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ON REDUCTIVE ALGEBRAS CONTAINING COMPACT OPERATORS

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ABSTRACT. It is shown that a reductive operator algebra containing an injective compact operator is selfadjoint.

A subalgebra $\mathfrak{A}$ of the algebra of bounded linear operators on a Hilbert space is *reductive* if it is weakly closed, contains the identity operator, and has the property that $\mathfrak{M}^{\perp}$ is invariant under $\mathfrak{A}$ whenever $\mathfrak{M}$ is an invariant subspace of $\mathfrak{A}$ (cf. [4]). *The reductive algebra problem* (raised in [3]) is the question: Is every reductive algebra a von Neumann algebra? An affirmative answer to this question would be a very powerful result which would imply, in particular, that every operator has an invariant subspace (cf. [3], [4]).

Partial results on the reductive algebra problem have been found by a number of people: The known results are all discussed in [4]. The purpose of this note is to record the observation that Lomonosov's remarkable result [1] implies another special case of the reductive algebra problem. W. B. Arveson, Carl Pearcy, Heydar Radjavi, Allen Shields (and undoubtedly others) have observed that the following lemma is the essence of Lomonosov's result, and that the lemma shows, in particular, that a transitive algebra (i.e., an algebra whose only invariant subspaces are $\{0\}$ and $\mathcal{H}$) containing a nonzero compact operator is strongly dense. I am grateful to Carl Pearcy for describing these results to me. A full discussion is contained in [4].

Lomonosov's Lemma [1]. *If* $\mathfrak{A}$ *is a transitive algebra of operators (not necessarily closed in any topology), and if* K *is any nonzero compact operator, then there exists an* $A \in \mathfrak{A}$ *such that* 1 *is in the point spectrum of* AK.

The result on reductive algebras which can easily be obtained using this beautiful lemma is the following.

Theorem. *A reductive algebra which contains an injective compact operator is a von Neumann algebra.*

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The proof of this theorem requires two other known lemmas in addition to Lomonosov's.

Lemma 1 [2]. *If* $\mathfrak{A}$ *is a reductive algebra, and if the span of the ranges of the finite-rank operators in* $\mathfrak{A}$ *is the entire space, then* $\mathfrak{A}$ *is a von Neumann algebra.*

We show that the hypothesis of the Theorem implies the hypothesis of Lemma 1. For this we need

Lemma 2 [3]. *If* $\mathfrak{A}$ *satisfies the hypothesis of the Theorem, then there exists a collection* $\{\mathfrak{M}_i\}_{i=1}^{\infty}$ *of mutually orthogonal reducing subspaces of* $\mathfrak{A}$ *such that* $\sum_{i=1}^{\infty} \oplus \mathfrak{M}_i$ *is the entire space and, for each* i, $\mathfrak{A}|\mathfrak{M}_i = \{A|\mathfrak{M}_i: A \in \mathfrak{A}\}$ *is a transitive algebra of operators on* $\mathfrak{M}_i$.

Proof of the Theorem. Given a reductive algebra $\mathfrak{A}$ of operators on the Hilbert space $\mathcal{H}$ and an injective compact operator $K \in \mathfrak{A}$, find a collection $\{\mathfrak{M}_i\}_{i=1}^{\infty}$ by Lemma 2. Since $\mathfrak{A}|\mathfrak{M}_i$ is transitive, Lomonosov's lemma implies that, for each i, there is an operator $A_i \in \mathfrak{A}$ and a unit vector $x_i \in \mathfrak{M}_i$ such that $A_iKx_i = x_i$. Now, as observed by Arveson, Pearcy and Shields, the Riesz projection

$$P_i = \frac{1}{2\pi i}\int_{\gamma_i} (z - A_iK)^{-1}dz$$

(where γ_i is a circle about 1 which is disjoint from $\sigma(A_iK)$ and whose interior intersects $\sigma(A_iK)$ in $\{1\}$) of the compact operator A_iK is in the uniformly closed algebra generated by 1 and A_iK, and hence $P_i \in \mathfrak{A}$. Clearly P_i is a finite-rank operator, and so is AP_i for any $A \in \mathfrak{A}$. Since $\mathfrak{A}|\mathfrak{M}_i$ is transitive, for each $x \in \mathfrak{M}_i$ and each $\epsilon > 0$ there exists an operator $A \in \mathfrak{A}$ such that $\|Ax_i - x\| < \epsilon$. Then $\|AP_ix_i - x\| < \epsilon$ (since $P_ix_i = x_i$), and it follows that the span of the ranges of the finite-rank operators in $\mathfrak{A}$ is $\mathcal{H}$. Thus the Theorem follows from Lemma 1.

Remark. Replacing the hypothesis of the Theorem that the compact operator be injective by the assumption that it is not 0 would solve the reductive algebra problem. For if $\mathfrak{A}$ is a reductive algebra and $\mathfrak{B}(\mathcal{H})$ is the algebra of all operators on $\mathcal{H}$, then $\mathfrak{A} \oplus \mathfrak{B}(\mathcal{H})$ is a reductive algebra which contains many compact operators.

Added in proof. Since this paper was submitted, several related results have been obtained:

E. Azoff, *Compact operators in reductive algebras*, Canad. J. Math. (to appear).

H. Radjavi and P. Rosenthal, *On transitive and reductive operator algebras*, Math. Ann. 209 (1974), 43–56.

P. Rosenthal, *On commutants of reductive operator algebras*, Duke Math. J. (to appear).

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2. Eric Nordgren and Peter Rosenthal, *Algebras containing unilateral shifts or finite-rank operators*, Duke Math. J. 40 (1973), 419–424.

3. Heydar Radjavi and Peter Rosenthal, *A sufficient condition that an operator algebra be self-adjoint*, Canad. J. Math. 23 (1971), 588–597.

4. ———, *Invariant subspaces*, Springer-Verlag, Berlin and New York, 1973.

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