

ON ϕ -RECURRENT SASAKIAN MANIFOLDS

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Abstract. The objective of the present paper is to study ϕ -recurrent Sasakian manifolds.

AMS Mathematics Subject Classification (2000): 53C05, 53C20, 53C25

Key words and phrases: ϕ -recurrent Sasakian manifold, Einstein manifold, sectional curvature

1. Introduction

In 1977, T. Takahashi [2] introduced the notion of locally ϕ -symmetric Sasakian manifolds and studied their several interesting properties. In this paper we introduce the notion of ϕ -recurrent Sasakian manifolds which generalizes the notion of locally ϕ -symmetric Sasakian manifolds. After preliminaries, in Section 3, we study ϕ -recurrent Sasakian manifolds and show that such a manifold is always an Einstein manifold. Again, it is proved that in a ϕ -recurrent Sasakian manifold, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are codirectional. Also we obtain some other interesting results of this manifold.

2. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold with the structure (ϕ, ξ, η, g) . Then the following relations hold [1]:

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi,$$

$$(2.2) \quad a) \ \eta(\xi) = 1, \quad b) \ g(X, \xi) = \eta(X), \quad c) \ \eta(\phi X) = 0,$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.4) \quad R(\xi, X)Y = (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X,$$

$$(2.5) \quad a) \ \nabla_X \xi = -\phi X, \quad b) \ (\nabla_X \eta)(Y) = g(X, \phi Y),$$

$$(2.6) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$(2.7) \quad R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi,$$

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$$(2.8) \quad \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$$

$$(2.9) \quad S(X, \xi) = 2n \eta(X),$$

$$(2.10) \quad S(\phi X, \phi Y) = S(X, Y) - 2n \eta(X)\eta(Y),$$

for all vector fields X, Y, Z , where ∇ denotes the operator of covariant differentiation with respect to g , ϕ is a skew-symmetric tensor field of type (1,1), S is the Ricci tensor of type (0,2) and R is the Riemannian curvature tensor of the manifold.

Definition 2.1. [2] *A Sasakian manifold is said to be a locally ϕ -symmetric manifold if*

$$(2.11) \quad \phi^2 ((\nabla_W R)(X, Y)Z) = 0$$

for all vector fields X, Y, Z, W orthogonal to ξ .

Definition 2.2. *A Sasakian manifold is said to be a ϕ -recurrent manifold if there exists a non-zero 1-form A such that*

$$(2.12) \quad \phi^2 ((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z$$

for arbitrary vector fields X, Y, Z, W .

If the 1-form A vanishes, then the manifold reduces to a ϕ -symmetric manifold.

3. ϕ -recurrent Sasakian manifolds

Let us consider a ϕ -recurrent Sasakian manifold. Then by virtue of (2.1) and (2.12) we have

$$(3.1) \quad -(\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi = A(W)R(X, Y)Z,$$

from which it follows that

$$(3.2) \quad \begin{aligned} -g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\ = A(W)g(R(X, Y)Z, U). \end{aligned}$$

Let $\{e_i\}$, $i = 1, 2, \dots, 2n + 1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.2) and taking summation over i , $1 \leq i \leq 2n + 1$, we get

$$(3.3) \quad -(\nabla_W S)(Y, Z) + \sum_{i=1}^n \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = A(W)S(Y, Z).$$

The second term of (3.3) by putting $Z = \xi$ takes the form

$$g((\nabla_W R)(e_i, Y)\xi, \xi)g(e_i, \xi)$$

which is denoted by E . In this case E vanishes. Namely we have

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ &\quad - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) \end{aligned}$$

at $p \in M$. Since $\{e_i\}$ is an orthonormal basis, $\nabla_X e_i = 0$ at p . Using (2.2) and (2.4) we obtain

$$g(R(e_i, \nabla_W Y)\xi, \xi) = g(\nabla_W Y, \xi)g(e_i, \xi) - g(\xi, e_i)g(\nabla_W Y, \xi) = 0.$$

Thus we obtain

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

In virtue of $g(R(e_i, Y)\xi, \xi) = g(R(\xi, \xi)Y, e_i) = 0$, we have

$$g(\nabla_W(R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0,$$

which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi).$$

Using (2.5) and applying the skew-symmetry of R we get

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(R(\phi W, \xi)Y, e_i) + g(R(\xi, \phi W)Y, e_i).$$

Hence we reach

$$\begin{aligned} E &= \sum_{i=1}^n \{g(R(\phi W, \xi)Y, e_i)g(\xi, e_i) + g(R(\xi, \phi W)Y, e_i)g(\xi, e_i)\} \\ &= g(R(\phi W, \xi)Y, \xi) + g(R(\xi, \phi W)Y, \xi) = 0. \end{aligned}$$

Replacing Z by ξ in (3.3) and using (2.9) we have

$$(3.4) \quad -(\nabla_W S)(Y, \xi) = 2n A(W)\eta(Y).$$

Now we have $(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi)$. Using (2.9) and (2.5) in the above relation, it follows that

$$(3.5) \quad (\nabla_W S)(Y, \xi) = 2n g(W, \phi Y) + S(Y, \phi W).$$

In view of (3.4) and (3.5) we obtain

$$(3.6) \quad -[2n g(W, \phi Y) + S(Y, \phi W)] = 2n A(W)\eta(Y).$$

Replacing Y by ϕY in (3.6) and then using (2.1), (2.2) and (2.10) we get

$$(3.7) \quad S(Y, W) = 2n g(Y, W) \quad \text{for all } Y, W.$$

This leads to the following:

Theorem 3.1. *A ϕ -recurrent Sasakian manifold (M^{2n+1}, g) is an Einstein manifold.*

Now from (3.1) we have

$$(3.8) \quad (\nabla_W R)(X, Y)Z = \eta((\nabla_W R)(X, Y)Z)\xi - A(W)R(X, Y)Z.$$

From (3.8) and the Bianchi identity we get

$$(3.9) \quad A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) = 0.$$

By virtue of (2.8) we obtain from (3.9)

$$(3.10) \quad \begin{aligned} & A(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\ & + A(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\ & + A(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)] = 0. \end{aligned}$$

Putting $Y = Z = e_i$ in (3.10) and taking summation over i , $1 \leq i \leq 2n + 1$, we get

$$(3.11) \quad A(W)\eta(X) = A(X)\eta(W) \quad \text{for all vector fields } X, W.$$

Replacing X by ξ in (3.11), it follows that

$$(3.12) \quad A(W) = \eta(W)\eta(\rho) \quad \text{for any vector field } W,$$

where $A(\xi) = g(\xi, \rho) = \eta(\rho)$, ρ being the vector field associated to the 1-form A , i.e., $g(X, \rho) = A(X)$. From (3.11) and (3.12) we can state the following:

Theorem 3.2. *In a ϕ -recurrent Sasakian manifold (M^{2n+1}, g) , $n > 1$, the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (3.12).*

Next, in view of (2.5) and (2.6) it can be easily seen that in a Sasakian manifold the following relation holds:

$$(3.13) \quad (\nabla_W R)(X, Y)\xi = g(W, \phi Y)X - g(W, \phi X)Y + R(X, Y)\phi W.$$

By virtue of (2.8), it follows from (3.13) that

$$(3.14) \quad \eta((\nabla_W R)(X, Y)\xi) = 0.$$

Again from Tanno [3] we have

$$(3.15) \quad \begin{aligned} R(X, Y)\phi Z &= g(\phi X, Z)Y - g(Y, Z)\phi X - g(\phi Y, Z)X \\ &+ g(X, Z)\phi Y + \phi R(X, Y)Z \end{aligned}$$

for any $X, Y, Z \in T_pM$. From (3.13) and (3.15), it follows that

$$(3.16) \quad (\nabla_W R)(X, Y)\xi = g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X, Y)W.$$

In view of (3.16) and (3.14), we obtain from (3.1) that

$$(3.17) \quad g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X, Y)W = -A(W)R(X, Y)\xi.$$

Using (2.6) and (3.12) in (3.17) we have

$$(3.18) \quad \begin{aligned} g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X, Y)W \\ = -\eta(W)\eta(\rho)[\eta(Y)X - \eta(X)Y]. \end{aligned}$$

Hence if X and Y are orthogonal to ξ , then (3.18) reduces to

$$(3.19) \quad \phi R(X, Y)W = g(Y, W)\phi X - g(X, W)\phi Y.$$

Operating ϕ on both sides of (3.19) and using (2.1) we get

$$(3.20) \quad R(X, Y)W = g(Y, W)X - g(X, W)Y \quad \text{for all } X, Y, W.$$

Hence we can state the following:

Theorem 3.3. *A ϕ -recurrent Sasakian manifold (M^{2n+1}, g) , $n > 1$, is a space of constant curvature, provided that X and Y are orthogonal to ξ .*

We now suppose that a Sasakian manifold (M^{2n+1}, g) , $n > 1$, is ϕ -recurrent. Then from (3.8) and (3.16), it follows that

$$(3.21) \quad \begin{aligned} (\nabla_W R)(X, Y)Z = \{g(Y, W)g(\phi X, Z) - g(X, W)g(\phi Y, Z) \\ - g(\phi R(X, Y)W, Z)\}\xi - A(W)R(X, Y)Z. \end{aligned}$$

This leads to the following:

Theorem 3.4. *If a Sasakian manifold (M^{2n+1}, g) , $n > 1$, is ϕ -recurrent then the relation (3.21) holds.*

Let us now suppose that in a Sasakian manifold, the relation (3.21) holds. Then from (3.21) it follows that

$$(3.22) \quad \begin{cases} \phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z - A(W)\{g(Y, Z)\eta(X) \\ - g(X, Z)\eta(Y)\}\xi, \end{cases}$$

which yields

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z,$$

if X and Y are orthogonal to ξ . Hence we can state the following:

Theorem 3.5. *A Sasakian manifold (M^{2n+1}, g) , $n > 1$, satisfying the relation (3.21) is ϕ -recurrent provided that X and Y are orthogonal to ξ .*

Next, we suppose that in a ϕ -recurrent Sasakian manifold, the sectional curvature of a plane $\pi \subset T_p M$ defined by

$$K_p(\pi) = g(R(X, Y)Y, X)$$

is a non-zero constant k , where $\{X, Y\}$ is any orthonormal basis of π . Then we have

$$(3.23) \quad g((\nabla_Z R)(X, Y)Y, X) = 0.$$

By virtue of (3.23) and (3.1) we obtain

$$(3.24) \quad g((\nabla_Z R)(X, Y)Y, \xi)\eta(X) = A(Z)g(R(X, Y)Y, X).$$

Since in a ϕ -recurrent Sasakian manifold, the relation (3.21) holds good, using (3.21) in (3.24) we get

$$(3.25) \quad \begin{cases} \eta(X)[g(Y, Z)g(\phi X, Y) - g(X, Z)g(\phi Y, Y) - g(\phi R(X, Y)Z, Y)] \\ -A(Z)[g(Y, Y)\eta(X) - g(X, Y)\eta(Y)] = kA(Z). \end{cases}$$

Putting $Z = \xi$ in (3.25) we obtain

$$\eta(\rho)[k + \{g(Y, Y)\eta(X) - g(X, Y)\eta(Y)\}] = 0,$$

which implies that

$$\eta(\rho) = 0.$$

Hence by (3.12) we obtain from (2.12) that

$$\phi^2((\nabla_W R)(X, Y)Z) = 0.$$

This leads to the following:

Theorem 3.6. *If a ϕ -recurrent Sasakian manifold (M^{2n+1}, g) , $n > 1$, has a non-zero constant sectional curvature, then it reduces to a locally ϕ -symmetric manifold in the sense of Takahashi.*

References

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Received by the editors September 25, 2003