ON RECURRENCE IN ZERO DIMENSIONAL FLOWS

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Abstract. For an action of a finitely generated group G on a compact space X we define recurrence at a point and then show that, when X is zero dimensional, the conditions (i) pointwise recurrence, (ii) X is a union of minimal sets and (iii) the orbit closure relation is closed in $X \times X$, are equivalent. As a corollary we get that for such flows distality is the same as equicontinuity. In the last part of the paper we describe an example of a $\mathbb{Z}$-flow where all points are positively recurrent, but there are points which are not negatively recurrent.

Introduction

Let X be a compact metric space and let $T : X \to X$ be a homeomorphism. Consider the following four conditions:

(1) Every point of X is positively recurrent.
(2) Every point of X is positively and negatively recurrent.
(3) The system (X, T) is pointwise almost periodic. (Equivalently, X is a union of minimal sets.)
(4) The orbit closure relation is closed. (The orbit closure relation R is defined by $(x, y) \in R$ if $y \in \overline{O(x)}$.)

Obviously, (3) $\implies$ (2) $\implies$ (1). It is not difficult to show that (4) $\implies$ (3), and in fact this will be shown below for general group actions. Moreover no converse is valid. The almost equicontinuous (but not equicontinuous) systems have all points positively and negatively recurrent but (X, T) is not pointwise almost periodic. Such examples were first constructed by Katznelson and Weiss [6], and studied by Glasner and Weiss [3] and Akin, Auslander, and Berg [1]. An example of a $\mathbb{Z}$-flow where all points are positively recurrent, but there are points which are not negatively recurrent is described in section 2.

As for the relation between (3) and (4), let (X, T) be a distal flow. By a classical theorem of Ellis [2], the product system $(X \times X, T)$ is pointwise almost periodic. It is not difficult to see that the orbit closure relation in $X \times X$ is closed if and only if the flow (X, T) is equicontinuous. Hence if the distal system (X, T) is not equicontinuous, then the product system $(X \times X, T)$ provides an example which is pointwise almost periodic but the orbit closure relation is not closed.

Now suppose that the space X is zero dimensional. Then all four conditions are equivalent. (In particular, it follows that a distal integer action on a zero dimensional space is equicontinuous.) The proofs are rather easy, and in any case (except for (1) $\implies$ (2)) they will be subsumed by more general proofs given below. For completeness we show that (1) $\implies$ (2). Suppose $x \in X$ is not negatively recurrent. Then there is an open closed ("clopen") neighborhood V of x such that $T^{-j}(x) \notin V$

2000 Mathematical Subject Classification 54H20.

for all $j > 0$. We show that for every $m > 0$ there is a $y_m \in V$ such that $T^j(y_m) \notin V$ for $1 \le j \le m$. Then if (a subsequence of) $y_m \to y$ we have $y \in V$ but $T^j(y) \notin V$ for all $j > 0$ so Y is not positively recurrent. In fact, we can find such y_m in the positive semi orbit of x. Let $m > 0$, and let $k > 3m$ such that $T^k(x)$ is so close to x so that $T^{k-i}(x) \notin V$ for $1 \le i \le m$. For such i, $k - i > 0$, so these $T^{k-i}(x)$ are in the positive semi orbit of x. Just choose y_m to be the last point in the positive semi orbit of x which is in V before a string of m iterates of y_m which are not in V.

Now consider actions of more general discrete countable groups. A flow (X, G) is a continuous action of the group G on X ($(g, x) \mapsto gx$). The notion of an almost periodic point is easy to define even in this general setup. Recall that a subset $S \subset G$ is called *(left) syndetic* if there is a finite set $F \subset G$ with $FS = G$. For a point $x \in X$ and a neighborhood U of x let $N(x, U) = \{g \in G : gx \in U\}$. We say that x is an *almost periodic point* if $N(x, U)$ is syndetic for every neighborhood U of x. It is well known, and not hard to see, that x is almost periodic iff its orbit closure in X is a minimal subset (see [4]).

We can therefore ask: for which acting groups conditions (3) and (4) are equivalent in zero dimensional flows? Now without some restriction on the group the equivalence does not hold. That is, there is an example (due to McMahon and Wu [7]) of a countable group G acting on a zero dimensional space X such that (X, G) is distal and not equicontinuous.

In [4] Gottschalk and Hedlund formulate a definition of recurrence in terms of *replete semigroups*. With this notion of recurrence they show that for a commutative compactly generated topological group G (2) and (4) are equivalent in zero dimensional G-flows, ([4, Theorem 7.07]). According to [4] a subset S of a commutative topological group T is replete in T if S contains some bilateral translate of each compact subset of T. For non commutative groups the existence of replete semigroups seems to be quite rare (even if the requirement is, say, to contain just a right translate of each compact set), and the definition of recurrence in these terms appears to be inadequate. The purpose of the present note is to show that with an appropriate more restrictive definition of recurrence, a satisfactory analogue of the equivalence of the conditions (2), (3), and (4) holds for the actions of a general finitely generated group on zero dimensional metric compact spaces. We note that for commutative finitely generated abelian groups our definition of recurrence coincides with that of Gottschalk and Hedlund.

The authors would like to thank Ethan Akin for a careful reading of the paper and for suggesting several improvements which made it more readable.

1. The main results

First we attend to the matter of the relation between the orbit closure relation being closed and pointwise almost periodicity. If (X, G) is a flow, the *orbit closure relation* R is defined by $(x, y) \in R$ if $y \in \overline{Gx}$. R is reflexive and transitive, and is symmetric if and only if (X, G) is pointwise almost periodic.

1.1. **Proposition.** *Let (X, G) be a flow, and suppose the orbit closure relation R is closed. Then (X, G) is pointwise almost periodic.*

Proof. Suppose (X, G) is not pointwise almost periodic. Let $x \in X$ be a non almost periodic point, and let y be an almost periodic point with $y \in \overline{Gx}$. Then $x \notin \overline{Gy}$ so $(y, x) \notin R$. Let $g_n \in G$ such that $g_n x \to y$. Then $(g_n x, x) \in R$, and $(g_n x, x) \to (y, x)$ so R is not closed. □

Let G be a finitely generated group, and let Γ be a finite subset of generators. We assume that $\Gamma^{-1} = \Gamma$. For $e \neq g \in G$ let $|g|$, the *length of* g, be the smallest integer r such that $g = \gamma_1\gamma_2 \cdots \gamma_r$ with $\gamma_i \in \Gamma$. Let B_r denote the set of elements of G of length $\leq r$. Let $\{K_n\}$ be a sequence of subsets of G. We say that a subset C of G is the *limit of the* K_n and write $C = \lim_{n\to\infty} K_n$ if, for each $r > 0$ there exists n_r such that $K_n \cap B_r$ is independent of n for $n > n_r$, and C is defined by $\mathbf{1}_C(g) = \lim_{n\to\infty} \mathbf{1}_{K_n}(g)$.

1.2. **Proposition.** *Any sequence of sets $\{K_n\}$ in a finitely generated group G has a convergent subsequence.*

Proof. Each B_r is finite, so we may choose a subsequence (and relabel) so that each $K_n \cap B_r$ is eventually constant. The desired subsequence can then be obtained by a diagonal process. □

For $g \in G$ let $K(g) = B_{|g|-1} \cdot g$.

1.3. **Definition.** A subset C of G is a *cone* if there is a sequence $g_n \in G$ with $|g_n| \to \infty$ and $C = \lim_{n\to\infty} K(g_n)$.

1.4. **Examples.**
- If $G = \mathbb{Z}$, a cone is either $\mathbb{N}$ or $-\mathbb{N}$.
- More generally for $G = \mathbb{Z}^d$ every cone contains a subset of the form $Q \pm e_j$, where Q is an orthant and $e_j = (0, \dots, 1, \dots, 0)$ with 1 at the j-th coordinate.
- For $G = F_d$, the free group on $d \geq 2$ generators, if $C = \lim_{n\to\infty} K_n$ is a cone with $K_n = B_{m_n} d_n$, then we can assume that the sequence d_n converges to a point ξ in the boundary and then $C = Q_\xi \cdot \gamma$ is the "horoball" Q_ξ at ξ translated by one of the generators $\gamma \in \Gamma$.

The next proposition says that a cone is always *replete* (i.e. it contains a right translate of every finite set) and that it contains a "capturing" sequence.

1.5. **Proposition.** *If C is a cone then there exists a sequence $\{c_n\}$ in C such that $B_n c_n \subset C$. Thus for each $g \in G$, gc_n is eventually in C.*

Proof. Suppose $C = \lim_{k\to\infty} K(g_k)$. Given n set $r = 2n + 1$, then choose k so that $|g_k| > n + 1$ and so that $K(g_k) \cap B_r = C \cap B_r$. Write $g_k = hc$, where $|c| = n + 1$ and $|c| + |h| = |g_k|$, so that $|h| = |g_k| - (n + 1)$. Now set $c_n = c$. If $x \in B_n$ then $xc \in B_r$ and $xc = xh^{-1}hc = xh^{-1}g_k \in K(g_k)$ since $|xh^{-1}| \leq |x| + |h^{-1}| = |x| + |h| \leq n + |g_k| - (n+1) = |g_k| - 1$. Therefore $xc \in K(g_k) \cap B_r = C \cap B_r \subseteq C$ as required. □

We can now define our notion of recurrence.

1.6. **Definition.** Let (X, G) be a flow and let C be a subset of G such that $e \notin C$. We say that a point $x \in X$ is *C-recurrent*, if, for every neighborhood U of x, $Cx \cap U \neq \emptyset$. We say that x is *recurrent*, if it is C-recurrent for every cone C.

1.7. **Proposition.** *Let (X, G) be a flow, with G a finitely generated group. Suppose $x \in X$ is an almost periodic point. Then x is recurrent.*

Proof. Let U be a neighborhood of x, and let C be a cone in G. By assumption there exists a finite set $F \subset G$ such that $FN(x,U) = G$, hence we have $F^{-1}g \cap N(x,U) \neq \emptyset$ for every $g \in G$. Choose an n_0 such that $F \subset B_{n_0}$. Let $\{c_n\}$ be the sequence constructed in the previous proposition. Then $F^{-1}c_{n_0} \subset B_{n_0}c_{n_0} \subset C$ and we conclude that $C \cap N(x,U) \neq \emptyset$, as required. □

1.8**. Theorem.** *Let (X,G) be a flow, with X a zero dimensional compact metric space, and G a finitely generated group. Then the following are equivalent.*

(*i*) *(X,G) is pointwise recurrent.*
(*ii*) *(X,G) is pointwise almost periodic.*
(*iii*) *The orbit closure relation is closed.*

Proof. The implications $(ii) \implies (i)$ and $(iii) \implies (ii)$ always hold, by the propositions above, so we need only prove $(i) \implies (iii)$. Suppose the orbit closure relation R is not closed. Then there are $(x_n, y_n) \in R$ such that $(x_n, y_n) \to (x, y)$ with $(x,y) \notin R$. It follows that there is a clopen set U containing $\overline{Gx}$ and a sequence $\{x_n\}$ in U with $x_n \to x$ but $\overline{Gx_n} \not\subset U$ (so $Gx_n \not\subset U$). Let m_n be defined by $B_k x_n \subset U$ for $k \le m_n$ but $B_{m_n+1}x_n \not\subset U$. Note that $m_n \to \infty$. Then there is a g_n with $|g_n| = m_n + 1$ such that $g_n x_n \notin U$. Let (a subsequence of) $g_n x_n \to y$ so $y \notin U$. Consider the sets $B_{m_n} g_n^{-1}$. By choosing a subsequence and relabelling, we may assume $\lim_{n\to\infty} B_{m_n} g_n^{-1} = C$, a cone. We show that $Cy \subset U$, from which it will follow that y is not recurrent. Let $c \in C$. Then, for all sufficiently large n, $c \in B_{m_n} g_n^{-1}$, so $c = \beta_n g_n^{-1}$ with $\beta_n \in B_{m_n}$. Then $cg_n x_n = \beta_n g_n^{-1} g_n x_n = \beta_n x_n \in U$. Therefore $cy \in \overline{U} = U$. □

1.9**. Corollary.** *Let (X,G) be a distal flow, where X is zero dimensional, and G is finitely generated. Then (X,G) is equicontinuous.*

1.10**. Example.** We will next show that even for $G = \mathbb{Z}^2$ one can not replace "recurrence on every cone" by "recurrence on some cones". Here is a simple example; of course it can be modified to get many similar examples.

Let u and v be elements of $\{-1,1\}^{\mathbb{Z}}$. Define an element w of $\{-1,1\}^{\mathbb{Z}^2}$ as follows. Set $w(n,0) = u(n)$, then let $w(n,k) = w(n,0)v(k)$; i.e.

$$w(n,k) = \begin{cases} u(n) & \text{if } v(k) = 1 \\ -u(n) & \text{if } v(k) = -1 \end{cases}$$

We let $X \subset \{-1,1\}^{\mathbb{Z}^2}$ be the orbit closure of w under the natural $\mathbb{Z}^2$ action.

If in the flow $(\{-1,1\}^{\mathbb{Z}}, \mathbb{Z})$, generated by the one-dimensional shift, u is not recurrent and v is minimal then clearly the $\mathbb{Z}^2$ action is recurrent at each point of X for the vertical direction, and there are nonrecurrent points for any other direction. Thus for the zero-dimensional flow $(X, \mathbb{Z}^2)$ there is recurrence along a cone C iff C contains one of the two vertical rays $\{(0,t) : t > 0\}$ or $\{(0,t) : t < 0\}$.

1.11**. Remark.** A commutative finitely generated group G is a direct product $G = F \times \mathbb{Z}^d$, where F is a finite group and d is a positive integer (see e.g. [5], page 451). From this structure theorem and Example 1.10 above one can easily deduce that every cone in G contains a replete semigroup. It then follows that a dynamical system (X,G) is recurrent in our sense iff it is recurrent in the sense of Gottschalk and Hedlund. Thus for these groups Theorem 7.07 of [4] is a special case of Theorem 1.8.

2. Pointwise positive but not negative recurrence

In this section we construct a $\mathbb{Z}$-flow (X,T) where all points of X are positively recurrent, but there are points which are not negatively recurrent.

The space X will be the orbit closure of a point x in the shift space on the unit interval $[0,1]$, i.e. if $\Omega = [0,1]^{\mathbb{Z}}$ with T the shift then we will construct a point $x \in \Omega$ and X will be the closure of $\{T^n(x) : n \in \mathbb{Z}\}$. We begin with x_0 which is defined as follows:

$$x_0(n) = 1 \text{ if } n \geq 0 \text{ and } 10|n \text{ , and is 0 otherwise.} \tag{2.1}$$

To continue we will need a sequence of positive even integers p_k, where for p_0 we take 10, and for each succeeding k we assume that $p_k|p_{k+1}$. In addition we will need to assume that the sequence grows sufficiently rapidly so that $\lim_{k\to\infty} p_k/p_{k+1} = 0$. For the next point x_1 in the inductive construction we first define an auxiliary point z_1 as follows. It will be p_1-periodic, $z_1(n)$ will be equal to 1 when $n = 0$ and decrease linearly to 0 at $n = p_1/2$, and then continue to be 0 until $n = p_1 - 1$. With this in hand set

$$x_1(n) = \min\,\{x_0(n), z_1(n)\}, \quad \text{for all } n. \tag{2.2}$$

In general the definition of z_k is similar. It is defined to be p_k-periodic and $z_k(n)$ equals 1 when $n = 0$, decreases linearly to 0 at $n = p_k/2$, and continues to be 0 until $n = p_k - 1$. With this for z_k and assuming that x_{k-1} has already been defined , x_k is defined by

$$x_k(n) = \min\,\{x_{k-1}(n), z_k(n)\}, \quad \text{for all } n. \tag{2.3}$$

Since for fixed n, $x_k(n)$ is monotonically non-increasing there is a limit which we take to be our desired point x. We need now to describe the periodicity properties of x, or more precisely $x(n), n \geq 0$. It is clear that under negative powers of the shift acting on x we get as a limit point only the fixed point which is identically zero, so that only the part of x defined for nonnegative integers needs to be examined. In the sequel we shall consider only this part of the x_k which were used to define x, so that we will, for example, say that x_0 is periodic with period 10, since $x_0(n) = x_0(n+10)$ is valid for all nonnegative n, even though it is not valid for all n.

Notice that most of the time $x_1(n+10)$ differs from $x_1(n)$ only by a small amount which is controlled by $1/p_1$. In fact the only exception occurs when $n = mp_1 - 10$ for some integer m, and in that case $x_1(n) = 0$. Further changes will only decrease this discrepancy overall, and no new exceptional positions will appear by our assumption that p_1 divides all subsequent p_k's. Notice too that for these values, $n = mp_1 - 10$, it is also the case that $x_1(n-k) = 0$ for all $10 \leq k \leq p_1/2$.

This property will extend to any limit point of the positive orbit of x. Thus any such limit point y, will either satisfy

$$|y(10) - y(0)| \leq 20/p_1, \tag{2.4}$$

or $y(k) = 0$ for all $-p_1/2 + 10 \leq k < 10$. Furthermore, in this case we know that y is the limit of a sequence of shifts of x by s_i where $s_i \equiv -10 \pmod{p_1}$.

We consider the next approximation to x and note that the positive part of x_1 is periodic with period p_1. This is maintained most of the time by x_2 up to an error which is at most $2p_1/p_2$ and which we may assume of course is strictly less than $20/p_1$. In fact the only n's for which this estimate for $|x_2(n+p_1) - x_2(n)|$ fails are n in intervals of the form $[mp_2 - p_1, mp_2 - p_1/2]$. Thus we will be able to conclude just as before that any limit point y of shifts of x will either satisfy

$$|y(p_1 + j) - y(j)| \leq 2p_1/p_2, \quad \text{all } |j| \leq 10, \tag{2.5}$$

or $y(k) = 0$ for all $-p_2/2 + p_1 \leq k \leq p_1/2$. Furthermore, we know that y is approximated by a sequence of shifts of x by s_i where the $s_i \equiv -a \pmod{p_2}$, for some $-p_1 \leq a \leq -p_1/2$.

Returning to (2.4) we claim that in case (2.4) fails to hold then in fact $y(p_1) = y(0)$. This is a consequence of the fact that if $0 = x_1(n) = x_1(n + p_1)$ then also $0 = x(n) = x(n + p_1)$ and this will carry over to any y in the orbit closure.

We consider the next approximation to x and note that the positive part of x_2 is periodic with period p_2. This is maintained most of the time by x_2 up to an error controlled by p_2/p_3 which we may assume of course is strictly less than p_1/p_2. Thus we can conclude just as before that any limit point y of shifts of x will either satisfy

$$|y(p_2 + j) - y(j)| \leq 2p_2/p_3, \quad \text{all } |j| \leq p_1, \tag{2.6}$$

or $y(k) = 0$ for all $-p_2/2 + p_1 \leq k \leq p_1/2$. Furthermore, we know that y is approximated by a sequence of shifts of x by s_i where the $s_i \equiv -a \pmod{p_2}$, for some $-p_1 \leq a \leq -p_2/2$.

Returning to (2.5) we claim that in case (2.5) fails to hold then in fact

$$|y(p_2 + j) - y(j)| \leq 2p_2/p_3, \quad \text{all } |j| \leq 10. \tag{2.7}$$

This is a consequence of the fact that when passing from x_1 to x_2 the p_1-periodicity is maintained up to $2p_2/p_3$ in a symmetric interval around the multiples of p_2 of size p_1 and this will carry over to any y in the orbit closure.

It should be clear now how to continue this analysis and show that any point y in the orbit closure of x is indeed recurrent under the p_i's. In fact if the inequalities fail for infinitely many p_j's then the point would have to be zero on larger and larger intervals around the origin and hence identically zero. Formally one establishes the following lemma:

2.1. **Lemma.** *For any $n > 0$ and $i \geq 1$, either*

$$|x(n + p_i + j) - x(n + j)| \leq 2p_i/p_{i+1}, \ \textit{for all } |j| \leq p_{i-1},$$

or

$$|x(n + p_{i+1} + j) - x(n + j)| \leq 2p_i/p_{i+1}, \ \textit{for all } |j| \leq p_{i-1}.$$

Proof. The proof is carried out by an examination of the construction. The first alternative is clearly valid for x_i and all n. When it breaks down in the passage to x_{i+1} it does so only in the p_{i-1} vicinity of multiples of p_{i+1} and then the second alternative will hold for x_{i+1} and continue to do so throughout all the later stages for the same reason, since we assume that for all k $p_k | p_{k+1}$. □

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