Math. J. Okayama Univ. 32 (1990), 207–221

# ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM

U-Hang Ki* and Young Jin Suh

**Introduction.** A complex $n$-dimensional Kaehler manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. Let $F$ be its complex structure. The complete and simply connected complex space form consists of a complex projective space $CP^n$, a complex Euclidean space $C^n$ or a complex hyperbolic space $CH^n$, according as $c > 0$, $c = 0$ or $c < 0$.

In the study of real hypersurfaces of a complex projective space $CP^n$, Takagi [11] classified all homogeneous real hypersurfaces of $CP^n$. He showed also that real hypersurfaces of $CP^n$ with 2 or 3 distinct constant principal curvatures are homogeneous.

On the other hand, Cecil and Ryan [2] studied pseudo-Einstein real hypersurfaces of $CP^n$ on which $\xi = -FC$ is principal, where $C$ is the unit normal vector field on $M$. They showed that if $\xi$ is principal, then $M$ lies on a tube over a Kaehler submanifold. By making use of this notion and the results of Takagi's classification, Kimura [3] proved the following.

**Theorem A.** *Let $M$ be a connected real hypersurface of $CP^n$. Then $M$ has constant principal curvatures and $\xi$ is principal if and only if $M$ is locally congruent to one of the following*

($A_1$) *a tube over a hyperplane $CP^{n-1}$.*

($A_2$) *a tube over a totally geodesic $CP^k (1 \leqq k \leqq n-2)$.*

(B) *a tube over a complex quadric $Q_{n-1}$.*

(C) *a tube over $CP^1 \times CP^{(n-1)/2}$ and $n (\geqq 5)$ is odd.*

(D) *a tube over a complex Grassmann $G_{2,5}(C)$ and $n = 9$.*

(E) *a tube over a Hermitian symmetric space $SO(10)/U(5)$, and $n = 15$.*

According to Takagi's classification [11], the principal curvatures and their multiplicities of the above homogeneous real hypersurfaces are given.

On the other hand, real hypersurfaces of a complex hyperbolic space $CH^n$ have also been investigated by Berndt [1], Montiel [8], Montiel and

*This research was partially supported by KOSEF.

Romero [9]. In particular, by using the notion of the tube in Cecil and Ryan [2], Montiel [8] also classified the real hypersurface of complex hyperbolic space with at most two distinct principal curvatures. Recently, Berndt [1] classified all real hypersurfaces with constant principal curvature of $CH^n$ under the condition such that $\xi$ is principal. Namely he proved the following.

**Theorem B.** *Let $M$ be a connected real hypersurface of $CH^n (n \geqq 2)$. Then $M$ has constant principal curvatures and $\xi$ is principal if and only if $M$ is locally congruent to one of the following*

($A_1$) *a horosphere in $CH^n$.*

($A_2$) *a tube over $CH^k$ for a $k = 0, 1, \ldots, n-1$.*

(B) *a tube over $RH^n$.*

For the principal curvatures and their multiplicities of the above hypersurfaces are also given in [1].

The purpose of this paper is to characterize some real hypersurfaces of $M_n(c)$, $c \neq 0$, by using above classification theorems. The authors would like to express their thanks to the referee for his valuable comments.

**1. Preliminaries.** Let $M$ be a real hypersurface of a complex $n$ dimensional complex space form $M_n(c)$, and let $C$ be a unit normal vector field on a neighborhood of a point $x$ in $M$. Let us denote by $F$ the almost complex structure of $M_n(c)$. For any local vector field $X$ on a neighborhood of $x$ in $M$, the transformations of $X$ and $C$ under $F$ can be given by

$$FX = \phi X + \eta(X)C, \; FC = -\xi,$$

where $\phi$ defines a skew-symmetric transformation on the tangent bundle $TM$ of $M$, while $\eta$ and $\xi$ denote a 1-form and a vector field on a neighborhood of $X$ in $M$ respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on $M$. The set of tensors $(\phi, \xi, \eta, g)$ is called an almost contact structure on $M$. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \; \phi\xi = 0, \; \eta(\phi X) = 0, \; \eta(\xi) = 1, \tag{1.1}$$

where $I$ denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$(\nabla_X \phi) Y = \eta(Y)AX - g(AX, Y)\xi, \; \nabla_X \xi = \phi AX, \tag{1.2}$$

where $\nabla$ is the Riemannian connection of $g$ and $A$ denotes the shape operator with respect to the unit normal $C$ on $M$.