2',... there are enough k' s in the numerator to make the integral convergent when any of the k' s goes to zero with all its components.

In a later publication [7] it will be shown that if $\bar{M}$ is expressed in a perturbation series in α, its effective expansion parameter is at most

$$\frac{\alpha}{\pi} \ln^{4/3}\left(\frac{q^2}{m^2}\right) \tag{76}$$

Comparison between (76) and the effective expansion parameter for F, (74), then shows that the effective expansion parameter for the transition probability density of an electromagnetic scattering process is at most $\ln^{4/3}\left(\frac{q^2}{m^2}\right)$.

This result and its implications for the reliability of radiative correction calculations was stated in a note by Petermann and the author [8].

One implication mentioned in ref. [8] was that at energies which are used in present experiments, calculations of first order radiative corrections to scattering cross-sections give a sufficient accuracy. To first order in α (71) and (60) are identical and we have

$$P(\Delta E) = |M^{(0)}|^2 \left[1 + C \ln\left(\frac{\Delta E}{E}\right) + B\right] + 2\mathrm{Re}\left\{M^{(0)*}\bar{M}^{(1)}\right\} \tag{77}$$

One must then add to (77) the probability density for emission of one photon with an energy $> \Delta E$.

* * *

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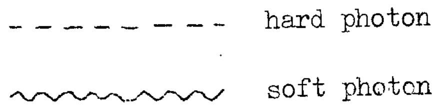

Fig. 1

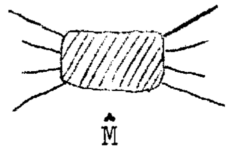

Fig. 2

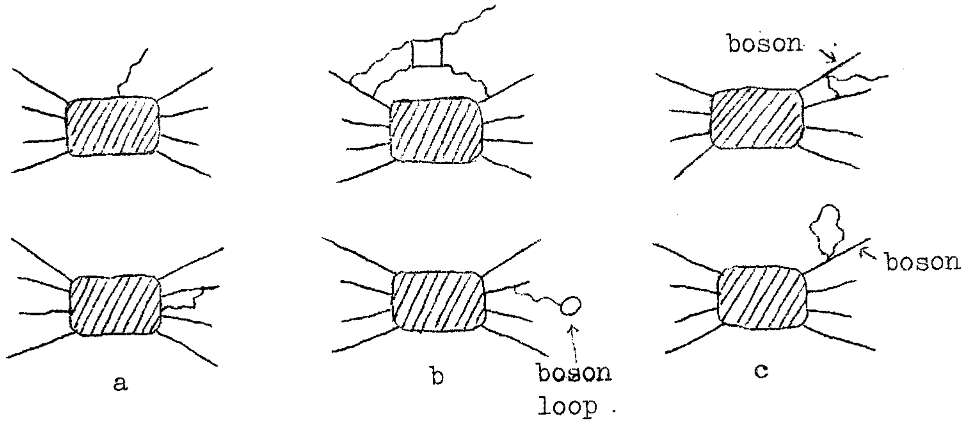

Fig. 3

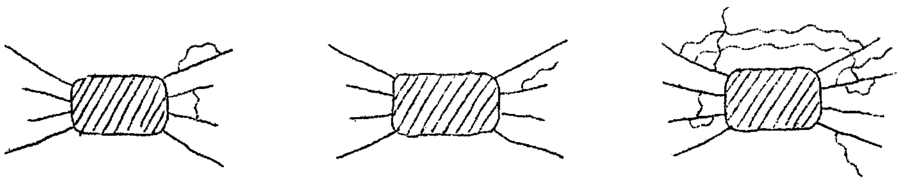

Fig. 4