TSUKUBA J. MATH.
Vol. 14 No. 1 (1990), 27—37

ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM WITH η-PARALLEL RICCI TENSOR

By

Young Jin SUH*

Introduction.

Let $M_n(c)$ denote an n-dimensional complex space form with constant holomorphic sectional curvature c. It is well known that a complete and simply connected complex space form consists of a complex projective space $\boldsymbol{CP}^n$, a complex Euclidean space $\boldsymbol{C}^n$ or a complex hyperbolic space $\boldsymbol{CH}^n$, according as $c>0$, $c=0$ or $c<0$. In this paper we consider a real hypersurface M of $\boldsymbol{CP}^n$ or $\boldsymbol{CH}^n$.

The study of real hypersurfaces of $\boldsymbol{CP}^n$ was initiated by Takagi [10], who proved that all homogeneous hypersurfaces of $\boldsymbol{CP}^n$ could be divided into six types which are said to be of type A_1, A_2, B, C, D and E. Moreover, he showed that if a real hypersurface M of $\boldsymbol{CP}^n$ has two or three distinct constant principal curvatures, then M is locally congruent to one of the homogeneous ones of type A_1, A_2 and B ([11]). Recently, to give another charac terization of homogeneous hypersurfaces of type A_1, A_2 and B in $\boldsymbol{CP}^n$ Kimura and Maeda [6] introduced the notion of an η-parallel second fundamental form, which was defined by $g((\nabla_X A)Y, Z)=0$ for any vector fields X, Y and Z orthogonal to the structure vector field ξ, where A means the second fundamental form of M in $\boldsymbol{CP}^n$, and g and ∇ denote the induced Riemannian metric and the induced Riemannian connection, respectively.

On the other hand, real hypersurfaces of $\boldsymbol{CH}^n$ have also been investigated by many authors (Berndt [1], Montiel [8], Montiel and Romero [9]).

Using some results about focal sets, Berndt [1] proved the following.

THEOREM A. *Let M be a connected real hypersurface of $CH^n (n\geqq 2)$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following.*
(A_0) *a horosphere in CH^n.*

Received September 16, 1988. Revised January 24, 1989.
*) Partially supported by TGRC-KOSEF.

(A_1) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane* CH^{n-1}.

(A_2) *a tube over a totally geodesic submanifold* CH^k *for* $k=1, \cdots, n-2$.

(B) *a tube over a totally real hyperbolic space* RH^n.

It is necessary to remark that real hypersurfaces of type A_0 or A_1 appearing in Theorem A, are *totally* η*-umblical* hypersurfaces with two distinct constant principal curvatures. In the paper of Montiel [7] the real hypersurface of type A_0 in Theorem A is said to be self-tube.

In §3 we also consider the η*-parallel* second fundamental form in $\boldsymbol{CH}^n$ and give a further characterization of type A_0, A_1, A_2, and B in $\boldsymbol{CH}^n$. Now we introduce the notion of an η*-parallel* Ricci-tensor of M in $M_n(c)$, $c\neq 0$, which is defined by $g((\nabla_X S)Y, Z)=0$ for any X, Y, and Z orthogonal to ξ, where S is the Ricci-tensor of M in $M_n(c)$, $c\neq 0$. It is easily seen that if the second fundamental form is η*-parallel*, then so is the Ricci-tensor, under the condition such that ξ is principal. Thus the purpose of this paper is to investigate this converse problem. By using the classification theorem due to Takagi [10] and Kimura and Maeda [6], we get the following.

THEOREM B. *Let* M *be a real hypersurface of* $\boldsymbol{CP}^n$. *Then the Ricci-tensor of* M *is* η*-parallel and* ξ *is principal if and only if* M *is locally congruent to one of homogeneous real hypersurfaces of type* A_1, A_2 *and* B.

By applying the Theorem A we can also prove the following.

THEOREM C. *Let* M *be a real hypersurface of* $\boldsymbol{CH}^n (n\geqq 2)$. *Then the Ricci-tensor of* M *is* η*-parallel and* ξ *is principal if and only if* M *is locally congruent to one of type* A_0, A_1, A_2 *and* B.

§1. Preliminaries.

Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$, and let C be its unit normal vector field. Since $M_n(c)$ admits an almost complex structure, let us denote by F its almost complex structure. For any tangent vector field X and normal vector field C on M, the transformations of X and C under F can be given by

$$FX=\phi X+\eta(X)C, \qquad FC=-\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of

M, while η and ξ denote a 1-form and a vector field on a neighborhood of x in M, respectively. In which it is seen that $g(\xi, X)=\eta(X)$, where g denotes the induced Riemannian metric on M. By the properties of the almost complex structure F, they satisfy the following

$$(1.1) \qquad \phi^2=-I+\eta\otimes\xi, \qquad \phi\xi=0, \qquad \eta(\phi X)=0, \qquad \eta(\xi)=1,$$

where I denotes the identity transformation. The set of tensors (ϕ, ξ, η, g) is called an almost contact structure on M.

Furthermore, the covariant derivatives of the structure tensors are given by

$$(1.2) \qquad (\nabla_X\phi)Y=\eta(Y)AX-g(AX, Y)\xi, \qquad \nabla_X\xi=\phi AX,$$

where ∇ is the induced Riemannian connection of g and A denotes the shape operator with respect to C on M.

Since the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equation of Gauss and Codazzi are respectively given as follows:

$$(1.3) \qquad R(X, Y)Z$$
$$=c\{g(Y, Z)X-g(X, Z)Y+g(\phi Y, Z)\phi X-g(\phi X, Z)\phi Y-2g(\phi X, Y)\phi Z\}/4$$
$$+g(AY, Z)AX-g(AX, Z)AY,$$

$$(1.4) \qquad (\nabla_X A)Y-(\nabla_Y A)X=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

The Ricci-tensor S' of M is the tensor of type (0, 2) given by $S'(X, Y)=tr\{Z\to R(Z, X)Y\}$. Also it may be regarded as the tensor of type (1, 1) and denoted by $S: TM\to TM$; it satisfies $S'(X, Y)=g(SX, Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$(1.5) \qquad S=c\{(2n+1)I-3\eta\otimes\xi\}/4+hA-A^2,$$

where we have put $h=Tr\,A$. The covariant derivative of (1.5) are given as follows

$$(1.6) \qquad (\nabla_X S)Y=\frac{c}{4}\Big\{-3(\nabla_X\eta)(Y)\xi-3\eta(Y)\nabla_X\xi\Big\}+(Xh)AY+h(\nabla_X A)Y-(\nabla_X A^2)Y.$$

The Ricci-tensor on the real hypersurface of $M_n(c)$, $c\neq 0$, is said to be η-*parallel* if it satisfies $g((\nabla_X S)Y, Z)=0$ for any tangent vector fields X, Y, and Z in $\xi^\perp$. In the sequel, assume that the hypersurface M is with η-*parallel* Ricci-tensor. Thus for any X, Y, and Z in $\xi^\perp$, (1.6) gives

$$(1.7) \qquad g((\nabla_X S)Y, Z)=(Xh)g(AY, Z)+hg((\nabla_X A)Y, Z)-g((\nabla_X A^2)Y, Z)=0.$$

It follows from (1.7) that if ξ is principal and if the second fundamental form is *η-parallel*, then the Ricci-tensor is *η-parallel.*

§ 2. Certain lemmas.

Let M be a real hypersurface of a complex space form $M_n(c)$, $c\neq 0$. The shape operator A of M can be considered as a symmetric $(2n-1, 2n-1)$-matrix. Now we suppose that the structure vector ξ is a principal curvature vector of A, that is, $A\xi=\alpha\xi$, where α is the principal curvature corresponding to ξ.

Then the covariant derivative gives

$$(\nabla_X A)\xi=(X\alpha)\xi+\alpha\phi AX-A\phi AX,$$

where we have used the second formular of (1.2). Thus it follows that

$$g((\nabla_X A)Y, \xi)=(X\alpha)\eta(Y)+\alpha g(Y, \phi AX)-g(Y, A\phi AX), \tag{2.1}$$

for any tangent vector fields X, and Y on M. By using the equation of Codazzi to (2.1) and using the fact $X\alpha=(\xi\alpha)\eta(X)$, we have

$$2A\phi AX-c\phi X/2=\alpha(\phi A+A\phi)X. \tag{2.2}$$

We now introduce the following fact without proof.

LEMMA 2.1. ([3]) *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. If ξ is a principal curvature vector of A, then its principal curvature α is locally constant.*

REMARK. Maeda [7] proved that α is constant for the real hypersurface of $\boldsymbol{CP}^n$.

Since $\boldsymbol{CP}^n$ has constant holomorphic sectional curvature $c=4$, (2.2) gives the following.

LEMMA 2.2. ([7]) *Let M be a real hypersurface of CP^n. Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX=\lambda X$ for any X in $\xi^{\perp}$, then $A\phi X=((\alpha\lambda+2)/(2\lambda-\alpha))\phi X$.*

§ 3. Real hypersurfaces of CH^n with η-parallel second fundamental form.

It is well known that the complex hyperbolic space $\boldsymbol{CH}^n$ admits the Bergmann metric normalized so that the constant holomorphic sectional curvature c is -4.

Thus (2.2) gives the following equation for the real hypersurface of $\boldsymbol{CH}^n$ whose structure vector field ξ is principal.

(3.1) $$2A\phi AX+2\phi X=\alpha(\phi A+A\phi)X$$

for any tangent vector field X in M. It follows that if $AX=\lambda X$ for any X in $\xi^{\perp}$, then

(3.2) $$(2\lambda-\alpha)A\phi X=(\alpha\lambda-2)\phi X.$$

Now we need the following lemmas which will be used in the later.

LEMMA 3.1. (Montiel and Romero [9]) *Let M be a real hypersurface of CH^n. Then*

(3.3) *$A\phi=\phi A$ holds on M if and only if M is of type A_0, A_1 or A_2.*

LEMMA 3.2. *Let M be a real hypersurface of CH^n. Then*

(3.4) *$A\phi+\phi A=k\phi$ ($k\neq 0$: constant) holds on M if and only if M is of type A_0, A_1 or B.*

PROOF. From (3.4) we have that $A\xi=\alpha\xi$, that is, ξ is the principal curvature vector. If $AX=\lambda X$ for any X in $\xi^{\perp}$, then $A\phi X=(k-\lambda)\phi X$.

By Lemma 2.1 α is constant. Thus we can consider the following two cases: $\alpha^2-4\neq 0$ and $\alpha^2-4=0$.

For $\alpha^2-4\neq 0$ we then have $2\lambda-\alpha\neq 0$ by (3.2). Thus also from (3.2) it follows that $k-\lambda=(\alpha\lambda-2)/(2\lambda-\alpha)$. Hence it follows that $2\lambda^2-2k\lambda+\alpha k-2=0$. Since λ satisfies the above quadratic equation with constant coefficients, all principal curvatures are constant on M. Thus due to Theorem A, M is of type A_1, A_2 or B. Suppose that M is of type A_2. By Lemma 3.1 $A\phi=\phi A$ holds on M. This fact and (3.4) imply $2A\phi=k\phi$. Thus from the almost contact structure it follows that $A=aI+b\eta\otimes\xi$, that is, M *is totally η-umblical*. Then it is seen by Montiel and Romero [9] that M is of type A_0 or A_1, a contradicts. Thus the type of A_2 can not occur.

Now we consider for the case $\alpha^2-4=0$. Let $M_0=\{x\in M|(2\lambda-\alpha)_x\neq 0\}$. Then λ also satisfies $2\lambda^2-2k\lambda+\alpha k-2=0$. Thus λ is constant on M_0. On the other hand, we have $2\lambda-\alpha=0$ on $M-M_0$. Then (3.2) gives $\alpha\lambda=2$. Thus $\lambda=\pm 1$ on $M-M_0$.

The continuity of principal curvatures implies that if the set $M-M_0$ is not empty, then $\lambda=\pm 1$ on M. Hence M is of type A_0.

For the case where M_0 coincides with the whole M, it is of type A_1, A_2 or B and therefore it must be of type A_1 or B by the same argument as that of the above half, a contradiction.

Conversely, suppose that M is of type A_0, A_1 or B. It is seen by Montiel and Romero [9] that the type of A_0 and A_1 are the only totally η-umblical real hypersurfaces of $\boldsymbol{C}H^n$. Thus it naturally satisfies $A\phi+\phi A=k\phi$.

For the type of B we can take an orthonormal basis $\{X_1, \cdots, X_{n-1}, \phi X_1, \cdots, \phi X_{n-1}, \xi\}$ of $T_x(M)$ such that $AX_i=\coth\theta X_i$, $A\phi X_i=\tanh\theta\phi X_i$, $i=1, \cdots, n-1$, and $A\xi=2\tanh 2\theta\xi$. Then we have $A\phi X+\phi AX=(\tanh\theta+\coth\theta)\phi X$ for any X in $T_x(M)$. Thus we complete the above lemma.

LEMMA 3.3. *Let M be a real hypersurface of $M_n(c)$, $c\neq 0$. If the structure vector field ξ is principal and if the second fundamental form A satisfies the following quadratic formula:*

$$A^2+aA+cI=0 \ (a^2-4b\neq 0,\ a, b: \text{constant}) \text{ on } \xi^\perp, \tag{3.5}$$

then the second fundamental form A is η-parallel.

PROOF. By taking covariant derivative of (3.5), we get

$$g((\nabla_X A)AY, Z)+g(A(\nabla_X A)Y, Z)+ag((\nabla_X A)Y, Z)=0 \tag{3.6}$$

for any X, Y, and Z in $\xi^\perp$.

Taking the skew-symmetric part of (3.6) and using the equation of Codazzi, we have

$$g((\nabla_X A)AY, Z)=g((\nabla_Y A)AX, Z),$$

from which together with $g(AX, (\nabla_Z A)Y)=g((\nabla_Z A)AX, Y)=g((\nabla_X A)AZ, Y)$, we get

$$g((\nabla_X A)AY, Z)=g(A(\nabla_X A)Y, Z) \tag{3.7}$$

for any X, Y, and Z in $\xi^\perp$, where we have used the fact that $\xi^\perp$ is invariant under the transformation of A because ξ is the principal curvature vector.

Combining (3.6) and (3.7), we obtain for any X, Y, and Z in $\xi^\perp$

$$2g(A(\nabla_X A)Y, Z)+ag((\nabla_X A)Y, Z)=0\,. \tag{3.8}$$

Transforming (3.8) with A and using (3.5) again, we get

$$2bg((\nabla_X A)Y, Z)=-ag(A(\nabla_X A)Y, Z). \tag{3.9}$$

From which, substituting into (3.8), we have

$$g(A(\nabla_X A)Y, Z)=0\,, \tag{3.10}$$

where we have used the fact $a^2-4b\neq 0$. Thus (3.9) gives $g((\nabla_X A)Y, Z)=0$ for $b\neq 0$.

For the case where $b=0$, $a^2-4b\neq 0$ implies $a\neq 0$. From which together with (3.8) and (3.10) it follows that $g((\nabla_X A)Y, Z)=0$ for any X, Y and Z in $\xi^\perp$. Hence we get the above lemma.

These Lemmas 3.1, 3.2 and 3.3 and Theorem A enable us to prove the following.

THEOREM 3.4. *Let M be a real hypersurface of CH^n. Then the second fundamental form of M is η-parallel and the structure vector field ξ is principal if and only if M is locally congruent to one of type A_0, A_1, A_2 or B.*

PROOF. First we shall show that the second fundamental form of type A_0, A_1, A_2 or B is *η-parallel.*

Now let M be a of type A_0, A_1 or A_2. Then by Lemma 3.1 $A\phi=\phi A$ holds on M. Thus $\phi A\xi=0$ implies that ξ is principal, that is, $A\xi=\alpha\xi$. From which and (3.1) it follows that

$$A^2-\alpha A+I=0 \qquad \text{on } \xi^\perp.$$

Thus Lemma 3.3 gives that the second fundamental form is *η-parallel* for the case $\alpha^2-4\neq 0$. For the case where $\alpha^2=4$ all the principal curvatures λ are ± 1. Thus M is of type A_0 and totally η-umblical. Hence the second fundamental form is also *η-parallel* in this case.

Now we consider that M is of type B. Then by Lemma 3.2 $A\phi+\phi A=k\phi$ ($k\neq 0$: constant) holds on M. From which we also get $A\xi=\alpha\xi$. Thus from (3.1) it follows that

$$A^2-kA-(1-\alpha k/2)I=0 \qquad \text{on } \xi^\perp.$$

On the other hand, due to Berndt's classification [1] all the principal curvatures of type B are given as follows: $\lambda=\coth\theta$, $\mu=\tanh\theta$, $\alpha=2\tanh 2\theta$. Since $\lambda+\mu=2\coth 2\theta=4/\alpha$, $A\phi+\phi A=k\phi$ implies $k=4/\alpha$. Hence we conclude that $k^2+4(1-\alpha k/2)\neq 0$. Hence by Lemma 3.3 we also get our result.

Conversely, it suffices to show that all the principal curvatures are constant on M. If $AX=\lambda X$ for any X in $\xi^\perp$, then $g((\nabla_Y A)X, X)=(Y\lambda)g(X, X)$. Thus from the assumption we have that $Y\lambda=0$ for any Y in $\xi^\perp$.

On the other hand, using equation of Codazzi and making use of (2.1) and Lemma 2.1, we get the following.

$$\xi\lambda=g((\nabla_\xi A)X, X)=g((\nabla_X A)\xi, X)=0\,.$$

From these facts and Theorem A, we conclude that M is of type A_0, A_1, A_2, and B. This completes the above Theorem.

REMARK. Kimura and Maeda [6] showed that a real hypersurface of $\boldsymbol{CP}^n$ with η-*parallel* second fundamental form and principal structure vector field ξ is locally congruent to one of homogeneous real hypersurfaces of type A_1, A_2 and B.

§4. Real hypersurfaces of $M_n(c)$, $c\neq 0$, with η-parallel Ricci-tensor.

Let M be a real hypersurface of $M_n(c)$ with η-*parallel* Ricci-tensor, that is, $g((\nabla_X S)Y, Z)=0$ for any X, Y and Z in $\xi^{\perp}$. It is easily seen that if ξ is principal, then the second fundamental form A of M in $M_n(c)$ is η-*parallel* implies that the Ricci-tensor S is η-*parallel.* In this section we are investigated to study this converse problem by using Kimura and Maeda's [6] result and Theorem 3.4. Then we can state another characterization as the following.

THEOREM 4.1. *Let M be a real hypersurface of CP^n. Then the Ricci-tensor is η-parallel and the structure vector field ξ is principal if and only if M is of type A_1, A_2 and B.*

PROOF. For any X, Y in $\xi^{\perp}$, the fact that the Ricci-tensor is η-*parallel* implies

$$(\nabla_X S)Y=-3(\nabla_X\eta)(Y)\xi+(Xh)AY+h(\nabla_X A)Y-(\nabla_X A^2)Y$$

belongs to $[\xi]$, where $[\xi]$ means 1-dimensional vector space spanned by ξ. Thus $g((\nabla_X S)Y, Y)=0$ for any Y in $\xi^{\perp}$. Hence if we put $AY=\lambda Y$, then

$$\lambda(Xh)+h(X\lambda)-(X\lambda^2)=0 \qquad \text{for any } X \text{ in } \xi^{\perp}. \tag{4.1}$$

Also for any Y in $\xi^{\perp}$ such that $AY=\lambda Y$ we have $(\nabla_\xi A)Y=(\xi\lambda)Y+(\lambda I-A)\nabla_\xi Y$. Thus $\xi\lambda=g((\nabla_\xi A)Y, Y)=g((\nabla_Y A)\xi, Y)=0$. Hence the mean curvature h is also constant on ξ-direction. Together with this fact and (4.1), we conclude that $\lambda h-\lambda^2$ is constant on M. Thus we can put as the following.

(4.2) $\lambda h-\lambda^2=a$, (4.3) $\mu h-\mu^2=b$.

By Lemma 2.2, (4.2) and (4.3) can be rewritten as follows

$$\lambda^2-h\lambda+a=0, \tag{4.4}$$

$$(2h\alpha-\alpha^2-4b)\lambda^2-\{(\alpha^2-4)h+4\alpha-4b\alpha\}\lambda-(2\alpha h+b\alpha^2+4)=0. \tag{4.5}$$

Substituting $h\lambda=\lambda^2+a$ into (4.5), we then have

$$2\alpha\lambda^4-(2\alpha^2+4b-4)\lambda^3+2(a\alpha+2b\alpha-3\alpha)\lambda^2-(a\alpha^2-4a+b\alpha^2+4)\lambda-2a\alpha=0. \tag{4.6}$$

From which we see that λ satisfies an algebraic equation with constant coefficients. Thus M has at most five constant principal curvatures. According to Kimura's theorem [4], M is homogeneous.

On the other hand, due to Takagi's classification of homogeneous real hypersurface of $\boldsymbol{CP}^n$, we conclude that M is of type A_1, A_2, B, C, D and E.

In order to prove this theorem we shall show that the shape operator is *η-parallel*.

Let $A(\lambda)$ be an eigenspace of A with eigenvalue λ. Then the subspace $\xi_X^\perp$ of the tangent space $T_x(M)$ at x can be decomposed as $\xi_X^\perp = A(\lambda_1) \oplus A(\lambda_2) \oplus \cdots \oplus A(\lambda_s)$. Now in what follows we consider the following eigenvector such that $X \in A(\lambda)$, $Y \in A(\mu)$ and $Z \in A(\sigma)$, where λ, μ and σ are corresponding constant principal curvatures. Then we have that

$$g((\nabla_X A)Y, Z) = (\mu - \sigma) g(\nabla_X Y, Z) . \tag{4.7}$$

On the other hand, from the *η-parallel* Ricci-tensor it follows that

$$(h - \mu - \sigma) g((\nabla_X A)Y, Z) = 0 . \tag{4.8}$$

For the case where $\mu = \sigma$, (4.7) implies that $g((\nabla_X A)Y, Z) = 0$. Thus it suffices to show that the shape operator is *η-parallel* for the case where $\mu \neq \sigma$.

In the case where $h - \mu - \sigma \neq 0$, (4.8) gives our result. Thus it remains to consider for the case where $h - \mu - \sigma = 0$. Thus the *η-parallel* Ricci-tensor gives

$$g((\nabla_Y S)X, Z) = (h - \lambda - \sigma) g((\nabla_Y A)X, Z) = 0. \tag{4.9}$$

If $\lambda \neq \mu$, then $h - \mu - \sigma = 0$ implies $h - \lambda - \sigma \neq 0$. From which together with (4.9) it follows $g((\nabla_X A)Y, Z) = g((\nabla_Y A)X, Z) = 0$. If $\lambda = \mu$, (4.7) gives $g((\nabla_X A)Y, Z) = g((\nabla_Z A)X, Y) = 0$. Summing up, we conclude that the shape operator is *η-parallel*. Thus, due to Kimura and Maeda's Theorem [6], M is of type A_1, A_2 and B.

Conversely, if M is of type A_1, A_2 or B, then by Kimura and Maeda's Theorem [6] the second fundamental form is *η-parallel* and its structure vector field ξ is principal. Since *η-parallel* second fundamental form with the principal structure vector ξ implies *η-parallel* Ricci-tensor, we get the above Theorem.

REMARK. Kimura [5] showed that a real hypersurface of $\boldsymbol{CP}^n$ with the condition $(\nabla_X S)Y = c\{g(\phi AX, Y) + \eta(Y)\phi AX\}$, where c is constant, is locally congruent to homogeneous hypersurfaces with 2 or 3 distinct principal curvatures. Thus this condition implies that the Ricci-tensor S is *η-parallel* and structure vector field ξ is principal.

On the other hand, for a real hypersurface of CH^n we get the following.

THEOREM 4.2. *Let M be a real hypersurface of CH^n, $n\geqq 2$. Then the Rcci-tensor is η-parallel and the structure vector field ξ is principal if and only if M is of type A_0, A_1, A_2 or B.*

PROOF. The converse is trivial by Theorem 3.4.

Let M be a real hypersurface of CH^n with *η-parallel* Ricci-tensor and principal structure vector field ξ. Then similarly as in Theorem 4.1 we can put

$$\lambda h-\lambda^2=a\ , \tag{4.13}$$

$$\mu h-\mu^2=b\ . \tag{4.14}$$

By Lemma 2.1 we can consider the following two cases.

CASE I. $\alpha^2-4\neq 0$.

Then $2\lambda-\alpha\neq 0$. In fact, suppose that $2\lambda-\alpha=0$. Then (3.2) gives $\alpha\lambda=2$. Together with this fact we have $\alpha^2-4=0$, a contradiction. Thus from (3.2) it follows $A\phi X=\mu\phi X$, $\mu=(\alpha\lambda-2)/(2\lambda-\alpha)$. From which, substituting (4.14), then we get

$$(2\alpha h-\alpha^2-4b)\lambda^2+\{4\alpha+4b\alpha-(\alpha^2+4)h\}\lambda+(2\alpha h-4-b\alpha^2)=0\ . \tag{4.15}$$

Substituting (4.13) into (4.15), then λ satisfies the following equation with constant coefficients

$$2\alpha\lambda^4-2(\alpha^2+2b+2)\lambda^3+2\alpha(a+2b+3)\lambda^2-(a\alpha^2+b\alpha^2+4a+4)\lambda+2a\alpha=0.$$

In the case where $\alpha=0$, $a=-1$ and $b=-1$, coefficients of the above equation are all vanishing. Thus it suffices to prove that principal curvatures are also constant on M in this case.

For the case where $a=-1$, and $b=-1$ it follows from (4.13) and (4.14) that $\lambda=\mu$ or $h=\lambda+\mu$. Since $\mu=-1/\lambda$ for $\alpha=0$, $\lambda=\mu$ implies $\lambda^2+1=0$. This contradicts. Thus we have $h=\lambda+\mu$. From which together with $h=m_1\lambda+m_2(-1/\lambda)$ for $\alpha=0$, it follows that $(m_1-1)\lambda^2-(m_2-1)=0$. Since $m_1\neq 1$, principal curvatures are constant on M in this case. Hence all principal curvatures are constant on M. Thus due to Theorem A we conclude that M is of type A_1, A_2 or B.

CASE II. $\alpha^2=4$.

Now we consider for the case $\alpha=2$. Then (3.2) gives

$$(\lambda-1)A\phi X=(\lambda-1)\phi X.$$

Let us take an open set $M_0=\{x\in M \mid \lambda\neq 1\}$. Then $A\phi X=\phi X$. Thus $\mu=1$. From which and (4.14) it follows $h=b+1$ on M_0. Since $\lambda=1$ on $M-M_0$, also from (4.13) it follows $h=a+1$. Hence h is constant and $a=b$ on M. Thus λ satisfies a quadratic equation with constant coefficients: $\lambda^2-h\lambda+a=0$. Hence all principal curvatures are constant on M.

Similarly, for the case $\alpha=-2$ we also get the same conclusion. By virtue of Theorem A, M is of type A_0, A_1, A_2 or B. Since $\alpha=\pm 2$, then M is of type A_0.

References

[1] Berndt, J., Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. reine angew, Math. **395** (1989), 132-141.

[2] Cecil T.E. and Ryan., P.J., Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. **269** (1982), 481-499.

[3] Ki, U-H. and Suh., Y.J., On real hypersurfaces of a complex space form, to appear in Okayama Math. J.

[4] Kimura, M., Real hypersurfaces and complex submanifolds in a complex projective space, Trans. Amer. Math. Soc. **296** (1986), 137-149.

[5] Kimura, M., Real hypersurfaces of a complex projective space, Bull. Austral. Math. Soc. Vol. **33** (1986), 383-387.

[6] Kimura, M. and Maeda, S., On real hypersurfaces of a complex projective space, Math. Z. **202** (1989), 299-311.

[7] Maeda, Y., On real hypersurfaces of a complex projective space, J. Math. Soc. Japan **28** (1976), 529-540.

[8] Montiel, S., Real hypersurfaces of a complex hyperbolic spaces, J. Math. Soc. Japan **37** (1985), 515-435.

[9] Montiel, S. and Romero, A., On some real hypersurfaces of a complex hyperbolic space, Geometriae Dedicata, **20** (1986), 245-261.

[10] Takagi, R., On homogeneous real hypersurfaces of a complex projective space, Osaka J. Math. **10** (1973), 495-506.

[11] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures. I, II. J. Math. Soc. Japan **27** (1975), 43-53, 507-516.

[12] Yano, K. and Kon, M., *CR*-submanifolds of Kaehlerian and Sasakian manifold, Birkhauser 1983.

University of Tsukuba
Institute of Mathematics
Ibaraki, 305
Japan
and
Andong University
Department of Mathematics
Andong, 760-749
Korea