## ON RECONSTRUCTING A GRAPH

## ROBERT L. HEMMINGER

1. Introduction. The term "graph" will here denote an unoriented finite graph without loops or multiple edges. $V(G)$ will denote the vertex set of $G$ and $E(G)$ will denote the edge set. If $a \in V(G)$, we will let $G_{a}$ denote the graph obtained from $G$ by deleting the vertex $a$ and the edges adjacent to $a$. If $e \in E(G)$ we will let $G^{e}$ denote the graph obtained from $G$ by deleting $e$. P. J. Kelly [3] has proven the following theorem: If $G$ and $H$ are trees and $\sigma: V(G) \rightarrow V(H)$ is a 1-1 onto function such that $G_{a} \cong H_{\sigma(a)}$ for all $a \in V(G)$, then $G \cong H$. He conjectures that this theorem is true for arbitrary graphs and has verified it for graphs on $n$ vertices where $2<n \leqq 6$. An equivalent formulation of Kelly's conjecture is as follows: $G$ is uniquely determined, up to isomorphism, by the collection $\left\{G_{a}\right\}_{a \in V(G)}$. We will refer to this as the vertex problem. If a graph $G$ is uniquely determined, up to isomorphism, by a given collection of subgraphs we will say that $G$ can be reconstructed from that collection of subgraphs. It needs to be emphasized that the given subgraphs have no labellings.

Harary and Palmer [1] generalized Kelly's theorem on trees by showing that a tree $G$ can be reconstructed from the $G_{a}$ with $a$ of degree one in $G$.

In [2], Harary asks if $G$ can be reconstructed from the collection $\left\{G^{e}\right\}_{e \in E(G)}$. We will refer to this as the edge problem. The purpose of this paper is to show that the edge problem is a special case of the vertex problem.

Undefined terms in the paper can be found in the above-mentioned papers or in [4].
2. The use of the line graph. If $G$ is a graph, then the line graph of $G$, denoted by $L(G)$, is the graph with $V(L(G))=E(G)$ and with $\left(e_{1}, e_{2}\right) \in E(L(G))$ if and only if $e_{1}$ and $e_{2}$ are adjacent in $G$.

Lemma. Let $G$ be a given graph. Then $L\left(G^{e}\right)=(L(G))_{e}$ for all $e \in E(G)$.
Proof. Both graphs have $E(G)-\{e\}$ as vertex set, and if $e_{1}$, $e_{2} \in E(G)-\{e\}$, then the criterion for $\left(e_{1}, e_{2}\right)$ to be an edge in either graph is the same; namely that $e_{1}$ and $e_{2}$ be adjacent in $G$.

Since the number of isolated vertices in $G$ can be discovered from the $\left\{G^{l}\right\}_{e \in E(G)}$ we assume in the following that $G$ and $H$ have no isolated vertices.

[^0]Theorem. The edge problem is equivalent to the vertex problem for line graphs; i.e., a solution to the edge problem would give a solution to the vertex problem for line graphs and conversely.

Proof. Suppose the vertex problem is true for line graphs. Let $G$ and $H$ be graphs and let $\tau: E(G) \rightarrow E(H)$ be a 1-1 onto function such that $G^{e} \cong H^{\tau(e)}$ for all $e \in E(G)$. By the Lemma we then have $(L(G))_{e}$ $=L\left(G^{e}\right) \cong L\left(H^{\tau(e)}\right)=(L(H))_{\tau(e)}$ for all $e \in E(G)$. But then $\tau: V(L(G))$ $\rightarrow V(L(H))$ is a 1-1 onto function such that $(L(G))_{e} \cong(L(H))_{\tau(e)}$ for all $e \in V(L(G))$ so by our assumption $L(G) \cong L(H)$. Now $G$ and $L(G)$ have the same number of components so $G$ and $H$ have the same number of components and by Whitney's Theorem [5], or see pp. 248 of [4] $G$ and $H$ have the same number of components of each isomorphism type with the possible exception of triangles and 3-pointed stars.

If for each $e \in E(G), e$ is from a triangle component of $G$ if and only if $\tau(e)$ is from a triangle component of $H$, then $G \cong H$ since they would have the same number of triangle components. If there is some $e \in E_{-}(G)$ such that $e$ is from a triangle component but $\tau(e)$ is not then $\tau(e)$ must be from a 3-pointed star component of $H$. But then $G^{e} \nsubseteq H^{\tau(e)}$ since the latter has one more component than the former. (Removing $\tau(e)$ from the star leaves a path of length two and an isolated vertex.) One gets the same contradiction if $e$ is not from a triangle component while $\tau(e)$ is.

The proof that the vertex problem for line graphs is valid if the edge problem is valid is omitted because of its similarity to the above proof.

Corollary. If $G$ is disconnected then $G$ can be reconstructed from the collection $\left\{G^{e}\right\}_{e \in E(G)}$.

Proof. $L(G)$ can be constructed from the collection $(L(G))_{e}$ by [2] since $L(G)$ is disconnected.

It should be pointed out that one can decide from the $G^{e}$ if $G$ is connected or not. This follows from the observation that $G$ is connected if and only if either $G^{e}$ is connected for some $e \in E(G), G^{e}$ is a forest with exactly two trees for all $e \in E(G)$ and for some $e \in E(G)$ neither component of $G^{e}$ is an isolated vertex, or else $G^{e}$ is a star plus an isolated vertex for each $e \in E(G)$.

## References

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Vanderbilt University


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