ON RANGES OF LYAPUNOV TRANSFORMATIONS IV†

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1. Introduction. Let $\mathbb{C}^{n,n}$ denote the space of $n \times n$ matrices with complex entries and let $\mathscr{H}_n$ denote the set of $n \times n$ hermitian matrices. Given any matrix $A \in \mathbb{C}^{n,n}$, the *Lyapunov transformation* corresponding to A is defined by $\mathscr{L}_A(H) = AH + HA^*$, where $H \in \mathscr{H}_n$. Let $PSD(n)$ be the set of all $n \times n$ hermitian positive semidefinite matrices. Taussky **[8, 9]** raised the problems of determining

$$\mathscr{L}_A(PSD(n)) = \{AH + HA^* : H \in PSD(n)\}$$

and

$$\mathscr{L}_A^{-1}(PSD(n)) = \{H \in \mathscr{H}_n : AH + HA^* \in PSD(n)\}.$$

Both of these problems seem to be difficult.

It was shown in **[4]** that if $A, B \in \mathbb{C}^{n,n}$ and $\mathscr{L}_A$ is invertible then, $\mathscr{L}_A(PSD(n)) = \mathscr{L}_B(PSD(n))$ if and only if

$$B = \mu(A + i\alpha I) \text{ for some real } \alpha, \mu \text{ such that } \mu > 0 \tag{1}$$

or

$$B = \mu[(A + i\alpha_1 I)^{-1} + i\alpha_2 I] \text{ for some real } \alpha_1, \alpha_2, \mu \text{ such that } \mu > 0. \tag{2}$$

This result answers the question to what extent does $\mathscr{L}_A(PSD(n))$ characterize A. The proof in **[4]** is by induction on n, the order of A, and involves several tedious computations. In Section 2 we give a simpler proof of this result based on a theorem by Schneider **[7]** which characterizes all linear transformations on the real space $\mathscr{H}_n$ that map $PSD(n)$ on to itself. It is not difficult to see that A and B satisfy (1) or (2) if and only if

$$B = (\mu I + i\nu A)(\varphi A + i\psi I)^{-1} \text{ for some real } \mu, \nu, \varphi, \psi \text{ with } \mu\varphi + \nu\psi = 1; \tag{3}$$

so we shall show that if $\mathscr{L}_A$ is invertible then, $\mathscr{L}_A(PSD(n)) = \mathscr{L}_B(PSD(n))$ if and only if (3) is satisfied.

In order to describe the results of Section 3 we need the following definition. Let $\mathbb{C}^n(\mathbb{R}^n)$ denote the vector space of all complex (real) column n-tuples. If $x, y \in \mathbb{C}^n$, let (x, y) denote the inner product of x and y.

DEFINITION. (i) A nonempty set $S \subseteq \mathbb{C}^n$ (or $\mathbb{R}^n$) is said to be a *cone* if $S + S \subseteq S$ and $\alpha S \subseteq S$ for every $\alpha \geqq 0$.

(ii) If $S \subseteq \mathbb{C}^n$ is a cone then S^P, the *polar* of S, is defined by

$$S^P = \{y \in \mathbb{C}^n : \operatorname{Re}(x, y) \geqq 0 \text{ for every } x \in S\}.$$

The polar can be similarly defined for a cone in $\mathbb{R}^n$.

In Section 3 a theorem of Ben-Israel **[1]** on the solvability of linear equations over cones

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is used to show that if $\mathscr{L}_A$ is invertible then $\mathscr{L}_A^{-1}(PSD(n)) = [\mathscr{L}_{A^*}(PSD(n))]^P$. It is then proved that $\mathscr{L}_A^{-1}(PSD(n)) = \mathscr{L}_B^{-1}(PSD(n))$ if and only if (3) is satisfied.

We assume throughout that $A \in \mathbb{C}^{n,n}$ and $\mathscr{L}_A$ is invertible. This is equivalent (cf. [**2, 10**]) to $\prod_{1 \leq i, j \leq n} (\lambda_i + \bar{\lambda}_j) \neq 0$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. Let $\bar{A}$, A^t, A^* denote the conjugate, transpose and conjugate transpose of A, respectively. The Kronecker product of two matrices C and D is denoted by $C \otimes D$.

2. Necessary and sufficient conditions for $\mathscr{L}_A(PSD(n)) = \mathscr{L}_B(PSD(n))$. The main result in this section is Theorem 3, which characterizes the matrices B such that

$$\mathscr{L}_A(PSD(n)) = \mathscr{L}_B(PSD(n)).$$

To establish its proof we need the following two theorems on matrix equations, which may be of independent interest.

THEOREM 1. *Let $A, C, D \in \mathbb{C}^{n,n}$ and suppose that $\mathscr{L}_A$ is invertible. If*

$$AX + XA^* = DXC^* + CXD^*$$

for every $X \in \mathbb{C}^{n,n}$, then there exist real numbers $\theta, \mu, v, \varphi, \psi$ such that $\mu\varphi + v\psi = 1$ and $C = e^{i\theta}(\varphi A + i\psi I)$, $D = e^{i\theta}(\mu I + ivA)$.

Proof. We consider each matrix in $\mathbb{C}^{n,n}$ as an n^2 column vector. Thus, if $X_{(i)}$ denotes the ith row of X, we consider X as the column vector $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})^t$. The assumption of the theorem then implies (cf. [**5**]) that

$$A \otimes I + I \otimes \bar{A} = D \otimes \bar{C} + C \otimes \bar{D},$$

whence

$$a_{ij} I + \delta_{ij} \bar{A} = d_{ij} \bar{C} + c_{ij} \bar{D}, \quad i, j = 1, \ldots, n. \tag{4}$$

We may replace A, C, D by UAU^*, UCU^*, UDU^*, respectively, where U is any unitary matrix. Given real numbers α and r such that $r \neq 0$, we may replace A by $r(A + i\alpha I)$. Given real numbers w and t such that $t \neq 0$, we may replace C and D by $te^{iw}C$ and $t^{-1}e^{iw}D$, respectively. Hence we may assume that $a_{11} = 1$ and c_{11} is nonnegative real. Since $c_{11}(d_{11} + \bar{d}_{11}) = 2$, it follows that $c_{11} \neq 0$, so we may assume that $c_{11} = 1$, whence $d_{11} = 1 + iy$ for some real y.

It follows from (4), with $i = j = 1$, that $D = I + A - (1 - iy)C$. Substituting back into (4), we are led to

$$(a_{ij} + \delta_{ij} - (1 - iy)c_{ij})\bar{C} + c_{ij}(\bar{A} + I - (1 + iy)\bar{C}) = a_{ij} I + \delta_{ij} \bar{A}, \quad i, j = 1, \ldots, n,$$

and thence to

$$(\bar{a}_{ij} + \delta_{ij} - 2\bar{c}_{ij})C = (\bar{a}_{ij} - \bar{c}_{ij})I + (\delta_{ij} - \bar{c}_{ij})\bar{A}, \quad i, j = 1, \ldots, n. \tag{5}$$

There are now two cases.

Case I. Suppose that $2\bar{c}_{ij} = \bar{a}_{ij} + \delta_{ij}$, $i, j = 1, \ldots, n$. Thence $C = \frac{1}{2}(A + I)$, and it follows from (5) that $(\bar{a}_{ij} - \delta_{ij})(I - A) = 0$, $i, j = 1, \ldots, n$. Hence $A = I$, $C = I$ and $D = (1 + iy)I$, which completes the proof in this case.

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Case II. We may assume that there exist i_0, j_0 such that $\bar{a}_{i_0j_0}+\delta_{i_0j_0}-2\bar{c}_{i_0j_0}\neq 0$. Hence, by (5), $C=z_1I+z_2A$ for some $z_1, z_2\in\mathbb{C}$. Substituting back into (5), we get

$$((\bar{a}_{ij}+\delta_{ij}-2\bar{z}_1\delta_{ij}-2\bar{z}_2\bar{a}_{ij})z_1-\bar{a}_{ij}+\bar{z}_1\delta_{ij}+\bar{z}_2\bar{a}_{ij})I$$
$$+((\bar{a}_{ij}+\delta_{ij}-2\bar{z}_1\delta_{ij}-2\bar{z}_2\bar{a}_{ij})z_2-\delta_{ij}+\bar{z}_1\delta_{ij}+\bar{z}_2\bar{a}_{ij})A=0,\quad i,j=1,\ldots,n.$$

The matrix A is not a scalar matrix. If A were a scalar matrix then $a_{11}=1$ would imply $A=I$. Hence C would be scalar matrix, and $c_{11}=1$ would imply $C=I$, contrary to the assumption of Case II. Hence we conclude that

$$(z_1-2z_1\bar{z}_2-1+\bar{z}_2)\bar{a}_{ij}+(z_1-2z_1\bar{z}_1+\bar{z}_1)\delta_{ij}=0,\quad i,j=1,\ldots,n,$$

and

$$(z_2-2z_2\bar{z}_2+\bar{z}_2)\bar{a}_{ij}+(z_2-2\bar{z}_1z_2-1+\bar{z}_1)\delta_{ij}=0,\quad i,j=1,\ldots,n.$$

Since A is not a scalar matrix it follows that

$$z_1-2z_1\bar{z}_2-1+\bar{z}_2=0,\quad z_1-2z_1\bar{z}_1+\bar{z}_1=0\quad\text{and}\quad z_2-2z_2\bar{z}_2+\bar{z}_2=0.$$

Hence $z_1=\frac{1}{2}(1+e^{i\theta})$ and $z_2=\frac{1}{2}(1-e^{i\theta})$ for some real θ. It follows that

$$C=z_1I+z_2A=(\sin\tfrac{1}{2}\theta A+i\cos\tfrac{1}{2}\theta I)e^{i(\frac{1}{2}\theta-\frac{1}{2}\pi)}$$

and

$$D=I+A-(1-iy)C=((\sin\tfrac{1}{2}\theta-y\cos\tfrac{1}{2}\theta)I+i(\cos\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta)A)e^{i(\frac{1}{2}\theta-\frac{1}{2}\pi)},$$

which completes the proof.

Let A have eigenvalues $\lambda_1,\ldots,\lambda_n$ and define

$$\Delta(A)=\prod_{1\leq i,\,j\leq n}(\lambda_i+\bar{\lambda}_j).$$

Recall that $\mathcal{L}_A$ is invertible if and only if $\Delta(A)\neq 0$.

THEOREM 2. *Let $n\geq 2$ and let $A\in\mathbb{C}^{n,n}$ such that $\mathcal{L}_A$ is invertible. There exist no matrices $C, D\in\mathbb{C}^{n,n}$* such that

$$(AX+XA^*)^t=DXC^*+CXD^*\quad\textit{for every}\quad X\in\mathbb{C}^{n,n}.\tag{6}$$

Proof. We consider again each matrix in $\mathbb{C}^{n,n}$ as an n^2 column vector. Let $E_{ij}\in\mathbb{C}^{n,n}$ be the matrix with 1 in the i,j position and 0 elsewhere. Let $T\in\mathbb{C}^{n^2,n^2}$ be the matrix consisting of n^2 blocks $T_{ij}\in\mathbb{C}^{n,n}$ such that $T_{ij}=E_{ji}$, $i,j=1,\ldots,n$. It is easy to show that (6) is equivalent to

$$T(A\otimes I+I\otimes\bar{A})=D\otimes\bar{C}+C\otimes\bar{D}.\tag{7}$$

Hence it suffices to show that there exist no matrices $C, D\in\mathbb{C}^{n,n}$ that satisfy (7).

Suppose that C and D satisfy (7). Then

$$d_{ij}\bar{C}+c_{ij}\bar{D}=\sum_{k=1}^{n}E_{ki}(a_{kj}I+\delta_{kj}\bar{A})=E_{ji}\bar{A}+\sum_{k=1}^{n}a_{kj}E_{ki},\quad i,j=1,\ldots,n.\tag{8}$$

The matrices C and D are nonsingular. For suppose there exists $x\in\mathbb{C}^n$ such that $Dx=0$.

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Let $X = xx^*$. Then $DXC^*+CXD^* = 0$, which implies that $\mathscr{L}_A(X) = 0$. Hence $X = 0$ and $x = 0$. Similarly one shows that C is nonsingular.

Given any real numbers α, r, t, w such that $r \neq 0$ and $t \neq 0$ we may replace A by $r(A+i\alpha I)$ and C, D by $te^{iw}C$, $t^{-1}e^{iw}D$, respectively. Hence we may assume that a_{11} and c_{11} are real and nonnegative.

Let

$$W = \begin{bmatrix} 2a_{11} & \bar{a}_{12} & \dots & \bar{a}_{1n} \\ a_{21} & 0 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{n1} & 0 & \dots & 0 \end{bmatrix} \in \mathbb{C}^{n,n}. \tag{9}$$

It follows from (8), with $i = j = 1$, and (9) that

$$d_{11}\bar{C}+c_{11}\bar{D} = W. \tag{10}$$

There are now two cases.

Case I. $n \geqq 3$. Suppose that $c_{11} = 0$. It follows from (9) and (10) that $a_{11} = 0$. Since A is nonsingular, at least one of $a_{12}, \dots, a_{1n}$ is nonzero, whence $d_{11} \neq 0$. It follows that rank C is at most 2, but C must be nonsingular, a contradiction. Hence $c_{11} \neq 0$ and we may assume that $c_{11} = 1$. It follows from (10) that $\bar{D} = W-d_{11}\bar{C}$. Substituting back into (8), we are led to

$$(d_{ij}-d_{11}c_{ij})\bar{C} = E_{ji}\bar{A}+\sum_{k=1}^{n} a_{kj}E_{ki}-c_{ij}W, \quad i,j = 1, \dots, n,$$

and hence to

$$(d_{ij}-d_{11}c_{ij})\bar{C} = \begin{bmatrix} 0 & 0 & \dots & 0 & a_{1j} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & a_{2j} & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \\ 0 & 0 & \dots & 0 & a_{j-1j} & 0 & \dots & 0 \\ \bar{a}_{i1} & \bar{a}_{i2} & \dots & \bar{a}_{ii-1} & \bar{a}_{ii}+a_{jj} & \bar{a}_{ii+1} & \dots & \bar{a}_{in} \\ 0 & 0 & \dots & 0 & a_{j+1j} & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 & a_{nj} & 0 & \dots & 0 \end{bmatrix} - c_{ij}W, \; i,j = 1, \dots, n. \tag{11}$$

Consider now a fixed pair (i,j) such that $i \geqq 2$ and $j \geqq 2$. We want to show that $d_{ij}-d_{11}c_{ij} \neq 0$. Suppose that $d_{ij}-d_{11}c_{ij} = 0$. It follows from (11) that $a_{ik} = 0$ for $k \neq 1, i$; $a_{kj} = 0$ for $k \neq 1, j$; and $\bar{a}_{ii}+a_{jj} = 0$. If also $c_{ij} = 0$ then $a_{1j} = a_{i1} = 0$, whence a_{ii} and a_{jj} are eigenvalues of A. But $\bar{a}_{ii}+a_{jj} = 0$, which implies that $\Delta(A) = 0$. Since this is not the case we conclude that $c_{ij} \neq 0$. Now, if $i \neq j$ it follows from (11) that $a_{1k} = 0$, $k = 1, \dots, n$, this is a contradiction. If $i = j$, it follows from (11) that $A_{(1)}$ and $A_{(j)}$, the first and jth rows of A, respectively, have the form

$$A_{(1)} = [0, 0, \dots, a_{1j}, 0, \dots, 0], \qquad A_{(j)} = [a_{j1}, 0, \dots, i\beta, 0, \dots, 0],$$

where in each case the jth entry of the row is the third displayed entry, β is real and

$\bar{a}_{1j}^{-1}a_{1j} = \bar{a}_{j1}a_{j1}^{-1} = c_{jj}$. This implies that $\Delta(A) = 0$, which is a contradiction. Hence $d_{ij} - d_{11}c_{ij} \neq 0$.

It follows from (11) that $c_{kl} = 0$ if $k \geqq 2$, $k \neq j$ and $l \geqq 2$, $l \neq i$. But since i and j were arbitrary $(i, j \geqq 2)$ it follows that $c_{kl} = 0$ for all $2 \leqq k$, $l \leqq n$. Hence rank $C \leqq 2$. This contradicts the fact that C must be nonsingular and completes the proof of this case.

Case II. $n = 2$. Suppose that $c_{11} = 0$. Then, by (9) and (10), $a_{11} = 0$. Since A is nonsingular, a_{12} and a_{21} are nonzero, whence $d_{11} \neq 0$. Hence, by (10),

$$\bar{C} = d_{11}^{-1}\begin{bmatrix} 0 & \bar{a}_{12} \\ a_{21} & 0 \end{bmatrix}.$$

It follows from (8), with $i = j = 2$, that

$$d_{22}d_{11}^{-1}\begin{bmatrix} 0 & \bar{a}_{12} \\ a_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ \bar{a}_{21} & a_{22} + \bar{a}_{22} \end{bmatrix},$$

and, by an easy computation, that $\Delta(A) = 0$, contrary to our assumption. Hence $c_{11} \neq 0$, and we may assume that $c_{11} = 1$. Thus, (11) holds also in this case, while (10) implies that

$$D = \begin{bmatrix} a_{11} + iy & a_{12} - (a_{11} - iy)c_{12} \\ \bar{a}_{21} - (a_{11} - iy)c_{21} & -(a_{11} - iy)c_{22} \end{bmatrix}, \tag{12}$$

for some real y. It follows from (12) and (11), with $i = 1, j = 2$ and $i = j = 2$, that

$$(a_{12} - 2a_{11}c_{12})\bar{C} = \begin{bmatrix} a_{12} - 2c_{12}a_{11} & -c_{12}\bar{a}_{12} \\ a_{11} + a_{22} - c_{12}a_{21} & \bar{a}_{12} \end{bmatrix}, \tag{13}$$

and

$$-2a_{11}c_{22}\bar{C} = \begin{bmatrix} -2c_{22}a_{11} & a_{12} - c_{22}\bar{a}_{12} \\ \bar{a}_{21} - c_{22}a_{21} & \bar{a}_{22} + a_{22} \end{bmatrix}. \tag{14}$$

The assumption $\Delta(A) \neq 0$ implies that $a_{11} \neq 0$ and $c_{22} \neq 0$, so we can assume $a_{11} = 1$. If we solve (14) for C and substitute into (13) we conclude, after some elementary calculations, that there exist real numbers p and q such that

$$A = \begin{bmatrix} 1 & a_{12} \\ a_{21} & -1 + iq \end{bmatrix},$$

where $a_{12}a_{21} = p + iq$. This implies that $\Delta(A) = 0$, contrary to our assumption. This completes the proof of the theorem.

Theorems 1 and 2 and a theorem of Schneider [7] which characterizes all linear transformations on $\mathscr{H}_n$ that map $PSD(n)$ on to itself are needed in the proof of the next theorem.

THEOREM 3. *Let $A, B \in \mathbb{C}^{n,n}$ and suppose that $\mathscr{L}_A$ is invertible. Then the following are equivalent:*

(i) $B = (\mu I + ivA)(\varphi A + i\psi I)^{-1}$ *for some real* μ, v, φ, ψ *with* $\mu\varphi + v\psi = 1$;
(ii) $\mathscr{L}_A(PSD(n)) = \mathscr{L}_B(PSD(n))$.

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Proof. (i) $\Rightarrow$ (ii). If $\varphi \neq 0$ then

$$B = (iv\varphi^{-1}(\varphi A + i\psi I) + (\mu + \varphi^{-1}v\psi)I)(\varphi A + i\psi I)^{-1} = iv\varphi^{-1}I + \varphi^{-1}(\varphi A + i\psi I)^{-1},$$

while if $\varphi = 0$ then $\psi^{-1} = v$ and $B = v^2A - i\mu vI$. Hence $\Delta(B) \neq 0$ and $\mathscr{L}_B$ is invertible.

Let H be positive semidefinite and let K be the unique solution of the matrix equation $\mathscr{L}_A(H) = \mathscr{L}_B(K)$. It is easily verified that $K = \varphi^2(A + i\varphi^{-1}\psi I)H(A + i\varphi^{-1}\psi I)^*$ if $\varphi \neq 0$, and $K = v^{-2}H$ if $\varphi = 0$. Hence K is positive semidefinite and $\mathscr{L}_A(PSD(n)) \subseteq \mathscr{L}_B(PSD(n))$, but we also have

$$A = (\mu I - i\psi B)(\varphi B - ivI)^{-1},$$

whence $\mathscr{L}_A(PSD(n)) \supseteq \mathscr{L}_B(PSD(n))$.

(ii) $\Rightarrow$ (i). Since $\mathscr{H}_n = PSD(n) - PSD(n)$, the assumption $\mathscr{L}_A(PSD(n)) = \mathscr{L}_B(PSD(n))$ implies that $\mathscr{L}_B$ is invertible and $\mathscr{L}_B^{-1}\mathscr{L}_A(PSD(n)) = PSD(n)$. Hence $\mathscr{L}_B^{-1}\mathscr{L}_A$ is a linear transformation on the real space $\mathscr{H}_n$ which maps $PSD(n)$ on to itself. It now follows by Schneider [7, Theorem 2] that there exists a nonsingular matrix $C \in \mathbb{C}^{n,n}$ such that either $\mathscr{L}_B^{-1}\mathscr{L}_A(H) = CHC^*$ for all $H \in \mathscr{H}_n$, or $\mathscr{L}_B^{-1}\mathscr{L}_A(H) = CH^tC^*$ for all $H \in \mathscr{H}_n$. Hence, either

$$AH + HA^* = BCHC^* + CHC^*B^* \text{ for all } H \in \mathscr{H}_n, \tag{15}$$

or

$$AH + HA^* = BCH^tC^* + CH^tC^*B^* \text{ for all } H \in \mathscr{H}_n. \tag{16}$$

We may replace H in (15) and (16) by any matrix $X \in \mathbb{C}^{n,n}$, because any matrix X can be written as $H_1 + iH_2$, where $H_1, H_2 \in \mathscr{H}_n$. If (16) is satisfied then

$$(\bar{A}X + X\bar{A}^*)^t = BCXC^* + CXC^*B^*$$

for all $X \in \mathbb{C}^{n,n}$. This is impossible for $n \geqq 2$, by Theorem 2, since $\mathscr{L}_{\bar{A}}$ is invertible (while for $n = 1$ (15) and (16) are the same). Hence it remains to consider the case that

$$AX + XA^* = BCXC^* + CXC^*B^*$$

for all $X \in \mathbb{C}^{n,n}$, but then, by Theorem 1, there exist real numbers $\theta, \mu, v, \varphi, \psi$ such that $\mu\varphi + v\psi = 1$ and $BC = e^{i\theta}(\mu I + ivA)$, $C = e^{i\theta}(\varphi A + i\psi I)$. This completes the proof of the theorem.

3. Necessary and sufficient conditions for $\mathscr{L}_A^{-1}(PSD(n)) = \mathscr{L}_B^{-1}(PSD(n))$. In this section we use the duality theory for cones to point out the relation between the image and inverse image of $PSD(n)$ under the Lyapunov transformation. We use the well-known fact (cf. [**1**], [**6**, Theorem 14.1]) that for a closed cone S in $\mathbb{C}^n$ or $\mathbb{R}^n$ $(S^P)^P = S$.

Ben-Israel [**1**, Theorem 2.4] proved the following *solvability theorem for linear equations over cones*. Let $T \in \mathbb{C}^{n,n}$, $b \in \mathbb{C}^n$. Further let S be a closed cone in $\mathbb{C}^n$ and suppose that $\text{Null}(T) + S$ is closed, where $\text{Null}(T)$ is the null space of T. Then the linear system $Tx = b$ has a solution $x \in S$ if and only if $T^*y \in S^P$ implies that $\text{Re}(b, y) \geqq 0$. This result can also be stated with the obvious modifications for cones in $\mathbb{R}^n$ and is applied in the proof of the next theorem, which is essentially Theorem 4 of [**3**].

THEOREM 4. *Let $A \in \mathbb{C}^{n,n}$ and suppose that $\mathscr{L}_A$ is invertible. Then*

$$\mathscr{L}_A(PSD(n)) = [\mathscr{L}_{A^*}^{-1}(PSD(n))]^P \quad \text{and} \quad \mathscr{L}_A^{-1}(PSD(n)) = [\mathscr{L}_{A^*}(PSD(n))]^P.$$

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Proof. It is known that the real linear space $\mathscr{H}_n$ can be made into an inner product space by defining the inner product $\langle H, K\rangle = \operatorname{trace}(HK)$ for any $H, K \in \mathscr{H}_n$. It is easily verified that $\langle \mathscr{L}_A(H), K\rangle = \langle H, \mathscr{L}_{A^*}(K)\rangle$ for any $H, K \in \mathscr{H}_n$, whence $\mathscr{L}_{A^*}$ is the adjoint of $\mathscr{L}_A$ with respect to the given inner product in $\mathscr{H}_n$.

The cone $PSD(n)$ is closed and self-polar, i.e., $PSD(n) = PSD(n)^P$, and since $\operatorname{Null}(\mathscr{L}_A) = \{0\}$, we may apply Ben-Israel's solvability theorem. Thus, $K \in \mathscr{L}_A(PSD(n))$ if and only if $\langle H, K\rangle \geqq 0$ for every $H \in \mathscr{L}_{A^*}^{-1}(PSD(n))$. Hence $\mathscr{L}_A(PSD(n)) = [\mathscr{L}_{A^*}^{-1}(PSD(n))]^P$. We may replace A by A^*, since $\mathscr{L}_{A^*}$ is also invertible, so $\mathscr{L}_{A^*}(PSD(n)) = [\mathscr{L}_A^{-1}(PSD(n))]^P$. Since $\mathscr{L}_A^{-1}(PSD(n))$ is a closed cone, it follows that

$$[\mathscr{L}_{A^*}(PSD(n))]^P = [\mathscr{L}_A^{-1}(PSD(n))]^{PP} = \mathscr{L}_A^{-1}(PSD(n)),$$

which completes the proof.

Theorems 3 and 4 imply the following theorem.

THEOREM 5. *Let $A, B \in \mathbb{C}^{n,n}$ and suppose that $\mathscr{L}_A$ is invertible. Then the following are equivalent:*

(i) $B = (\mu I + i\nu A)(\varphi A + i\psi I)^{-1}$ *for some real* μ, ν, φ, ψ *with* $\mu\varphi + \nu\psi = 1$;

(ii) $\mathscr{L}_A^{-1}(PSD(n)) = \mathscr{L}_B^{-1}(PSD(n))$.

Proof. (i) $\Rightarrow$ (ii). Since $B^* = (\mu I - i\nu A^*)(\varphi A^* - i\psi I)^{-1}$, the desired result follows from Theorems 3 and 4.

(ii) $\Rightarrow$ (i). Suppose that $\mathscr{L}_B(H) = 0$, where $H \in \mathscr{H}_n$. Then H and $-H$ are in $\mathscr{L}_A^{-1}(PSD(n))$, whence $AH + HA^* \in PSD(n)$ and $-(AH + HA^*) \in PSD(n)$. Hence $H = 0$ and $\mathscr{L}_B$ is invertible. Theorems 3 and 4 imply that (i) must hold, which completes the proof.

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