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ON RATES OF PROPAGATION OF HEAT ACCORDING TO FOURIER'S THEORY

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Introduction. It is the aim of this paper to draw attention to certain consequences of Fourier's theory of heat conduction [1]. It might be thought that so venerable a theory could no longer be capable of generating controversy, but this is not so. The question at issue is what the theory predicts about rates of propagation of heat and, in particular, whether those rates are finite or infinite.

We consider an isotropic and homogeneous conductor† which occupies all of n-dimensional space $\mathbf{R}^n$; the cases $n = 1, 2, 3$ are, of course, the ones of physical interest. A point x is identified with its position vector $(x_1, \ldots, x_n)$ with respect to the origin, and its distance from the origin is $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$. The scalar product $x \cdot y$, of $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, is the sum $x_1 y_1 + \cdots + x_n y_n$.

It will be recalled that Fourier's theory is founded upon two hypotheses, namely balance of heat and a constitutive assumption which relates the heat flux vector to the temperature gradient. Balance of heat says that

$$\frac{d}{dt} \int_B \rho c u \, dx + \int_{\partial B} q \cdot \nu \, d\sigma = 0,$$

where B may be any n-dimensional region to which the divergence theorem applies, t is the time, $u(x,t)$ is the temperature, $q(x,t)$ is the heat flux vector, the vector $\nu(x)$ is the outward unit normal on the boundary ∂B, $dx = dx_1 \cdots dx_n$ is the element of hypervolume, $d\sigma$ is the element of hypersurface area, and the positive constants ρ and c are, respectively, the mass density and the specific heat capacity.

Balance of heat implies that the temperature and the heat flux vector must satisfy the local differential equation

$$\rho c \frac{\partial u}{\partial t} + \frac{\partial q_1}{\partial x_1} + \cdots + \frac{\partial q_n}{\partial x_n} = 0.$$

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†A simple re-scaling of the spatial variables enables us to extend our conclusions to cover heat conduction in an orthotropic conductor which has different thermal conductivities in different coordinate directions.

This last is to be supplemented by the constitutive equations

$$q_1 = -k\frac{\partial u}{\partial x_1}, \ldots, q_n = -k\frac{\partial u}{\partial x_n},$$

the positive constant k being the thermal conductivity. It follows that the temperature must satisfy the heat equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \frac{\partial u}{\partial t},$$

where we have chosen units in which the thermal diffusivity $k/(\rho c)$ is equal to unity.

Let us suppose the initial temperature to be $f(x)$, say. Then the temperature at all subsequent times must be a solution of an initial-value problem consisting of the heat equation together with the initial condition

$$u(x,0) = f(x) \quad \text{for } x \in \mathbf{R}^n.$$

For the moment we require only that $f(x)$ be continuous and bounded.

The initial-value problem has infinitely many solutions; our interest lies in that (unique) solution that remains bounded for $x \in \mathbf{R}^n$ and $t > 0$. That solution is

$$u(x,t) = \int K(x-y,t)f(y)\,dy,$$

the kernel being

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(\frac{-|x|^2}{4t}\right).$$

For every $x \in \mathbf{R}^n$ and every $t > 0$, $u(x,t)$ is C^∞ and bounded and it is a solution of the heat equation. Furthermore, $u(x,t)$ satisfies the initial condition in the sense that $u(x,t) \to f(x_0)$ when $x \to x_0$ and $t \to 0$ through positive values.

We now restrict the initial temperature somewhat more severely by requiring that $f(x) \geq 0$ for every $x \in \mathbf{R}^n$, that $f(x)$ does not vanish identically, and that the support of $f(x)$ is contained within a compact region Ω. The requirement that $f(x)$ be continuous, which was imposed earlier, is retained.

The integration occurring in the definition of $u(x,t)$ extends over all of $\mathbf{R}^n$ but we can replace it by an integration over the region Ω whenever we find it convenient to do so. We shall write

$$Q = \int f(x)\,dx,$$

and, apart from a factor ρc, Q is the total heat energy associated with the initial temperature.

The kernel has the property that $K(x,t) > 0$ for $x \in \mathbf{R}^n$ and $t > 0$ and it follows immediately that

$$u(x,t) > 0 \quad \text{for } x \in \mathbf{R}^n \text{ and } t > 0.$$

Thus, even though the temperature is initially zero outside the compact set Ω, Fourier's theory predicts that at any subsequent instant t the temperature is positive everywhere, no matter how small t may be. This conclusion has led some authors to speak of Fourier's theory as predicting that temperature disturbances propagate with infinite speed. Moreover, some have inferred that such a conclusion is sufficiently objectionable as to warrant the construction of alternative theories in which disturbances propagate with finite speed. Thus, to take one example, Cattaneo [2] has proposed a theory in which the temperature satisfies a hyperbolic telegraph equation while, to take another, Gurtin and Pipkin [3] have proposed a theory in which the heat flux vector and the specific internal energy depend upon the thermal history of the conductor and the temperature is a solution of an integro-differential equation. Cattaneo's theory was generalized by Coleman, Fabrizio, and Owen [4] who investigated the implications of the second law of thermodynamics for such a theory.

By contrast Fichera [5] has sought to defend Fourier's theory from the "accusation" (sic) that it produces the paradox according to which heat propagates with infinite speed and maintains that "this accusation, when the Fourier theory is properly interpreted, is unfounded" [5, p. 5]. Fichera's argument involves the recognition that measurements of temperature are possible only to within a limited accuracy and he introduces a positive constant ϵ which depends on the degree of refinement of the thermometer and which is an upper bound on temperature differences which must be regarded as being negligible because they cannot be detected by measurement. Fichera claims [5, p. 14] that "when the results of Fourier's theory are properly interpreted, heat does not propagate with infinite speed, but, if ϵ is not excessively small, rather slowly." He goes on to say that this was known to J. C. Maxwell and he cites an observation of Maxwell's [5, p. 18] to the effect that the sensible propagation of heat, so far from being instantaneous, is an excessively slow process, and the time taken for heat to propagate is proportional to the square of the distance.

My purpose is to point out certain facts which support Maxwell's observation. The conclusions lend further support to Fourier's theory over and above the defence already made by Fichera. The line of defence is different, though, in that it makes no use of Fichera's upper bound ϵ. Although the conclusions lend support to Fourier's theory I do not see them as detracting in any way from the value of the theories of Cattaneo, of Gurtin and Pipkin, or of Coleman, Fabrizio, and Owen, or of other writers on this subject. Those theories continue to be of interest and importance as alternative descriptions of the propagation of heat; this paper should not be viewed as a polemical attempt to demonstrate the superiority of Fourier's theory to all rival proposals.

The time taken to attain the maximum temperature. One way of giving precision to Maxwell's observation is to study the dependence of the temperature upon the time t at a fixed point x lying outside the support of the initial temperature. At such a point the temperature is zero initially but thereafter it increases to its maximum value and then decays again to zero. The first time at which the maximum temperature is attained will be denoted by $\tau(x)$; thus

$$u(x, \tau(x)) = \max_t u(x, t)$$

and

$$u(x, t) < u(x, \tau(x)) \quad \text{if } 0 < t < \tau(x).$$

The exact determination of $\tau(x)$ and $\max_t u(x,t)$ is difficult, but it is nonetheless possible to determine the asymptotic behaviour of each as $|x| \to \infty$. In fact, it will be shown that, as $|x| \to \infty$,

$$\tau(x) \sim |x|^2/(2n) \tag{1}$$

and

$$\max_t u(x,t) \sim Q\left(\frac{n}{2\pi e}\right)^{n/2} \frac{1}{|x|^n}. \tag{2}$$

The asymptotic relation (1) is in line with the observation of Maxwell's that was cited by Fichera.

In order to verify the relations (1) and (2) we choose a positive constant ω which is large enough to ensure that the support Ω is contained within the ball $\{x : |x| < \omega\}$. Thus if $|x| > \omega$ and $y \in \Omega$, we have $|y| \le \omega$ and $0 < |x| - \omega \le |x-y| \le |x| + \omega$. Hence, the kernel satisfies the estimates

$$K^-(x,t) \le K(x-y,t) \le K^+(x,t) \quad \text{if } |x| > \omega \quad \text{and} \quad y \in \Omega, \tag{3}$$

where

$$K^{\pm}(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp(-(|x| \mp \omega)^2/(4t)).$$

On multiplying through by $f(y)(\ge 0)$, and integrating with respect to y over the set Ω, we deduce immediately from the inequalities (3) that

$$QK^-(x,t) \le u(x,t) \le QK^+(x,t). \tag{4}$$

Hence

$$Q \max_t K^-(x,t) \le \max_t u(x,t) \le Q \max_t K^+(x,t).$$

However, elementary calculus establishes that

$$\max_t K^{\pm}(x,t) = \left(\frac{n}{2\pi e}\right)^{n/2} \frac{1}{(|x| - \omega)^n}$$

and, therefore, that

$$\begin{aligned} Q\left(\frac{n}{2\pi e}\right)^{n/2} \frac{1}{(|x| + \omega)^n} &\le \max_t u(x,t) \\ &\le Q\left(\frac{n}{2\pi e}\right)^{n/2} \frac{1}{(|x| + \omega)^n} \end{aligned}$$

whenever $|x| > \omega$. It follows that the asymptotic relation (2) is correct.

In order to verify the asymptotic relation (1) we use the fact that

$$\frac{\partial}{\partial t} u(x, \tau(x)) = 0, \tag{5}$$

where

$$\frac{\partial u}{\partial t}(u,t) = \int_\Omega \frac{\partial K}{\partial t}(x-y,t) f(y)\, dy.$$

The kernel has the property that

$$\frac{\partial}{\partial t}K(x,t) = \left[-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right] K(x,t)$$

and if we replace x by $x - y$, where $|x| > \omega$ and $y \in \Omega$, and remember that in these circumstances $0 < |x| - \omega \leq |x - y| \leq |x| + \omega$ we deduce that

$$\left[-\frac{n}{2t} + \frac{(|x| - \omega)^2}{4t^2}\right] K(x-y,t) \leq \frac{\partial K}{\partial t}(x-y,t) \leq \left[-\frac{n}{2t} + \frac{(|x| + \omega)^2}{4t^2}\right] K(x-y,t).$$

On multiplying through by $f(y)(\geq 0)$, and integrating with respect to y over Ω, we arrive at the estimates

$$\left[-\frac{n}{2t} + \frac{(|x| - \omega)^2}{4t^2}\right] u(x,t) \leq \frac{\partial u}{\partial t}(x,t) \leq \left[-\frac{n}{2t} + \frac{(|x| + \omega)^2}{4t^2}\right] u(x,t), \qquad (6)$$

which hold whenever $|x| > \omega$. Thus on setting $t = \tau(x)$ in these estimates and invoking Eq. (5) we have

$$-\frac{n}{2\tau(x)} + \frac{(|x| - \omega)^2}{4\tau(x)^2} \leq 0$$

and

$$-\frac{n}{2\tau(x)} + \frac{(|x| + \omega)^2}{4\tau(x)^2} \geq 0,$$

or, in other words,

$$\frac{(|x| - \omega)^2}{2n} \leq \tau(x) \leq \frac{(|x| + \omega)^2}{2n}$$

and it is now clear that the relation (1) must hold.

For reasons which will shortly become apparent it is convenient to re-label $\tau(x)$ as $\tau_3(x)$ so that

$$u(x, \tau_3(x)) = \max_t u(x,t)$$

and

$$\tau_3(x) \sim |x|^2/(2n) \quad \text{when } |x| \to \infty. \qquad (7)$$

The times taken to attain the maximum and minimum temperature-rates. By using much the same methods we can study the dependence of the temperature-rate $\partial u(x,t)/\partial t$ on the time t at a fixed point x lying outside the support of the initial temperature. The temperature-rate is zero initially; it then increases to a positive maximum value, returns to zero when $t = \tau_3(x)$, decreases to a negative minimum value, and ultimately tends to zero again as $t \to \infty$. Thus we are interested in determining the asymptotic behaviour of the times $\tau_1(x)$ and $\tau_4(x)$ for which

$$\frac{\partial u}{\partial t}(x, \tau_1(x)) = \max_t \frac{\partial u}{\partial t}(x,t),$$
$$\frac{\partial u}{\partial t}(x, \tau_4(x)) = \min_t \frac{\partial u}{\partial t}(x,t),$$

where, for the sake of definiteness, we confine our attention to the first times with these properties, i.e.,

$$\frac{\partial u}{\partial t}(x,t) < \frac{\partial u}{\partial t}(x,\tau_1(x)) \quad \text{for } 0 < t < \tau_1(x)$$

and

$$\frac{\partial u}{\partial t}(x,t) > \frac{\partial u}{\partial t}(x,\tau_4(x)) \quad \text{for } 0 < t < \tau_4(x).$$

We are also interested in the asymptotic behaviours of the maximum and minimum temperature-rates themselves.

It will turn out that, as $|x| \to \infty$,

$$\tau_1(x) \sim \frac{1}{2n}\left(1 - \frac{1}{\sqrt{1+\frac{n}{2}}}\right)|x|^2, \tag{8}$$

$$\tau_4(x) \sim \frac{1}{2n}\left(1 + \frac{1}{\sqrt{1+\frac{n}{2}}}\right)|x|^2, \tag{9}$$

and that

$$\max_t \frac{\partial u}{\partial t}(x,t) \sim \frac{4Q(\sqrt{1+\frac{n}{2}}+1)(1+\frac{n}{2}+\sqrt{1+\frac{n}{2}})^{1+\frac{n}{2}}}{\pi^{n/2}\exp(1+\frac{n}{2}+\sqrt{1+\frac{n}{2}})}\frac{1}{|x|^{2+n}}, \tag{10}$$

$$\min_t \frac{\partial u}{\partial t}(x,t) \sim -\frac{4Q(\sqrt{1+\frac{n}{2}}-1)(1+\frac{n}{2}-\sqrt{1+\frac{n}{2}})^{1+\frac{n}{2}}}{\pi^{n/2}\exp(1+\frac{n}{2}-\sqrt{1+\frac{n}{2}})}\frac{1}{|x|^{2+n}}. \tag{11}$$

Each of the asymptotic relations (8) and (9) is in line with Maxwell's observation. Before we attempt to derive them we first prove the relations (10) and (11). To do so we combine the inequalities (4) and (6) to arrive at the estimates

$$v^-(x,t) \le \frac{\partial u}{\partial t}(x,t) \le v^+(x,t) \quad \text{for } |x| > \omega,$$

in which the lower bound $v^-(x,t)$ is

$$\frac{Q}{(4\pi t)^{n/2}}\left[-\frac{n}{2t} + \frac{(|x|-\omega)^2}{4t^2}\right]\exp(-(|x|+\omega)^2/(4t))$$

and the upper bound $v^+(x,t)$ is

$$\frac{Q}{(4\pi t)^{n/2}}\left[-\frac{n}{2t} + \frac{(|x|+\omega)^2}{4t^2}\right]\exp(-(|x|-\omega)^2/(4t)).$$

It follows that

$$\max_t v^-(x,t) \le \max_t \frac{\partial u}{\partial t}(x,t) \le \max_t v^+(x,t),$$

and that

$$\min_t v^-(x,t) \le \min_t \frac{\partial u}{\partial t}(x,t) \le \min_t v^+(x,t).$$

The maximum and minimum values $\max_t v^-$, $\min_t v^-$, $\max_t v^+$, $\min_t v^+$ can all be evaluated by elementary calculus and on doing so we find that, when $|x| \to \infty$, both $\max_t v^-(x,t)$ and $\max_t v^+(x,t)$ are asymptotically equal to

$$\frac{4Q(\sqrt{1+\frac{n}{2}}+1)(1+\frac{n}{2}+\sqrt{1+\frac{n}{2}})^{1+\frac{n}{2}}}{\pi^{n/2}\exp(1+\frac{n}{2}+\sqrt{1+\frac{n}{2}})}\frac{1}{|x|^{2+n}}.$$

Thus the relation (10) is correct.

In like manner elementary calculus also establishes that, when $|x| \to \infty$, both $\min_t v^-(x,t)$ and $\min_t v^+(x,t)$ are asymptotically equal to

$$-\frac{4Q(\sqrt{1+\frac{n}{2}}-1)(1+\frac{n}{2}-\sqrt{1+\frac{n}{2}})^{1+\frac{n}{2}}}{\pi^{n/2}\exp(1+\frac{n}{2}-\sqrt{1+\frac{n}{2}})}\frac{1}{|x|^{2+n}}$$

and so the relation (11) is correct.

In order to derive the asymptotic relations (8) and (9) we use the fact that $\tau_1(x)$ must satisfy the conditions

$$\frac{\partial^2 u}{\partial t^2}(x,\tau_1(x)) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,\tau_1(x)) > 0, \tag{12}$$

while $\tau_4(x)$ must satisfy the conditions

$$\frac{\partial^2 u}{\partial t^2}(x,\tau_4(x)) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,\tau_4(x)) < 0.$$

We also note the formula

$$\frac{\partial^2}{\partial t^2}K(x,t) = \frac{1}{2t^2}\left[n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})|x|^2}{t} + \frac{|x|^4}{8t^2}\right]K(x,t).$$

On replacing x by $x-y$, where $|x| > \omega$ and $y \in \Omega$, and recalling that in these circumstances $0 < |x| - \omega \le |x-y| \le |x| + \omega$, we see that

$$\begin{aligned} &\frac{1}{2t^2}\left[n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})(|x|+\omega)^2}{t} + \frac{(|x|-\omega)^4}{8t^2}\right]K(x-y,t) \\ &\le \frac{\partial^2}{\partial t^2}K(x-y,t) \\ &\le \frac{1}{2t^2}\left[n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})(|x|-\omega)^2}{t} + \frac{(|x|+\omega)^4}{8t^2}\right]K(x-y,t). \end{aligned}$$

Thus, if we multiply through by $f(y)(\ge 0)$, integrate with respect to y over the set Ω, and remember that

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \int_\Omega \frac{\partial^2 K}{\partial t^2}(x-y,t)f(y)\,dy$$

we deduce that if $|x| > \omega$ then

$$\begin{aligned}
&\frac{1}{2t^2}\left[n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})(|x|+\omega)^2}{t} + \frac{(|x|-\omega)^4}{8t^2}\right]u(x,t) \\
&\quad\le \frac{\partial^2 u}{\partial t^2}(x,t) \\
&\quad\le \frac{1}{2t^2}\left[n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})(|x|-\omega)^2}{t} + \frac{(|x|+\omega)^4}{8t^2}\right]u(x,t).
\end{aligned}$$

Hence, if we set $t = \tau_1(x)$ and invoke the first of the conditions (12) we see that

$$n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})(|x|+\omega)^2}{\tau_1(x)} + \frac{(|x|-\omega)^4}{8\tau_1(x)^2} \le 0 \tag{13}$$

and

$$n\left(1+\frac{n}{2}\right) - \frac{(1+\frac{n}{2})(|x|-\omega)^2}{\tau_1(x)} + \frac{(|x|+\omega)^4}{8\tau_1(x)^2} \ge 0. \tag{14}$$

On the other hand, if we set $t = \tau_1(x)$ in the estimates (6), invoke the second of the conditions (12), and cancel a factor $2\tau_1(x)$ we see that

$$-n + \frac{(|x|+\omega)^2}{2\tau_1(x)} \ge 0. \tag{15}$$

It follows from the inequalities (13), (14), (15) that as $|x| \to \infty$, the ratio $|x|^2/\tau_1(x)$ is asymptotically equal to a number λ which is a root of the quadratic equation

$$n\left(1+\frac{n}{2}\right) - \left(1+\frac{n}{2}\right)\lambda + \frac{1}{8}\lambda^2 = 0$$

and which has the property that $\lambda \ge 2n$. The roots of the quadratic are

$$4\left[1+\frac{n}{2} \pm \sqrt{1+\frac{n}{2}}\right]$$

and if we are to have $\lambda \ge 2n$ we must choose the positive sign. Thus

$$\frac{|x|^2}{4\tau_1(x)} \to 1 + \frac{n}{2} + \sqrt{1+\frac{n}{2}} \quad \text{when } |x| \to \infty$$

and this last is equivalent to the asymptotic relation (8). The asymptotic relation (11) is proved in essentially the same way, with obvious minor changes.

The time taken to attain the maximum radial heat flux. Similar methods can be used to study the dependence upon the time t of the radial component of the heat flux vector, which is

$$\frac{x}{|x|} \cdot q(x,t).$$

It proves to be possible to determine the asymptotic behaviour, as $|x| \to \infty$, of the maximum value

$$\max_t \left(\frac{x}{|x|} \cdot q(x,t) \right)$$

and of the first time, $\tau_2(x)$ say, at which this maximum value is attained; thus

$$\frac{x}{|x|} \cdot q(x, \tau_2(x)) = \max_t \left(\frac{x}{|x|} \cdot q(x,t) \right)$$

and

$$\frac{x}{|x|} \cdot q(x,t) < \frac{x}{|x|} \cdot q(x, \tau_2(x)) \quad \text{for } 0 < t < \tau_2(x).$$

It will be shown that, when $|x| \to \infty$,

$$\tau_2(x) \sim |x|^2/(2(2+n)) \tag{16}$$

and

$$\max_t \left(\frac{x}{|x|} \cdot q(x,t) \right) \sim \frac{Q}{(2\pi)^{n/2}} \left(\frac{2+n}{e} \right)^{1+\frac{n}{2}} \frac{1}{|x|^{1+n}}. \tag{17}$$

Once again, the asymptotic relation (16) is in line with Maxwell's observation.

In order to verify these relations we note that

$$\frac{\partial K}{\partial x_1}(x,t) = -\frac{x_1}{2t} K(x,t), \ldots, \frac{\partial K}{\partial x_n}(x,t) = -\frac{x_n}{2t} K(x,t).$$

Hence the heat flux vector

$$q(x,t) = \frac{k}{2t} \int_\Omega (x-y) K(x-y,t) f(y)\, dy$$

and its radial component

$$\frac{x}{|x|} \cdot q(x,t) = \frac{k}{2t} \int_\Omega \left(|x| - \frac{x \cdot y}{|x|} \right) K(x-y,t) f(y)\, dy. \tag{18}$$

If $y \in \Omega$ then $|y| \leq \omega$, $|x \cdot y| \leq \omega |x|$, and therefore

$$\frac{k}{2t}(|x| - \omega) \int_\Omega K(x-y,t) f(y)\, dy \leq \frac{x}{|x|} \cdot q(x,t) \leq \frac{k}{2t}(|x| + \omega) \int_\Omega K(x-y,t) f(y)\, dy$$

or, in other words,

$$\frac{k}{2t}(|x| - \omega) u(x,t) \leq \frac{x}{|x|} \cdot q(x,t) \leq \frac{k}{2t}(|x| + \omega) u(x,t).$$

Thus, if we appeal to the inequalities (4) we see that

$$\frac{1}{2}kQ(|x|-\omega)\frac{K^-(x,t)}{t} \le \frac{x}{|x|}\cdot q(x,t) \le \frac{1}{2}kQ(|x|+\omega)\frac{K^+(x,t)}{t}$$

and, therefore,

$$\begin{aligned}&\frac{1}{2}kQ(|x|-\omega)\max_t\left(\frac{K^-(x,t)}{t}\right)\\&\qquad\le \max_t\left(\frac{x}{|x|}\cdot q(x,t)\right) \le \frac{1}{2}kQ(|x|+\omega)\max_t\left(\frac{K^+(x,t)}{t}\right).\end{aligned}$$

The maxima occurring on the extreme left-hand and right-hand sides of this last inequality can be readily evaluated by means of elementary calculus, and it is found that when $|x| \to \infty$ the extreme left-hand and right-hand sides are both asymptotically equal to

$$\frac{kQ}{(2\pi)^{n/2}}\left(\frac{2+n}{e}\right)^{1+\frac{n}{2}}\frac{1}{|x|^{1+n}}.$$

Hence the asymptotic relation (17) is correct.

In order to check the relation (16) we use the fact that

$$\frac{\partial}{\partial t}\left(\frac{x}{|x|}\cdot q(x,t)\right) = 0 \quad \text{when } t = \tau_2(x). \tag{19}$$

If we return to Eq. (18) and differentiate both sides with respect to t we see that

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{x}{|x|}\cdot q(x,t)\right)\\&\qquad= \frac{k}{4t^2}\int_\Omega\left(|x|-\frac{x\cdot y}{|x|}\right)\left[-(2+n)+\frac{|x-y|^2}{2t}\right]K(x-y,t)f(y)\,dy.\end{aligned}$$

Hence, if $|x| > \omega$,

$$\begin{aligned}&\frac{k}{4t^2}(|x|-\omega)\left[-(2+n)+\frac{(|x|-\omega)^2}{2t}\right]\int_\Omega K(x-y,t)f(y)\,dy\\&\quad\le \frac{\partial}{\partial t}\left(\frac{x}{|x|}\cdot q(x,t)\right)\\&\quad\le \frac{k}{4t^2}(|x|+\omega)\left[-(2+n)+\frac{(|x|+\omega)^2}{2t}\right]\int_\Omega K(x-y,t)f(y)\,dy\end{aligned}$$

and, therefore,

$$\begin{aligned}&\frac{k}{4t^2}(|x|-\omega)\left[-(2+n)+\frac{(|x|-\omega)^2}{2t}\right]u(x,t)\\&\quad\le \frac{\partial}{\partial t}\left(\frac{x}{|x|}\cdot q(x,t)\right)\\&\quad\le \frac{k}{4t^2}(|x|+\omega)\left[-(2+n)+\frac{(|x|+\omega)^2}{2t}\right]u(x,t).\end{aligned}$$

Thus, on setting $t = \tau_2(x)$ and appealing to Eq. (19) we see that

$$-(2+n) + \frac{(|x|-\omega)^2}{2\tau_2(x)} \leq 0$$

and

$$-(2+n) + \frac{(|x|+\omega)^2}{2\tau_2(x)} \geq 0.$$

Hence

$$\frac{(|x|-\omega)^2}{2(2+n)} \leq \tau_2(x) \leq \frac{(|x|+\omega)^2}{2(2+n)}$$

and the asymptotic relation (16) is correct.

Examination of the asymptotic formulae (7), (8), (9), (16) discloses that when $|x|$ is large the times $\tau_1(x), \tau_2(x), \tau_3(x), \tau_4(x)$ occur in order of increasing magnitude, i.e.,

$$\tau_1(x) < \tau_2(x) < \tau_3(x) < \tau_4(x).$$

Thus, if observations of these four times are made at a point x that is sufficiently far from the support of the initial temperature then the maximum rate of increase of temperature occurs first, when $t = \tau_1(x)$, then the maximum radial heat flux occurs when $t = \tau_2(x)$, next the maximum temperature occurs when $t = \tau_3(x)$, and finally the maximum rate of decrease of temperature occurs when $t = \tau_4(x)$.

It is of interest to note that

$$\tau_2(x) \sim \frac{2}{\frac{1}{\tau_1(x)} + \frac{1}{\tau_4(x)}} \quad \text{when } |x| \to \infty$$

and so the time at which the maximum radial heat flux occurs is asymptotically equal to the harmonic mean of the times at which the maximum rates of increase and decrease of temperature occur. Moreover,

$$\tau_3(x) \sim \frac{1}{2}(\tau_1(x) + \tau_4(x)) \quad \text{when } |x| \to \infty$$

and so the time at which the maximum temperature occurs is asymptotically equal to the arithmetic mean of the times at which the maximum rates of increase and decrease of temperature occur.

In the limit as $|x| \to \infty$ the four times occur in the ratios

$$1 - \frac{1}{\sqrt{1+\frac{n}{2}}} : \frac{n}{2+n} : 1 : 1 + \frac{1}{\sqrt{1+\frac{n}{2}}}.$$

As the dimension n increases the times congregate more closely and, in the cases of physical interest, the ratios become

$$\begin{aligned} &0.1835 : 0.3333 : 1 : 1.8165 \quad (n=1),\\ &0.2929 : 0.5 \quad\;\; : 1 : 1.7071 \quad (n=2),\\ &0.3676 : 0.6 \quad\;\; : 1 : 1.6325 \quad (n=3). \end{aligned}$$

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