Nihonkai Math. J.
Vol.7 (1996), 185-195

ON REAL HYPERSURFACES IN QUATERNIONIC PROJECTIVE SPACE WITH $\mathcal{D}^{\perp}$-PARALLEL SECOND FUNDAMENTAL FORM

JUAN DE DIOS PÉREZ AND YOUNG JIN SUH*

ABSTRACT. In this paper we give a complete classification of real hypersurfaces in a quaternionic projective space QP^m satisfying certain conditions on the orthogonal distribution $\mathcal{D}$.

§1. Introduction

Throughout this paper let us denote by M a connected real hypersurface in a quaternionic projective space QP^m,$m\geq 3$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_i = -J_iN$, $i = 1,2,3$, where $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic structure of QP^m, [3]. Several examples of such real hypersurfaces are well known, see for instance ([1],[5],[6],[7]).

Now, let us consider the following conditions that the second fundamental tensor A of M in QP^m may satisfy

$$(\nabla_X A)Y = -\Sigma_{i=1}^{3}\{f_i(Y)\phi_i X + g(\phi_i X, Y)U_i\}, \tag{1.1}$$

$$g((A\phi_i - \phi_i A)X, Y) = 0, \tag{1.2}$$

for any $i = 1,2,3$, and any tangent vector fields X and Y of M, where the connection of M induced from the connection of QP^m is denoted by ∇.

Pak [7] investigated the above conditions and showed that they are equivalent to each other. Moreover he used the condition (1.1) to find a lower bound of $\|\nabla A\|$ for real hypersurfaces in QP^m. In fact, it was shown that $\|\nabla A\|^2 \geq 24(m-1)$ for such hypersurfaces and the equality holds if and only if the condition (1.1) holds. In this case it was also known that M is locally congruent to a real hypersurface of type A_1 or A_2, which means a tube of radius r over QP^k $(1\leq k\leq m-1)$ in the notion of Berndt [1], and Martinez and the first author [5].

Now let us define a distribution $\mathcal{D}$ by $\mathcal{D}(x) = \{X\in T_xM : X\perp U_i(x), i = 1,2,3\}$,$x\in M$, of a real hypersurface M in QP^m, which is orthogonal to the structure vector fields $\{U_1, U_2, U_3\}$ and invariant with respect to structure tensors $\{\phi_1, \phi_2, \phi_3\}$, and by $\mathcal{D}^{\perp} = Span\{U_1, U_2, U_3\}$ its orthogonal complement in TM.

1991 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C15.

* The second author was supported by NON DIRECTED RESEARCH FUND, Korea Research Foundation, 1995 and partly by TGRC-KOSEF.

There exist many studies about real hypersurfaces of quaternionic projective space QP^m. Among them Martinez and the first author [6] have classified real hypersurfaces of QP^m with constant principal curvatures when the distribution $\mathcal{D}$ is invariant by the second fundamental tensor, that is , the shape operator A. It was shown that these real hypersurfaces of QP^m could be divided into three types which are said to be of type A_1, A_2, and B, where a real hypersurface of type B denotes a tube over a complex projective space CP^m. Hereafter, let us say *A-invariant* when the distribution $\mathcal{D}$ is invariant by the shape operator A.

As a further improvement of this result, Berndt [1], recently showed that any real hypersurfaces of QP^m satisfying $g(A\mathcal{D}, \mathcal{D}^\perp) = 0$ must be of one of the above three types, avoiding the additional assumption about constancy of principal curvatures.

If we restrict the above properties (1.1) and (1.2) to the orthogonal distribution $\mathcal{D}$, then the second fundamental tensor A of M satisfies the following conditions

$$g((\nabla_X A)Y, Z) = 0 \tag{1.3}$$

and

$$g((A\phi_i - \phi_i A)X, Y) = 0 \tag{1.4}$$

for any $i = 1, 2, 3$ and for any vector fields X, Y and Z in $\mathcal{D}$. Then the second fundamental form of M satisfying the condition (1.3) is said to be *$\mathcal{D}^\perp$-parallel* . Moreover, the condition (1.3) is weaker than (1.1) (respectively, (1.4) is weaker than (1.2)). Thus it is natural that real hypersurfaces of type A_1, and A_2 should satisfy (1.3) and (1.4). Moreover, by a theorem of Berndt [1] it was known that the orthogonal distribution $\mathcal{D}$ of this type is *A-invariant.*

About a ruled real hypersurface of QP^m some properties are investigated by Martinez [5] and the first author [9]. It will be shown in section 3 that the second fundamental form of the ruled real hypersurfaces is *$\mathcal{D}^\perp$-parallel.* Contrary to real hypersurfaces of type A_1, A_2, and B given by Berndt [1] and Martinez and the first author [6], it can be easily seen that the orthogonal distribution $\mathcal{D}$ of any ruled real hypersurface of QP^m is not *A-invariant.* From this point of view we give a classification theorem as the following

Theorem. *Let M be a real hypersurface in QP^m,$m \geq 3$. If it satisfies (1.3) and (1.4), then M is congruent to one of the following spaces:*

(A_1) a tube of radius r over a hyperplane QP^{m-1}, where $0 < r < \frac{\pi}{2}$,

(A_2) a tube of radius r over a totally geodesic QP^k ($1 \leq k \leq m-2$), where $0 < r < \frac{\pi}{2}$.

(R) a ruled real hypersurface foliated by totally geodesic quaternionic hyperplanes QP^{m-1}.

§2. Preliminaries

Let X be a tangent vector field to M. We write $J_iX = \phi_iX + f_i(X)N, i = 1,2,3$, where ϕ_iX is the tangent component of J_iX and $f_i(X) = g(X, U_i), i = 1,2,3$. As $J_i^2 = -id$, $i = 1,2,3$, where id denotes the identity endomorphism on TQP^m, we get

$$\phi_i^2X = -X + f_i(X)U_i, \quad f_i(\phi_iX) = 0, \quad \phi_iU_i = 0, \quad i = 1,2,3 \tag{2.1}$$

for any X tangent to M. As $J_iJ_j = -J_jJ_i = J_k$, where (i,j,k) is a cyclic permutation of $(1,2,3)$ we obtain

$$\phi_iX = \phi_j\phi_kX - f_k(X)U_j = -\phi_k\phi_jX + f_j(X)U_k \tag{2.2}$$

and

$$f_i(X) = f_j(\phi_kX) = -f_k(\phi_jX) \tag{2.3}$$

for any vector field X tangent to M, where (i,j,k) is a cyclic permutation of $(1,2,3)$. It is also easy to see that for any X, Y tangent to M and $i = 1,2,3$

$$g(\phi_iX, Y) + g(X, \phi_iY) = 0, \quad g(\phi_iX, \phi_iY) = g(X,Y) - f_i(X)f_i(Y) \tag{2.4}$$

and

$$\phi_iU_j = -\phi_jU_i = U_k \tag{2.5}$$

(i,j,k) being a cyclic permutation of $(1,2,3)$. From the expression of the curvature tensor of QP^m, $m \geq 2$, we have the equations of Gauss and Codazzi respectively given by

(2.6)
$$\begin{aligned} R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + \Sigma_{i=1}^3\{g(\phi_iY,Z)\phi_iX - g(\phi_iX,Z)\phi_iY \\ &+ 2g(X,\phi_iY)\phi_iZ\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned}$$

and

$$(\nabla_XA)Y - (\nabla_YA)X = \Sigma_{i=1}^3\{f_i(X)\phi_iY - f_i(Y)\phi_iX + 2g(X,\phi_iY)U_i\} \tag{2.7}$$

for any X, Y, Z tangent to M, where R denotes the curvature tensor of M, see [7].

From the expressions of the covariant derivatives of J_i, $i = 1,2,3$, it is easy to see that

$$\nabla_XU_i = -p_j(X)U_k + p_k(X)U_j + \phi_iAX \tag{2.8}$$

and

$$(\nabla_X\phi_i)Y = -p_j(X)\phi_kY + p_k(X)\phi_jY + f_i(Y)AX - g(AX,Y)U_i \tag{2.9}$$

for any X, Y tangent to M, (i,j,k) being a cyclic permutation of $(1,2,3)$ and p_i, $i = 1,2,3$, local 1-forms on QP^m.

§3. $\mathcal{D}^{\perp}$-parallel second fundamental form

Let M be a real hypersurface in a quaternionic projective space QP^m, and let $\mathcal{D}$ be a distribution defined by $\mathcal{D}(x) = \{X \in T_xM : X \perp U_i(x), i = 1,2,3\}$. Then a real hypersurface M in QP^m is said to be $\mathcal{D}^{\perp}$-*parallel* if $g((\nabla_X A)Y, Z) = 0$ for any X,Y and $Z \in \mathcal{D}$.

In this section we define the notion of ruled real hypersurfaces in QP^m. By investigating some fundamental properties of these ruled real hypersurfaces we can prove that its second fundamental form is $\mathcal{D}^{\perp}$-*parallel*. Moreover, from the condition (1.1) we know that real hypersurfaces of type A_1 or A_2 in QP^m have its second fundamental form $\mathcal{D}^{\perp}$-*parallel*.

Now in order to prove our theorem in the introduction we need a lemma obtained from the restricted condition (1.4) as the following

Lemma 3.1. *Let M be a real hypersurface of QP^m. If it satisfies the condition (1.4) for any $i = 1,2,3$ and for any vector fields X, Y in $\mathcal{D}$, then we have*

$$g((\nabla_X A)Y, Z) = \mathfrak{S} g(AX, Y) g(Z, V_i), \quad i = 1,2,3 \tag{3.1}$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z in $\mathcal{D}$ and V_i stands for the vector field defined by $\phi_i A U_i$.

Proof. Taking the covariant derivative of (1.4), for any vector fields X, Y and Z in $\mathcal{D}$ we get

$$\begin{aligned} g((\nabla_X A)\phi_i Y + A(\nabla_X \phi_i)Y + A\phi_i \nabla_X Y - (\nabla_X \phi_i)AY - \phi_i(\nabla_X A)Y, Z) \\ - g(\phi_i A \nabla_X Y, Z) + g((A\phi_i - \phi_i A)Y, \nabla_X Z) = 0. \end{aligned}$$

Now let us consider the following for the case where $i = 1$

$$\begin{aligned} g((\nabla_X A)Y, \phi_1 Z) + g((\nabla_X A)Z, \phi_1 Y) &= -g((\nabla_X \phi_1)Y, AZ) - g(\phi_1 \nabla_X Y, AZ) \\ &\quad + g((\nabla_X \phi_1)AY, Z) - g(A\nabla_X Y, \phi_1 Z) + \Sigma_i \theta_i(Y) g(\phi_i AX, Z), \end{aligned}$$

where $g((A\phi_1 - \phi_1 A)Y, U_i)$ is denoted by $\theta_i(Y)$ and we have used the fact that

$$\begin{aligned} g((A\phi_1 - \phi_1 A)Y, \nabla_X Z) &= \Sigma_i \theta_i(Y) g(U_i, \nabla_X Z) \\ &= -\Sigma_i \theta_i(Y) g(\nabla_X U_i, Z) \\ &= -\Sigma_i \theta_i(Y) g(\phi_i AX, Z). \end{aligned}$$

Then by taking account of (2.8) and (2.9) and using the condition (1.4) again, we have

(3.2)

$$\begin{aligned} g((\nabla_X A)Y, \phi_1 Z) + g((\nabla_X A)Z, \phi_1 Y) &= f_1(AZ) g(AX, Y) + f_1(AY) g(AX, Z) \\ &\quad + \Sigma_i \theta_i(Z) g(\phi_i AX, Y) + \Sigma_i \theta_i(Y) g(\phi_i AX, Z). \end{aligned}$$

In this equation we shall replace X, Y and Z in $\mathcal{D}$ cyclically and we shall then add the second equation to (3.2), from which we subtract the third one. Then by means of Codazzi equation (2.7) we get

$$\begin{aligned} g((\nabla_X A)Y, \phi_1 Z) =& f_1(AZ)g(AX, Y) + \Sigma_i \theta_i(X) g(A\phi_i Y, Z) \\ &+ \Sigma_i \theta_i(Y) g(A\phi_i X, Z) \end{aligned}$$

From this, replacing Z by $\phi_1 Z$, we have

$$\begin{aligned} g((\nabla_X A)Y, Z) =& g(V_1, Z)g(AX, Y) - \Sigma_i \theta_i(X) g(A\phi_i Y, \phi_1 Z) \\ &- \Sigma_i \theta_i(Y) g(A\phi_i X, \phi_1 Z). \end{aligned} \tag{3.3}$$

where V_1 denotes $\phi_1 AU_1$ and the second term of the right hand side is given by the following

$$\begin{aligned} \Sigma_i \theta_i(X) g(A\phi_i Y, \phi_1 Z) =& - g(X, \phi_1 AU_1) g(AY, Z) + \{g(A\phi_1 X, U_2) \\ &+ g(AX, U_3)\} g(AY, \phi_3 Z) - \{g(A\phi_1 X, U_3) \\ &- g(AX, U_2)\} g(AY, \phi_2 Z), \end{aligned}$$

from this, the third term can be given by exchanging X and Y. Thus substituting this into (3.3), we have

$$g((\nabla_X A)Y, Z) = \mathfrak{S} g(V_1, Z) g(AX, Y) + \alpha(X, Y, Z) + \alpha(Y, X, Z) \tag{3.4}$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z in $\mathcal{D}$ and $\alpha(X, Y, Z)$ denotes

$$-\{g(A\phi_1 X, U_2) + g(AX, U_3)\} g(AY, \phi_3 Z) + \{g(A\phi_1 X, U_3) - g(AX, U_2)\} g(AY, \phi_2 Z)$$

so that, it is skew-symmetric with respect to Y and Z. Thus taking cyclic sum of (3.4) and using the skew-symmetry of $\alpha(X, Y, Z)$ and the equation of Codazzi (2.7), we have the above result for $i = 1$. For cases where $i = 2$ or 3 by using the similar method we can also prove the above result. □

Remark 3.1. Let us denote by S^{4m+3} a $(4m+3)$-dimensional unit sphere. Given a real hypersurface of QP^m, one can construct a hypersurface N of S^{4m+3} which is a principal S^3 − *bundle* over M with totally geodesic fibres and the projection $\pi : N \to M$ in such a way that the diagram

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[d, "\pi"'] & S^{4m+3} \arrow[d, "\pi"] \\ M \arrow[r, "\iota'"'] & QP^m \end{tikzcd}$$

is commutative (ι, ι' being the isometric immersions). Then it is seen ([1],[7]) that the second fundamental tensor A' of N is parallel if and only if the second fundamental tensor A of M satisfies the condition (1.1) or (1.2). Thus M is congruent to real hypersurfaces of type A_1 or A_2 in QP^m. Moreover, in this case it satisfies the condition (1.3), that is, its second fundamental form is $\mathcal{D}^{\perp}$*-parallel.*

Now let us define a ruled real hypersurface M of QP^m as follows: Let $\gamma : I \to QP^m$ be any regular curve. Then for any $t(\in I)$ let $QP^{m-1}_{(t)}$ be a totally geodesic quaternionic hypersurface of QP^m which is orthogonal to a quaternionic cubic spanned by $\gamma'(t)$, and $J_i\gamma'(t), i = 1,2,3$. Set $M = \{x \in QP^{m-1}_{(t)} : t \in I\}$. Then ,by the construction, M becomes a real hypersurface of QP^m, which is called a *ruled real hypersurface.* This construction gives us that there are many ruled real hypersurfaces of QP^m. Let $\mathcal{D}$ be a distribution defined by $\mathcal{D}(x) = \{X \in T_xM : X \perp U_i(x), i = 1,2,3\}$ in the tangent space T_xM of M at any point x in M. Then from this construction it can be easily verified that

$$AU_i = \Sigma_j \alpha_{ij} U_j + \epsilon_i X_i,\ AX_i = \Sigma_j \epsilon_j g_{ij} U_j,\ AX = 0 \tag{3.5}$$

for any vector field X orthogonal to U_i and X_i, where $g_{ij} = g(X_i, X_j)$ and $X_i, i = 1,2,3$, denote unit vector fields in $\mathcal{D}$, and $\epsilon_i(\epsilon_i \neq 0)$, α_{ij} are smooth functions on M. Moreover, a unit vector field X_i can be defined only for a non vanishing ϵ_i. But for such a ruled real hypersurface we know that at least one of $\epsilon_i, i = 1,2,3$, can not vanish.

From Remark 3.1 we know that real hypersurfaces of type A_1, or A_2 in QP^m are $\mathcal{D}^{\perp}$*-parallel*, because the condition (1.3) is weaker than the condition (1.1). Further, we can verify that ruled real hypersurfaces in QP^m are $\mathcal{D}^{\perp}$*-parallel* by the following

Proposition 3.2. *Let M be a ruled real hypersurface in QP^m. Then the second fundamental form of M is $\mathcal{D}^{\perp}$-parallel.*

Proof. Let M be a ruled real hypersurface. Then the expression of its second fundamental form is given by (3.5).

Now let us consider for the case where all of $\epsilon_i, i = 1,2,3$ do not vanish. Then we denote by $\mathcal{D}_1$ a subdistribution of the tangent space $T_xM, x \in M$, defined by $\mathcal{D}_1(x) = \{U_i(x), X_i(x) : i = 1,2,3\}$.

Now by (3.5) we have

$$\begin{aligned} g((\nabla_X A)Y, Z) &= g(\nabla_X(AY) - A\nabla_X Y, Z) \\ &= -g(A\nabla_X Y, Z) \\ &= 0 \end{aligned}$$

for any $X, Y, Z \in \mathcal{D}_1^{\perp}$. Also from (2.8) for any $X, Y \in \mathcal{D}_1^{\perp}$ we have

$$\begin{aligned} g((\nabla_X A)Y, X_j) &= g((\nabla_X(AY) - A\nabla_X Y, X_j) \\ &= -g(\nabla_X Y, AX_j) \\ &= \Sigma \epsilon_j g_{kj} g(Y, \nabla_X U_k) \\ &= 0, \end{aligned}$$

so that, by using the equation of Codazzi (2.7)

$$g((\nabla_{X_j} A)Y, Z) = g((\nabla_Y A)X_j, Z) = 0.$$

Moreover, for any $X, Z \in \mathcal{D}_1^{\perp}$ we have the following

$$\begin{aligned} g((\nabla_{X_j} A)X_k, Z) &= g((\nabla_{X_j}(AX_k) - A\nabla_{X_j}X_k, Z) \\ &= g(\Sigma_l X_j(\epsilon_l g_{kl})U_l + \Sigma_l \epsilon_l g_{kl} \nabla_{X_j} U_l, Z) \\ &= 0, \end{aligned}$$

$$g((\nabla_X A)X_k, X_l) = g((\nabla_{X_k} A)X, X_l) = g((\nabla_{X_k} A)X_l, X) = 0,$$

and

$$\begin{aligned} g((\nabla_{X_i} A)X_j, X_k) =& g(\nabla_{X_i}(AX_j) - A\nabla_{X_i}X_j, X_k) \\ =& g(\Sigma_l X_i(\epsilon_l g_{jl})U_l + \Sigma_l \epsilon_l g_{jl} \nabla_{X_i} U_l, X_k) \\ & - g(\nabla_{X_i}X_j, AX_k) \\ =& \Sigma_l \epsilon_l g_{kl} g(X_j, \nabla_{X_i} U_l) \\ =& 0. \end{aligned}$$

From these formulas we have the above assertion. Using the same method for the cases where one or two of ϵ_i do not vanish, we can also obtain the above assertion. □

Also for the case where M is a real hypersurface of type B in QP^m we have the following

Proposition 3.3. *Let M be a real hypersurface of type B in QP^m. Then the second fundamental form of M is $\mathcal{D}^{\perp}$-parallel.*

Proof. The tangent space T_xM of M can be decomposed as follows

$$T_xM = V_{\lambda_1} \oplus V_{\lambda_2} \oplus V_{\mu_1} \oplus V_{\mu_2}$$

where V_{λ_i}, $V_{\mu_i}, i = 1,2$ are eigenspaces of principal vectors with principal curvatures $\lambda_1 = cotr$, $\lambda_2 = -tanr$, $\mu_1 = 2cotr$, and $\mu_2 = -2tan2r$, where $0 < r < \frac{\pi}{4}$ or $\frac{\pi}{4} < r < \frac{\pi}{2}$, respectively. Then we can take an orthonormal basis $\{e_1, ..., e_{m-1} : \phi_1 e_1, ..., \phi_1 e_{m-1} : \phi_2 e_1, ..., \phi_2 e_{m-1} : \phi_3 e_1, ..., \phi_3 e_{m-1}, U_1, U_2, U_3\}$ with principal curvatures $cotr$,$-tanr$, $2cot2r$, and $-2tan2r$ of multiplicity *2(m-1), 2(m-1),1* and *2* respectively.

Firstly we know

$$\begin{aligned} g((\nabla_{e_j} A)e_k, e_l) &= g(\nabla_{e_j}(Ae_k) - A\nabla_{e_j} e_k, e_l) \\ &= cotrg(\nabla_{e_j} e_k, e_l) - cotrg(\nabla_{e_j} e_k, e_l) \\ &= 0, \end{aligned}$$

where the indices j, k, l run over the range $1, 2, ..., n$.

Secondly we want to calculate

$$\begin{aligned} g((\nabla_{e_j} A)e_k, \phi_2 e_l) &= g(\nabla_{e_j}(Ae_k) - A\nabla_{e_j} e_k, \phi_2 e_l) \\ &= (cotr + tanr)g(\nabla_{e_j} e_k, \phi_2 e_l) \end{aligned} \tag{3.6}$$

Thus it suffices to show $g(\nabla_{e_j} e_k, \phi_2 e_l) = 0$. In fact, by using the equation of Codazzi (2.7) we have

$$(3.7) \qquad \begin{aligned}(\nabla_{\phi_2 e_j} A)e_k - (\nabla_{e_k} A)\phi_2 e_j &= 2\Sigma_k g(\phi_2 e_j, \phi_l e_k)U_l \\ &= 2\delta_{jk} U_2,\end{aligned}$$

from which, the left side becomes

$$(\nabla_{\phi_2 e_j} A)e_k - (\nabla_{e_k} A)\phi_2 e_j = (cotrI - A)\nabla_{\phi_2 e_j} e_k - (-tanrI - A)\nabla_{e_k} \phi_2 e_j.$$

From these formulas, taking the inner product with e_l, we have

$$\begin{aligned}0 &= g((cotrI - A)\nabla_{\phi_2 e_j} e_k, e_l) + g((tanrI + A)\nabla_{e_k} \phi_2 e_j, e_l) \\ &= (cotr + tanr) g((\nabla_{e_k} \phi_2 e_j, e_l),\end{aligned}$$

where we have used the fact that the first term in the right side of the first equality vanishes. From this, together with (3.6) we know

$$(3.8) \qquad g((\nabla_{e_j} A)e_k, \phi_2 e_l) = 0.$$

Finally we can also obtain the following

$$\begin{aligned}g((\nabla_{e_j} A)\phi_2 e_k, \phi_3 e_l) &= -tanrg(\nabla_{e_j} \phi_2 e_k, \phi_3 e_l) + tanrg(\nabla_{e_j} \phi_2 e_k, \phi_3 e_l) \\ &= 0, \\ g((\nabla_{e_j} A)\phi_2 e_k, \phi_2 e_l) &= -tanrg(\nabla_{e_j} \phi_2 e_k, \phi_2 e_l) + tanrg(\nabla_{e_j} \phi_2 e_k, \phi_2 e_l) \\ &= 0.\end{aligned}$$

Therefore these formulas and (3.8) imply that the second fundamental form of M is $\mathcal{D}^{\perp}$-*parallel.* □

Remark 3.4. For real hypersurfaces of type B in QP^m it can be easily seen that they do not satisfy the condition (1.4). In fact, for the case $i = 2$ we have

$$A\phi_2 e_k - \phi_2 A e_k = -(tanr + cotr)\phi_2 e_k,$$

so that $g(A\phi_2 e_k - \phi_2 A e_k, \phi_2 e_k) = -(tanr + cotr) \neq 0$ for $0 < r < \frac{\pi}{4}$ or $\frac{\pi}{4} < r < \frac{\pi}{2}$.

§4. Proof of the Theorem

The purpose of this section is to prove the main theorem in the introduction. Now let us denote by $\mathcal{D}$ the distribution in M orthogonal to $\mathcal{D}^{\perp} = Span\{U_1, U_2, U_3\}$, where $U_i = -J_i N$, N is a unit normal to M. Now we prove the main theorem case by case. Thus firstly we consider the following.

Case 1) $\mathcal{D}$ is *A-invariant* on M. That is, $g(A\mathcal{D}, \mathcal{D}^{\perp}) = 0$ on M. Then by a theorem of Berndt [1] M is congruent to one of either type A_1, or A_2 or B. For the case where M is of type either A_1 or A_2 we know that the conditions (1.3) and (1.4) hold on it (see Remark 3.1). Though Proposition 3.3 gives that real hypersurfaces of type B satisfy the condition (1.3), but as was shown in Remark 3.4 they do not satisfy the condition (1.4). Thus real hypersurfaces of type B do not appear among them.

Case 2) $\mathcal{D}$ is not *A-invariant* on M, that is, a set $M_0 = \{p\in M : g(A\mathcal{D}, \mathcal{D}^\perp)_p \neq 0\}$ is not empty. Then we can prove this case in two steps. First one is to show that $g(A\mathcal{D}, \mathcal{D}) = 0$ on M_0, which means that M_0 is congruent to a ruled real hypersurface. And the other one is to show that the set M_0 can be extended to the whole set M.

On M_0 we can put $AU_i = \epsilon_i X_i + \Sigma_j \alpha_{ij} U_j$. Then from the definition of the set M_0 we know that at least one of $\epsilon_i, i = 1, 2, 3$ must not vanish. Thus for convienience sake let us put $\epsilon_1 \neq 0$. Then we can write

$$\begin{aligned} V_1 &= \phi_1 A U_1 \\ &= \epsilon_1 \phi_1 X_1 + \Sigma_j \alpha_{1j} \phi_1 U_j, \ X_1 \in \mathcal{D}. \end{aligned}$$

From the assumption of (1.3) and Lemma 3.1 we have the following

$$\epsilon_1 g(\phi_1 X_1, Z) g(AX, Y) + \epsilon_1 g(\phi_1 X_1, Y) g(AZ, X) + \epsilon_1 g(\phi_1 X_1, X) g(AZ, Y) = 0,$$

for any X, Y and Z in $\mathcal{D}$. From this, putting $Z = \phi_1 X_1$, then

$$g(AX, Y) + g(\phi_1 X_1, Y) g(A\phi_1 X_1, X) + g(\phi_1 X_1, X) g(A\phi_1 X_1, Y) = 0, \tag{4.1}$$

where we have used $\epsilon_1 \neq 0$. So also by taking $Y = \phi_1 X_1$ in (4.1) we have

$$2g(AX, \phi_1 X_1) + g(\phi_1 X_1, X) g(A\phi_1 X_1, \phi_1 X_1) = 0. \tag{4.2}$$

From this, putting $X = \phi_1 X_1$, we have

$$g(A\phi_1 X_1, \phi_1 X_1) = 0.$$

From this and (4.2) we have for any X in $\mathcal{D}$

$$2g(AX, \phi_1 X_1) = 0.$$

Thus it can be written

$$A\phi_1 X_1 \in \mathcal{D}^\perp.$$

From this and (4.1) it follows that for any X, Y in $\mathcal{D}$

$$g(AX, Y) = 0.$$

From this we know $AX \in \mathcal{D}^\perp$ for any $X \in \mathcal{D}$. That is $g(A\mathcal{D}, \mathcal{D}) = 0$ on M_0. Accordingly, the distribution $\mathcal{D}$ is integrable on M_0.

In fact for any $X, Y \in \mathcal{D}$ we have $[X, Y] = \nabla_X Y - \nabla_Y X \in \mathcal{D}$, because

$$g(\nabla_X Y, U_i) = -g(Y, \nabla_X U_i) = -g(Y, -p_j(X) U_k + p_k(X) U_j + \phi_i AX) = 0.$$

Now we want to show that M_0 coincides with M. Thus let us suppose that the interior of $M - M_0$ is not empty. Then on this open subset $g(A\mathcal{D}, \mathcal{D}^{\perp}) = 0$. So by a theorem of Berndt [1] this open set is congruent to an open part of one of real hypersurfaces of type A_1, A_2 and B. From this we know that all of its principal curvatures are constant on $Int(M - M_0)$. Thus by the continuity of principal curvatures $Int(M - M_0)$ must be closed and open. Since we have assumed the set M_0 is not empty and M is connected, $Int(M - M_0)$ must be empty and therefore by the continuity of principal curvatures again we can conclude that M_0 coincides with M. Accordingly, the distribution $\mathcal{D}$ is integrable on M.

Moreover, any integral manifold of $\mathcal{D}$ is totally geodesic in QP^m. In fact, for any $X, Y \in \mathcal{D}$ we write

$$D_X Y = \nabla'_X Y + \Sigma_i \sigma_i(X, Y) U_i + \rho(X, Y) N,$$

where D and ∇' denote the connection of QP^m and the induced connection from ∇ defined on an integral manifold of the distribution $\mathcal{D}$ respectively.

For this if we take the inner product with U_i, we have

$$g(D_X Y, U_i) = g(\nabla_X Y, U_i) = -g(Y, \phi_i AX) = 0.$$

This means $\sigma_i(X, Y) = 0$. Also taking the inner product with the unit normal N, we have $\rho(X, Y) = 0$. Moreover, it can be easily verified that $\mathcal{D}$ is J_i-invariant,$i = 1, 2,$ and 3, and its integral manifold is a quaternionic manifold and therefore quaternionic hyperplane QP^{m-1} of QP^m. Thus M is locally congruent to a ruled real hypersurface. From this we complete the proof of our theorem. $\square$

References

[1] J. Berndt, *Real hypersurfaces in quaternionic space forms*, J. Reine Angew. Math. **419** (1991), 9–26.

[2] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc. **269** (1982), 481–499.

[3] S. Ishihara, *Quaternion Kaehlerian manifolds*, J. Diff. Geom. **9** (1974), 483–500.

[4] U-H. Ki and Y.J. Suh, *On a characterization of real hypersurfaces of type A in a complex space form*, Canadian Math. Bull. **37** (1994), 238–244.

[5] A. Martinez, *Ruled real hypersurfaces in quaternionic projective space*, Anal. Sti. Univ. Al I Cuza **34** (1988), 73–78.

[6] A. Martinez and J.D. Pérez, *Real hypersurfaces in quaternionic projective space*, Ann. Math. Pura Appl. **145** (1986), 355–384.

[7] J.S. Pak, *Real hypersurfaces in quaternionic Kaehlerian manifolds with constant Q-sectional curvature*, Kodai Math. Sem. Rep. **29** (1977), 22–61.

[8] J.D. Pérez, *Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} A = 0$*, J. Geom. **49** (1994), 166–177.

[9] J.D. Pérez, *A characterization of real hypersurfaces of quaternionic projective space*, Tsukuba J. Math. **15** (1991), 315–323.

[10] J.D. Pérez and Y.J. Suh, *Real hypersurfaces of quaternionic projective space satisfying $\nabla_{U_i} R = 0$*, to appear in Differential Geom. and its Appl.

[11] R. Takagi, *On homogeneous real hypersurfaces of a complex projective space*, Osaka J. Math. **10** (1973), 495–506.

JUAN DE DIOS PÉREZ
DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA,
FACULTAD DE CIENCIAS,
UNIVERSIDAD DE GRANADA,
18071-GRANADA, SPAIN
E-mail: jdperez@goliat.ugr.es

YOUNG JIN SUH
DEPARTMENT OF MATHEMATICS,
KYUNGPOOK NATIONAL UNIVERSITY,
TAEGU 702-701, KOREA
E-mail: yjsuh@bh.kyungpook.ac.kr

Received April 3, 1996 Revised July 12, 1996