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On Regular Fréchet-Lie Groups IV

Definition and Fundamental Theorems

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Introduction

In the previous papers [10], [11], [12], we have seen that Fourier integral operators on a compact manifold have some group theoretical characters. Indeed, one of the purposes of this series is to show that the group of all invertible Fourier integral operators of order 0 on a C^∞ compact riemannian manifold is an *infinite dimensional Lie group.* It should, however, be remarked that we have not given in the previous papers the definition of infinite dimensional Lie groups. It will be given in this paper, hence *one may read this paper without knowing anything about the previous papers.*

Now, it continues to be a basic question when one may call a group G an infinite dimensional Lie group. However, taking the basic properties of finite dimensional Lie groups in mind, we suggest the following (L1)$\sim$(L3) are necessary at least, where

(L1) G is a C^∞ infinite dimensional manifold and the tangent space $\mathfrak{g}$ at the identity has a Lie algebra structure, called the Lie algebra of G.

(L2) There exists the exponential mapping exp of $\mathfrak{g}$ into G such that $\{\exp tu;\ t\in \mathbf{R}\}$ is a smooth one parameter subgroup of G for every $u\in\mathfrak{g}$.

(L3) Local group structures of G (i.e., a neighborhood of the identity) can be determined by its Lie algebra $\mathfrak{g}$.

Hilbert or Banach-Lie groups [1], [5] satisfy these conditions and so do strong *ILB*- (or strong *ILH*-) Lie groups defined by Omori [8], [9].

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Note that strong *ILB*-Lie groups include Banach-Lie groups, hence finite dimensional Lie groups. In this paper, we shall define a wider concept of infinite dimensional Lie groups, which will be called regular Fréchet-Lie groups throughout this series. Roughly speaking, a regular Fréchet-Lie group is a C^∞ manifold modeled on a locally convex Fréchet space, having a C^∞ group structure on which product integrals can be well-defined and have some smoothness properties.

§ 1. Several remarks on differentiability.

In this paper, we shall use the notion of differentiability defined in [3]. Let U be an open subset of a Fréchet space $\mathbf{E}$, and $\mathbf{F}$ another Fréchet space, where all Fréchet spaces in this series are assumed to be locally convex. A mapping $f: U \rightarrow \mathbf{F}$ is a C^0 mapping, if it is continuous. f is called to be *r-times differentiable at* $x \in U$, if f is C^{r-1} on a neighborhood of x and there exists a continuous symmetric r-linear mapping

$$(D^r f)(x): \underbrace{\mathbf{E} \times \cdots \times \mathbf{E}}_{r} \longrightarrow \mathbf{F}$$

such that

$$F(v) = f(x+v) - f(x) - (Df)(x)(v) - \cdots - \frac{1}{r!}(D^r f)(x)(v, \cdots, v)$$

satisfies the property that

$$R(t, v) = \begin{cases} F(tv)/t^r, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

is continuous on a neighborhood of $(0, 0)$. f is called a C^r *mapping*, if f is r-times differentiable at each $x \in U$ and the mapping

$$D^r f: U \times \mathbf{E} \times \cdots \times \mathbf{E} \rightarrow \mathbf{F}$$

is continuous. (Cf. See [2], [13] for the various definitions of the differentiability.)

Let U, V be open subsets of $\mathbf{E}, \mathbf{F}$ respecitively, and $\mathbf{G}$ another Fréchet space. $f: U \times V \rightarrow \mathbf{G}$ will be called a C^r *mapping with respect to the first variable* if for each fixed $v \in V$, $f: U \rightarrow \mathbf{G}$ is C^r and every $D_1^s f$ for $0 \leqq s \leqq r$ is a continuous mapping of $U \times V \times \mathbf{E} \times \cdots \times \mathbf{E}$ into $\mathbf{G}$, where the suffix 1 of D means the derivative with respect to the first variable. The paritial differentiability for the second variable is defined by the similar manner and the derivative is denoted by D_2, etc..

The following properties of C^r mappings are used very often in this series. (For the proof, see [3] and [4]):

(A) *A composition* $f\circ g$ *of* C^r *mappings* f, g *is a* C^r *mapping, and* $D(f\circ g)=Df\cdot Dg$, *more precisely*

$$D(f\circ g)(x)(v)=(Df)(g(x))((Dg)(x)(v)) .$$

(B) *If* $f:U\to\mathbf{F}$ *is* C^r, *then* $D^s f:U\times\mathbf{E}\times\cdots\times\mathbf{E}\to\mathbf{F}$ *is* C^{r-s} *for* $s\leqq r$, *and* $D_1^t D^s f=D^{t+s}f$, $t+s\leqq r$.

(C) $f:U\times V\to\mathbf{G}$ *is* C^r *if and only if* f *is* C^r *with respect to both first and second variables.*

As Fréchet spaces are assumed to be locally convex, there are a lot of continuous linear functionals. Let $\lambda\colon\mathbf{F}\to\mathbf{R}$ be one of them. For a C^1 mapping $f:U\to\mathbf{F}$, $h(t)=\lambda f(x+tv)$ is an $\mathbf{R}$-valued C^1-function on $[0, 1]$. Using the mean value theorem for h, we obtain

$$f(x+v)=f(x)+\int_0^1(Df)(x+tv)(v)dt . \tag{1}$$

Thus, using (B) successively, one can get the following Taylor's expansion theorem: For a C^r function f,

$$\begin{aligned} f(x+v)&=f(x)+(Df)(x)(v)+\cdots+\frac{1}{(r-1)!}(D^{r-1}f)(x)(v, \cdots, v)\\ &\quad+\int_0^1\frac{(1-t)^{r-1}}{(r-1)!}(D^r f)(x+tv)(v, \cdots, v)dt\\ &=f(x)+(Df)(x)(v)+\cdots+\frac{1}{r!}(D^r f)(x)(v, \cdots, v)\\ &\quad+\int_0^1\frac{(1-t)^{r-1}}{(r-1)!}\{(D^r f)(x+tv)-(D^r f)(x)\}(v, \cdots, v)dt . \end{aligned} \tag{2}$$

By this formula, one can get the converse of (B), namely,

(B′) $f:U\to\mathbf{F}$ *is* C^r *if and only if* f *is* C^s *for some* $s(\leqq r)$ *and* $D^s f:U\times\mathbf{E}\times\cdots\times\mathbf{E}\to\mathbf{F}$ *is* C^{r-s}.

REMARK. The above notion of differentiability coincides with the usual one if $\mathbf{E}, \mathbf{F}, \mathbf{G}$ are finite dimensional vector spaces. However, if $\mathbf{E}, \mathbf{F}, \mathbf{G}$ are infinite dimensional Banach spaces, the above notion is weaker than the usual definition of differentiability. For instance, the above definition requests only the continuity of $D^r f:U\times\mathbf{E}\times\cdots\times\mathbf{E}\to\mathbf{F}$, while in Banach spaces the continuity $D^r f:U\to L^r_{\mathrm{sym}}(\mathbf{E}, \mathbf{F})$ is usually requested, where the last one is the Banach space of symmetric r-linear mappings of $\mathbf{E}\times\cdots\times\mathbf{E}$ into $\mathbf{F}$ with the uniform topology. The continuity of

$D^r f: U\times \mathbf{E}\times\cdots\times\mathbf{E}\to\mathbf{F}$ ensures only that $D^r f: U\to L^r_{\mathrm{sym}}(\mathbf{E}, \mathbf{F})$ is locally bounded, namely for any $x\in U$ there is a neighborhood W of x, and a positive constant C such that

$$\|(D^r f)(y)(v_1, \cdots, v_r)\|\leqq C\|v_1\|\,\|v_2\|\cdots\|v_r\|$$

for every $y\in W, v_1, \cdots, v_r\in \mathbf{E}$. Thus, by $D_1 D^{r-1}f=D^r f$ and the mean value theorem (1), we have

$$\|(D^{r-1}f)(x+w)-(D^{r-1}f)(x)\|\leqq C\|w\| .$$

Therefore, *if $f: U\to\mathbf{F}$ is C^r in the above sense, then f is C^{r-1} in the usual sense in Banach spaces.* Thus, if we concern only C^∞ mappings, the above two notions of differentiability make no difference.

It should be remarked, however, that above two notions make a difference for the partial differentiability. In our definition, we request only the continuity of $D_1^s f: U\times V\times\mathbf{E}\times\cdots\times\mathbf{E}\to\mathbf{G}$ for $s\leqq r$. For a fixed $v\in V$, $D_1^{s-1}f: U\to L^{s-1}_{\mathrm{sym}}(\mathbf{E}, \mathbf{G})$ is continuous by the above remark, but this does not mean the continuity of $D_1^{s-1}f: U\times V\to L^{s-1}_{\mathrm{sym}}(\mathbf{E}, \mathbf{G})$.

Suppose $\mathbf{E}, \mathbf{F}$ are Fréchet spaces and U, V are open subsets of $\mathbf{E}, \mathbf{F}$ respectively. Let $f: U\to V$ be a C^r mapping ($r\geqq 1$). We define the differential map $df: U\times\mathbf{E}\to V\times\mathbf{F}$ by

$$(df)(x, v)=(f(x), (Df)(x)(v)) .$$

$Df(x)(v)$ is denoted sometimes by $(df)_x v$, $(Tf)_x v$ or $f_{*x}v$. Obviously, $df: U\times\mathbf{E}\to V\times\mathbf{F}$ is a C^{r-1} mapping by (B'). If f is a C^r-diffeomorphism (i.e., f^{-1} exists and C^r), then df is a C^{r-1} diffeomorphism.

LEMMA 1.1. *If $f: U\to V$ is invertible and C^r mapping $(r\geqq 1)$ and if f^{-1} is C^1 mapping, then f^{-1} is C^r mapping.*

PROOF. By using the composition rule (A), the derivative of f^{-1} at y is given by

$$(Df^{-1})(y)=(Df(x))^{-1} , \quad y=f(x)\in V .$$

Therefore, if $y, y+w\in V$

$$\begin{aligned}(Df^{-1})&(y+w)-(Df^{-1})(y)\\ &=-(Df^{-1})(y)((Df)(f^{-1}(y+w))-(Df)(f^{-1}(y)))(Df^{-1}(y+w)) .\end{aligned}$$

Hence by the continuity of $(Df^{-1})(y)w$ and the smoothness of f, we see that $(Df^{-1})(y)w$ is C^1, and hence f^{-1} is C^2. The desired regularity follows inductively by this manner. □

Now, we define the notion of C^∞ *Fréchet manifold* M modeled on E as usual, i.e., M admits an admissible atlas which is a collection of pairs (U, ϕ) of local charts with C^∞ coordinate transformations. *A tangent vector* X_x, at $x \in M$, is an equivalence class of triples (U_i, ϕ_i, X_i) where (U_i, ϕ_i) is an arbitrary chart of M at x and X_i is a vector of E; two triples are equivalent if

$$X_j = D(\phi_j \circ \phi_i^{-1})(\phi_i(x))X_i \ .$$

The representative $X_i \in \mathbf{E}$ of X_x in the triple (U_i, ϕ_i, X_i) plays the same role as the components in a local coordinate system. The space of vectors tangent at the point x, together with its natural vector space structure is *the tangent vector space* T_xM. It can be easily verified that the set of vectors tangent to M, $TM = \bigcup_{x \in M} T_xM$ has a structure of a C^∞ differentiable manifold modeled on $\mathbf{E} \times \mathbf{E}$ by the family of charts $\{(\bigcup_{x \in M} T_xM, T\phi)\}$ where $\{(U, \phi)\}$ is an atlas of M modeled on E and $T\phi$ the homeomorphism of $\bigcup_{x \in U} T_xM$ on $U \times \mathbf{E}$ defined by

$$T\phi(x, X_x) = (\phi(x), X)$$

X being the representative of X_x in the map (U, ϕ). Moreover, the space T_M is given a fibre bundle structure with the natural projection $\pi: T_M \to M$ of C^∞ mapping and is called the *Tangent bundle.* Thus, the $C^k(k \leqq \infty)$ vector fields are defined as usual as C^k sections of $\pi: T_M \to M$. However, we do not define the structure of vector bundle on T_M, for it is not easy to define a reasonable topology for $GL(\mathbf{E})$ as a topological group.

A subset N of M will be called a C^∞ Fréchet *submanifold,* if there is a closed subspace F of E and for every $x \in N$ there is a C^∞ local coordinate system $\phi: U \to M$ of M at x such that ϕ maps $U \cap \mathbf{F}$ homeomorphically onto the arcwise connected component of x of a neighborhood of x in N under the relative topology.

A subset $\mathscr{V}$ of T_M will be called a *subbundle* of T_M if the following conditions are satisfied:

(SB1) $\pi: \mathscr{V} \to M$ is surjective, and $\pi^{-1}(x) \cap \mathscr{V}$ is a closed linear subspace of T_xM for every $x \in M$. We denote $\pi^{-1}(x) \cap \mathscr{V}$ by $\mathbf{F}_x$ and call it *the fiber* of $\mathscr{V}$ at x.

(SB2) There exists an open neighborhood V_x of each $x \in M$, and a C^∞ diffeomorphism ψ_x of $V_x \times \mathbf{F}_x$ onto $\pi^{-1}(V_x) \cap \mathscr{V} \subset T_M$ such that $\pi\psi_x(y, v) \equiv y$, $y \in V_x$, and ψ_x is linear with respect to the second variable.

(SB3) If $V_x \cap V_y \neq \varnothing$, then $\psi_x^{-1}\psi_y: (V_x \cap V_y) \times \mathbf{F}_y \to (V_x \cap V_y) \times \mathbf{F}_x$ is C^∞.

Given a Fréchet space E and a unit interval $I = [0, 1]$, let $C^k(I, \mathbf{E})$

be the set of all C^k mappings from I into E ($k=0, 1, 2, \cdots$). The set $C^k(I, \mathrm{E})$ has the structure of vector space in the obvious way. Consider the evaluation map

$$\mathrm{ev}^k : I\times C^k(I, \mathrm{E}) \longrightarrow \underbrace{\mathrm{E}\times\cdots\times\mathrm{E}}_{k+1}$$

given by

$$\mathrm{ev}^k(t, c)=(c(t), (Dc)(t), \cdots, (D^k c)(t)) .$$

Putting a Fréchet structure into $\mathrm{E}\times\cdots\times\mathrm{E}$, we obtain the uniform topology of $C^k(I, \mathrm{E})$.

Let M be a C^∞ Fréchet manifold modeled on E and let $C^k(I, M)$ $k=0, 1, 2, \cdots$, be a space of all C^k mappings from I into M. Then, we have the following.

LEMMA 1.2. *$C^k(I, M)$ is a C^∞ Fréchet manifold modeled on $C^k(I, \mathrm{E})$ for each $k=0, 1, 2, \cdots$. For a fixed point $p\in M$,*

$$C^k_p(I, M)=\{c\in C^k(I, M);\ c(0)=p\}$$

is a C^∞ Fréchet submanifold of $C^k(I, M)$.

PROOF. Let U, V be open subsets of E and let $\phi: U\to V$ a C^∞ diffeomorphism. Define a mapping $f: C^0(I, U)\to C^0(I, V)$ by $f(\lambda)(t)=\phi(\lambda(t))$. Obviously, the crucial part of the proof is to show that f is of C^∞. For that purpose, set

$$F(\mu)(t)=\phi((\lambda+\mu)(t))-\phi(\lambda(t))-(D\phi)(\lambda(t))(\mu(t))-\cdots$$
$$-\frac{1}{r!}(D^r\phi)(\lambda(t))(\mu(t), \cdots, \mu(t)) .$$

By Taylor's theorem, we have

$$F(\mu)(t)=\int_0^1\frac{(1-\theta)^{r-1}}{(r-1)!}\{D^r\phi(\lambda(t)+\theta\mu(t))-D^r\phi(\lambda(t))\}(\mu(t), \cdots, \mu(t))d\theta .$$

Thus, using the uniform continuity of $D^r\phi(\lambda(t)+\theta\mu(t))$ we see that

$$R(s, \mu)=\begin{cases}F(s\mu)/s^r & s\neq 0\\ 0 & s=0\end{cases}$$

is continuous on a neighborhood of $(0, 0)$. Since $D^r\phi(\lambda(t))(\mu_1(t), \cdots, \mu_r(t))$ is continuous with respect to $\lambda\in C^0(I, U)$, $\mu_1, \cdots, \mu_r\in C^0(I, \mathrm{E})$, for all

$r \geqq 0$, we have that $f: C^0(I, U) \to C^0(I, V)$ is C^r and $D^r f(\lambda)(\mu_1, \cdots, \mu_r)(t) = D^r\phi(\lambda(t))(\mu_1(t), \cdots \mu_r(t))$.

To prove the differentiability for $k>0$, we have only to use $d^k\phi: U \times \mathbf{E} \times \cdots \times \mathbf{E} \to V \times \mathbf{E} \times \cdots \times \mathbf{E}$ instead of ϕ. □

§ 2. Several remarks on *FL*-groups.

An *FL*-group is the combined concept of a topological group and a C^∞ Fréchet manifold such that the group operations are C^∞. Therefore, one may call an *FL*-group a Fréchet-Lie group. However, we hesitate to use the name "Lie", for a general *FL*-group may not have the properties (L1)~(L3) mentioned in the introduction.

Now, let G be an *FL*-group. The tangent space $\mathfrak{g}$ of G at the identity e is naturally identified with its model space, and hence $\mathfrak{g}$ is a Fréchet space. By definition, there is a C^∞ diffeomorphism ξ of an open neighborhood U of 0 in $\mathfrak{g}$ onto an open neighborhood $\tilde{U}$ of e in G such that $\xi(0)=e$, $(d\xi)_0=\mathrm{Id}$. $\xi: U \to \tilde{U}$ will be called a *local coordinate system* at e. (Usually, $\xi^{-1}: \tilde{U} \to U$ is called a "local coordinate system". We use the above definition in accordance with the exponential map of the group.) As $\mathfrak{g}$ satisfies the first countability axiom, so also does G, and hence G has a right-invariant metric (cf [6] p. 34), where a metric ρ on G is called a *right-invariant metric* if $\rho(xa, ya)=\rho(x, y)$ for every $x, y, a \in G$.

As G is a topological group, there is an open neighborhood V of 0 in $\mathfrak{g}$ such $\xi(V)^2 \subset \tilde{U}$, $\xi(V)^{-1}=\xi(V)$. Therefore, by the local coordinate system ξ at e, the group operations of G are represented by

$$\eta(u, v)=\xi^{-1}(\xi(u)\xi(v)) , \quad \iota(u)=\xi^{-1}(\xi(u)^{-1}) .$$

For every $g \in G$, there is an open neighborhood W of 0 in $\mathfrak{g}$ such that $g\xi(W)g^{-1} \subset \tilde{U}$. We set

$$\tilde{A}_g(u)=\xi^{-1}(g\xi(u)g^{-1}) .$$

As G is *FL*-group, η, ι and $\tilde{A}_g$ are C^∞ on the domain on which they are defined.

Since $\eta(u, 0)\equiv u$, $\eta(0, v)\equiv v$, we see $(D_1\eta)_{(0,0)}=(D_2\eta)_{(0,0)}=\mathrm{Id}$ (the identity). Let $a(t), b(t)$ be C^1 curves in G such that $a(0)=b(0)=e$. Then, $\dot{a}(0), \dot{b}(0) \in \mathfrak{g}$, where $\dot{a}(0)=(d/dt)|_{t=0}a(t)$. The product $c(t)=a(t)hb(t)$ is a C^1 curve in G with $c(0)=h \in G$. The following is easy to prove.

LEMMA 2.1.

$$\frac{d}{dt}\Big|_{t=0} a(t)hb(t) = dR_h\dot{a}(0) + dL_h\dot{b}(0)$$

$$\frac{d}{dt}\Big|_{t=0} a(t)^{-1} = -\dot{a}(0)\ .$$

where R_h(resp. L_g) denotes the right- (resp. left-) translation.

Let G be an FL-group. For every $g \in G$, define the map A_g on G by $A_g h = ghg^{-1}$. Then the map $A: G \times G \to G$,

$$(3) \qquad A(g, h) = A_g h$$

is C^∞. We often use the notation $A(g)h$ instead of $A_g h$. Given $u \in \mathfrak{g}$, there is a C^∞ curve $c(t)$ in G such that $c(0)=e$, and $\dot{c}(0)=u$ (for instance, $c(t)=\xi(tu)$ satisfies this). Define the adjoint map $\mathrm{Ad}(g)$, $g \in G$ on $\mathfrak{g}$ by

$$(4) \qquad \mathrm{Ad}(g)u = \frac{d}{dt}\Big|_{t=0} gc(t)g^{-1}\Big(= \frac{d}{dt}\Big|_{t=0} A_g c(t)\Big)\ .$$

The adjoint map $\mathrm{Ad}(g)$ can be expressed in terms of the left and right auxiliary functions. If we denote the left- (resp. right-) translation by

$$L_g: h \longrightarrow gh \quad (\text{resp. } R_g: h \longrightarrow hg)\ , \quad g, h \in G\ ,$$

then we get $\mathrm{Ad}(g)u = (dR_g)^{-1}(dL_g)u$. Therefore, the definition (4) does not depend on the choice of the curve $c(t)$. Also, we put the following map of $G \times \mathfrak{g}$ to $\mathfrak{g}$ by

$$\mathrm{Ad}(g, u) = \mathrm{Ad}(g)u\ , \quad g \in G,\ u \in \mathfrak{g}\ .$$

Let T_G be the tangent bundle of FL-group G. Since $T_g G = \mathfrak{g}\cdot g (= (dR_g)_e \mathfrak{g})$, $g \in G$, we see that T_G is C^∞ diffeomorphic to $\mathfrak{g} \times G$. The differential map of the product operation on G gives the group structure on T_G. Namely, under the identification between T_G and $\mathfrak{g} \times G$, we obtain the product $*$ on $\mathfrak{g} \times G$ by

$$(5) \qquad (u, g)*(v, h) = (dR_{gh})(u + \mathrm{Ad}(g)v)(= (u + \mathrm{Ad}(g)v)\cdot gh)\ .$$

LEMMA 2.2. *By the product $*$ of* (5), *$\mathfrak{g} \times G$ turns out to be an FL-group with $(0, e)$ as the identity and can be identified with T_G as FL-group. Moreover, $\mathfrak{g} \times \{e\}$ is a normal abelian subgroup of $\mathfrak{g} \times G$.*

The above FL-group will be denoted by $\mathfrak{g} * G$

Let $g(t)$ be a C^∞ curve in G such that $g(0)=e$, $\dot{g}(0)=u$. We define the bracket $[u, v]$ on $\mathfrak{g}$ by

$$[u, v]=\frac{d}{dt}\bigg|_{t=0}\operatorname{Ad}(g(t))v\,, \quad v\in \mathfrak{g}\,. \tag{6}$$

To obtain that (6) satisfies the properties of the Lie bracket, we give the other description of (6). Denote for every $u\in\mathfrak{g}$ by u^* the right-invariant C^∞ vector field on G such that $u^*(e)=u$, i.e.,

$$u^*(g)=u\cdot g(=(dR_g)_e u)\,. \tag{7}$$

Then, we have the following by taking a C^∞ curve such that $c(0)=e$, $\dot c(0)=v$:

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\operatorname{Ad}(g(t))v&=\frac{\partial^2}{\partial t\partial s}\bigg|_{\substack{t=0\\s=0}} g(t)c(s)g(t)^{-1}\\ &=\frac{d}{ds}\bigg|_{s=0}\{dR_{c(s)}u-dL_{c(s)}u\}\quad \text{(Lemma 2.1)}\\ &=\partial_v u^*-\frac{\partial^2}{\partial s\partial t}\bigg|_{\substack{t=0\\s=0}} c(s)g(t)\\ &=\partial_v u^*-\frac{d}{dt}\bigg|_{t=0} dR_{g(t)}v\\ &=\partial_v u^*-\partial_u v^*\,,\end{aligned}$$

where in the above computations, differentials are computed by taking a local coordinate expression. Though $\partial_v u^*$, $\partial_u v^*$ depend on the choice of a local coordinate system, $\partial_v u^*-\partial_u v^*$ does not depend on it. Moreover, we see that $(d/dt)|_{t=0}\operatorname{Ad}(g(t))v$ does not depend on the choice of $g(t)$. Hence, we get

$$[u, v]=\partial_v u^*-\partial_u v^*\,. \tag{8}$$

Since $(d^2u^*)_e(v, w)=(d^2u^*)_e(w, v)$, $u, v, w\in\mathfrak{g}$, the above bracket product satisfies the Jacobi identity, and hence $\mathfrak{g}$ is a Lie algebra such that $[\,,\,]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is a continuous bi-linear mapping, i.e., $\mathfrak{g}$ is a Fréchet-Lie algebra. $\mathfrak{g}$ will be called the *Lie algebra* of G.

LEMMA 2.3. *Let $g(t)$ be a C^1 curve in G. Then,*

$$\frac{d}{dt}\operatorname{Ad}(g(t))v=[u(t), \operatorname{Ad}(g(t))v]\,,$$

$$\frac{d}{dt}\operatorname{Ad}(g(t)^{-1})v=-\operatorname{Ad}(g(t)^{-1})[u(t), v]\,,$$

where $u(t)=(dg(t)/dt)\cdot g(t)^{-1}\in\mathfrak{g}$.

PROOF. Remark at first that $\mathrm{Ad}(g(t))v$ is C^1 with respect to t. Note that $\mathrm{Ad}(g)\mathrm{Ad}(h)=\mathrm{Ad}(gh)$ and $(dg/dt)\cdot g(t)^{-1}=d\xi(su(t))/ds|_{s=0}$. We compute as follows:

$$\begin{aligned}\frac{d}{dt}\mathrm{Ad}(g(t))v&=\frac{d}{ds}\Big|_{s=0}\mathrm{Ad}(g(t+s))v\\&=\frac{d}{ds}\Big|_{s=0}\mathrm{Ad}(\xi(su(t))\cdot g(t))v\\&=\frac{d}{ds}\Big|_{s=0}\mathrm{Ad}(\xi(su(t)))\,\mathrm{Ad}(g(t))v\\&=[u(t),\ \mathrm{Ad}(g(t))v]\,.\end{aligned}$$

(Remark that $\xi(su(t))$ is C^∞ w.r.t. s and use (6)). Similarly, by Lemma 2.1 and (6),

$$\begin{aligned}\frac{d}{dt}\mathrm{Ad}(g(t)^{-1})v&=\frac{d}{ds}\Big|_{s=0}\mathrm{Ad}(g(t)^{-1})\mathrm{Ad}(\xi(su(t))^{-1})v\\&=\mathrm{Ad}(g(t))^{-1}\frac{d}{ds}\Big|_{s=0}\mathrm{Ad}(\xi(su(t))^{-1})v\\&=-\mathrm{Ad}(g(t)^{-1})[u(t),\ v]\,.\end{aligned}$$

□

LEMMA 2.4. *$\mathrm{Ad}(g)\colon \mathfrak{g}\to\mathfrak{g}$ is an automorphism of the Lie algebra and $\mathrm{Ad}\colon G\times\mathfrak{g}\to\mathfrak{g}$, $\mathrm{Ad}(g, u)=\mathrm{Ad}(g)u$, $g\in G$, $u\in\mathfrak{g}$, is a C^∞ mapping.*

PROOF. As $G\times G\to G$, $(g, h)\mapsto ghg^{-1}$ is C^∞, the second assertion is obvious by definition. For every $g\in G$, $\mathrm{Ad}(g)\colon\mathfrak{g}\to\mathfrak{g}$ is obviously a linear isomorphism. Thus, we have only to show $\mathrm{Ad}(g)[u, v]=[\mathrm{Ad}(g)u, \mathrm{Ad}(g)v]$. This is given as follows:

$$\begin{aligned}\mathrm{Ad}(g)[u,\ v]&=\mathrm{Ad}(g)\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(c(t))v\\&=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(gc(t))v\\&=\frac{d}{dt}\Big|_{t=0}\mathrm{Ad}(gc(t)g^{-1})\mathrm{Ad}(g)v\\&=[\mathrm{Ad}(g)u,\ \mathrm{Ad}(g)v]\,,\end{aligned}$$

where $c(t)$ is a C^∞ curve in G such that $c(0)=e$ and $\dot c(0)=u$. The last equality follows from (6), for $(d/dt)gc(t)g^{-1}|_{t=0}=\mathrm{Ad}(g)u$.

Remark that it is not known whether there is the exponential mapping $\exp\colon\mathfrak{g}\to G$ on every *FL*-group. Namely one might not be able to solve the equation $(d/dt)g(t)=u(t)\cdot g(t)$. However, one can get the following uniqueness theorem:

LEMMA 2.5. *Let $u(t)$ be a $\mathfrak{g}$-valued continuous function on* $[0, 1]$. *Then, the uniqueness holds for the equation*

$$\frac{d}{dt}g(t)=u(t)\cdot g(t)\ , \quad g(0)=e\ .$$

PROOF. Let $h(t)$ be another solution such that $h(0)=e$. Then, by

$$\frac{d}{dt}h(t)^{-1}g(t)=-dL_{h(t)^{-1}}dR_{g(t)}u(t)+dL_{h(t)^{-1}}dR_{g(t)}u(t)\equiv 0\ .$$

Hence by the mean value theorem (1), we have $h(t)^{-1}g(t)\equiv e$. □

Let $g(t)$ be a C^1 curve in G defined on $[0, 1]$. We set $u(t)=(dg(t)/dt)\cdot g(t)^{-1}\in\mathfrak{g}$.

LEMMA 2.6. *Let $w(t)$ be a $\mathfrak{g}$-valued C^0 function on* $[0, 1]$. *For the above $u(t)$, the differential equation*

$$\frac{d}{dt}v(t)-[u(t), v(t)]=w(t)\ , \quad v(0)=0$$

has a unique solution $\mathrm{Ad}(g(t))\int_0^t \mathrm{Ad}(g(s)^{-1})w(s)ds$.

PROOF. By using Lemma 2.3, we see easily that

$$\mathrm{Ad}(g(t))\int_0^t \mathrm{Ad}(g(s)^{-1})w(s)ds$$

is a solution. Suppose there is another solution $\bar{v}(t)$. Then $v(t)-\bar{v}(t)$ satisfies $(d/dt)(v(t)-\bar{v}(t))=[u(t), v(t)-\bar{v}(t)]$ and $v(0)-\bar{v}(0)=0$. Compute $(d/dt)\,\mathrm{Ad}(g(t)^{-1})(v(t)-\bar{v}(t))$, by using Lemma 2.3, and we see easily that it is identically 0. Hence by the mean value theorem (1), we have $v(t)=\bar{v}(t)$. □

§ 3. Product integrals and the definition of regular Fréchet-Lie groups.

Now, we start with considering a division Δ: $a=t_0<t_1<\cdots<t_m=b$, of a closed interval $J=[a, b]$. By Δ we indicate also the set of dividing points $\{t_0, \cdots, t_m\}$. For a division Δ of $[a, b]$, we denote by $|\Delta|$ the maximum of $|t_{j+1}-t_j|$.

Let G be an FL-group and $\mathfrak{g}$ its Lie algebra.

DEFINITION 3.1. A *step function* defined on $[0, \varepsilon]\times[a, b]$ is a pair (h, Δ) of a division Δ of $[a, b]$ such that $|\Delta|<\varepsilon$ and a mapping $h: [0, \varepsilon]\times[a, b]\to G$ satisfying the following:

(i) $h(0, t)=e$ for all $t\in[a, b]$, and $h(s, t)$ is C^1 in s for each fixed t.

(ii) $h(s, t)=h(s, t_j)$ for $(s, t)\in[0, \varepsilon]\times[t_j, t_{j+1})$.

EXAMPLE. Let $\xi: U\to\tilde{U}$ be a local coordinate system at e such that U is a convex neighborhood of 0 in $\mathfrak{g}$. Let $u: [a, b]\to\mathfrak{g}$ be a mapping which is piecewise constant. Then $h(s, t)=\xi(su(t))$ is a step function defined on $[0, \varepsilon]\times[a, b]$ for an appropriately small ε.

Denote $[a, b]$ by J. By $\mathcal{S}_{\varepsilon,J}(G)$ we denote the space of all step functions on $[0, \varepsilon]\times J$, and by $S_{\varepsilon,J}(G)$ the space of all mappings $h: [0, \varepsilon]\times J\to G$ such that (h, Δ) is a step function for some Δ.

DEFINITION 3.2. A mapping $h: [0, \varepsilon]\times J\to G$ will be called a *C^1-hair* at e if

(i) $h(0, t)=e$ for all $t\in J$, and $h(s, t)$ is C^1 in s for each fixed t.

(ii) $h(s, t)$ and $(\partial h/\partial s)(s, t)$ is C^0 with respect to $(s, t)\in[0, \varepsilon]\times J$.

If $u: J\to\mathfrak{g}$ is a continuous mapping, then $\xi(su(t))$ is a C^1-hair at e defined on $[0, \varepsilon]\times J$ for a small ε. By $H^1_{\varepsilon,J}(G)$ we denote the space of all C^1-hairs at e defined on $[0, \varepsilon]\times J$.

Let ρ be a right-invariant metric on G mentioned in the previous section, and d a metric on $\mathfrak{g}$ by which $\mathfrak{g}$ is a Fréchet space. Define a metric $\tilde{\rho}$ on the space of the union of $S_{\varepsilon,J}(G)$ and $H^1_{\varepsilon,J}(G)$ as follows:

$$(9)\qquad \tilde{\rho}(h, h')=\max_{[0,\varepsilon]\times J}\rho(h(s, t), h'(s, t)) + \max_{[0,\varepsilon]\times J} d\Big(\frac{\partial h(s, t)}{\partial s}h(s, t)^{-1}, \frac{\partial h'(s, t)}{\partial s}h'(s, t)^{-1}\Big)\,.$$

Given $h\in H^1_{\varepsilon,J}(G)$ and a division $\Delta: a=t_0<t_1<\cdots<t_m=b$ of J, we define a step function $(\sigma_\Delta(h), \Delta)\in\mathcal{S}_{\varepsilon,J}(G)$ by

$$(10)\qquad \sigma_\Delta(h)(s, t)=h(s, t_j)\quad\text{for}\quad t\in[t_j, t_{j+1})\,.$$

LEMMA 3.3. *Let $\{\Delta_n\}$ be a sequence of a division of J such that* $\lim_{n\to\infty}|\Delta_n|=0$. *Then, for any $h\in H^1_{\varepsilon,J}(G)$, $\lim_{n\to\infty}\sigma_{\Delta_n}(h)=h$ with respect to $\tilde{\rho}$ defined by* (9).

PROOF. Since $h(s, t)$ and $(\partial h(s, t)/\partial s)h(s, t)^{-1}$ are uniformly continuous in (s, t), we get easily $\lim_{n\to\infty}\tilde{\rho}(\sigma_{\Delta_n}(h), h)=0$. □

Let K be a compact subset of $H^1_{\varepsilon,J}(G)$. Then it is easy to see that h and $(\partial h(s, t)/\partial s)h(s, t)^{-1}$ are equi-continuous whenever $h\in K$. Hence we have

COROLLARY 3.4. *For a fixed sequence of a division* Δ_n *of* J *such that* $\lim_{n\to\infty}|\Delta_n|=0$, $\lim_{n\to\infty}\sigma_{\Delta_n}$ *converges uniformly to the identity on every compact subset* K *of* $H^1_{\varepsilon,J}(G)$.

For a step function $(h, \Delta)\in\mathscr{S}_{\varepsilon,J}(G)$, we define a *product integral* $\prod_s^t(h, \Delta)(t\geqq s)$ by

$$\prod_s^t(h, \Delta)=h(t-t_k, t_k)h(\Delta t_k, t_{k-1})\cdots h(\Delta t_{l+1}, t_l)h(t_l-s, s) \tag{11}$$

where k, l are the numbers such that $t\in[t_k, t_{k+1})$, $s\in[t_{l-1}, t_l)$ respectively, and set $\Delta t_j=t_j-t_{j-1}$.

DEFINITION 3.5. An FL-group G will be called a *regular Fréchet-Lie group*, if the following condition is satisfied: Let $\{(h_n, \Delta_n)\}$ be any sequence in $\mathscr{S}_{\varepsilon,J}(G)$ for some $\varepsilon>0$ and $J=[a, b]$ such that $\lim_{n\to\infty}|\Delta_n|=0$ and $\lim_{n\to\infty}h_n=h\in H^1_{\varepsilon,J}(G)$ with respect to $\tilde{\rho}$. Then, the product integral $\prod_a^t(h_n, \Delta_n)$ converges uniformly in $t\in[a, b]$.

Once such a condition is satisfied, the limit $\lim_{n\to\infty}\prod_a^t(h_n, \Delta_n)$ depends only on $h\in H^1_{\varepsilon,J}(G)$. Hence we denote the limit by $\prod_a^t(h, d\tau)$ and call it *the product integral* of h. Since $g_n(t)=\prod_a^t(h_n, \Delta_n)$ are continuous curves in G such that $g_n(a)=e$, so is the limit $g(t)=\prod_a^t(h, d\tau)=\lim_{n\to\infty}g_n(t)$, for the uniformity of the convergence is assumed. (For the aspects of the product integral, see [7]. p. 15.)

If $d<c$, we define

$$\prod_c^d(h, d\tau)=\left\{\prod_d^c(h, d\tau)\right\}^{-1}.$$

With this convention, we always have

LEMMA 3.6. *Let* G *be a regular Fréchet-Lie group and let* h *be a* C^1*-hair at* e *defined on* $[0, \varepsilon]\times J$. *Then,*

$$\prod_\alpha^\gamma(h, d\tau)=\prod_\beta^\gamma(h, d\tau)\prod_\alpha^\beta(h, d\tau)$$

for every $\alpha, \beta, \gamma\in J$.

PROOF. We may prove only for case $\alpha\leqq\beta\leqq\gamma$. Notice that if we take a division Δ of J such that β is a dividing point of Δ, then for each step function (h, Δ) we get

$$\prod_{\alpha}^{\gamma}(h, \Delta)=\prod_{\beta}^{\gamma}(h, \Delta)\prod_{\alpha}^{\beta}(h, \Delta) .$$

Using the above equality, we get the desired equality. □

LEMMA 3.7. *Let $\Phi: J\times H^1_{\varepsilon,J}(G)\to G$ be a mapping defined by $\Phi(t, h)=\prod_a^t(h, d\tau)$. Then Φ is continuous, where the topology on $H^1_{\varepsilon,J}(G)$ is given by $\tilde{\rho}$ defined by* (9).

PROOF. Suppose that Φ is discontinuous at (t, h). Then there is a sequence (t_k, h_k) such that $|t-t_k|\to 0$, $\tilde{\rho}(h_k, h)\to 0$ but there is $\delta>0$ such that $\rho(\prod_a^{t_k}(h_k, d\tau), \prod_a^t(h, d\tau))\geqq\delta$. Use Corollary 3.4 for the compact subset $K=\{h, h_k, k=1, 2, 3, \cdots\}$. Note that if $m\to\infty$ as $k\to\infty$ then $(\sigma_m(h_k), \Delta_m)$'s are step functions such that $\lim_{k\to\infty}\tilde{\rho}(\sigma_m(h_k), h)=0$, where σ_m is the abbreviation of σ_{Δ_m}. Hence by the hypothesis of regular Fréchet-Lie groups, $\prod_a^t(\sigma_m(h_k), \Delta_m)$ must converge to $\prod_a^t(h, d\tau)$ uniformly as $k\to\infty$. This contradicts the assumption if we choose $n(k)$ for each k so that it may satisfy $\rho(\prod_a^{t_k}(\sigma_{n(k)}(h_k), \Delta_{n(k)}), \prod_a^{t_k}(h_k, d\tau))<\delta/3$. □

REMARK. In what follows, we often use the notation σ_m instead of σ_{Δ_m} for a typographical reason.

Note that the Lie algebra $\mathfrak{g}$ of G is an abelian FL-group. For sufficiently small $\varepsilon>0$, consider a C^1-hair $\lambda\in H^1_{\varepsilon,J}(\mathfrak{g})$. Then, $\xi\circ\lambda$ is a C^1-hair at e defined on $[0, \varepsilon]\times J$. Set

$$\xi\circ\lambda_r(s, t)=\xi\circ\lambda(rs, t)\quad (0\leqq r\leqq 1) .$$

Then, obviously $\lim_{r\to 0}\xi\circ\lambda_r=e$ with respect to $\tilde{\rho}$.

COROLLARY 3.8. *Let G be a regular Fréchet-Lie group. Notations being as above, $\lim_{r\to 0}\prod_a^t(\xi\circ\lambda_r, d\tau)$ converges to e uniformly on J.*

PROOF. If it does not, there must exist a sequence (t_k, r_k) such that $t_k\in J$, $\lim_{k\to\infty}r_k=0$ but $\rho(\prod_a^{t_k}(\xi\circ\lambda_{r_k}, d\tau), e)\geqq\delta(>0)$ for some δ. Since J is compact, one may suppose that $\{t_k\}$ converges to $t\in J$. Hence, this contradicts the continuity of Φ in the above lemma. □

Finally, we shall remark the following:

LEMMA 3.9. *Every Fréchet space* E *is a regular Fréchet-Lie group.*

PROOF. Let $\{(h_n, \Delta_n)\}$ be a sequence in $\mathcal{S}_{\varepsilon,J}(\mathrm{E})$ such that $\lim_{n\to\infty}|\Delta_n|=0$ and $\{h_n\}$ converges to $h\in H^1_{\varepsilon,J}(\mathrm{E})$ with respect to $\tilde{\rho}$. We set

$$v_n(s, t)=\frac{\partial h_n}{\partial s}(s, t) , \quad v(s, t)=\frac{\partial h}{\partial s}(s, t) .$$

Then, by (11)

$$\begin{aligned}\prod_a^t (h_n, \Delta_n) &= \int_0^{t-t_k} v_n(\theta, t_k)d\theta + \sum_{i=0}^{k-1}\int_0^{\Delta t_{i+1}} v_n(\theta, t_i)d\theta \\ &= \int_a^t v_n(0, t)dt + \int_0^{t-t_k}(v_n(\theta, t_k) - v_n(0, t_k))d\theta \\ &\quad + \sum_{i=0}^{k-1}\int_0^{\Delta t_{i+1}}(v_n(\theta, t_i) - v_n(0, t_i))d\theta ,\end{aligned}$$

where Δ_n: $a=t_0<t_1<\cdots<t_{m_n}=b$, $t\in[t_k, t_{k+1})$. Note that

$$\lim_{n\to\infty}\int_a^t v_n(0, t)dt = \int_a^t v(0, t)dt .$$

Now, remark that the topology of E can be given by countably many semi-norms. Let $|\ \ |_\alpha$ be any one of them. Then, we have

$$\begin{aligned}\left|\prod_a^t (h_n, \Delta_n) - \int_a^t v(0, t)\,dt\right|_\alpha &\leq \int_a^t |v_n(0, t) - v(0, t)|_\alpha dt \\ &\quad + (b-a)\max_{\substack{t\in J\\ 0\leq\theta\leq|\Delta_n|}} |v_n(\theta, t) - v_n(0, t)|_\alpha .\end{aligned}$$

Hence, we see $\lim_{n\to\infty}\prod_a^t(h_n, \Delta_n) = \int_a^t v(0, t)dt$ uniformly in $t\in J$. □

§ 4. The first fundamental theorem.

The goal of this section is as follows:

THEOREM 4.1 (First fundamental theorem). *Suppose G is a regular Fréchet-Lie group. For a C^1-hair h at e on $[0, \varepsilon]\times J$, the product integral $g(t)=\prod_a^t(h, d\tau)$ is C^1 in t and satisfies the equation*

$$\frac{dg(t)}{dt} = u(t)\cdot g(t) , \quad g(a) = e ,$$

where $u(t)=(\partial h/\partial s)(0, t)$.

REMARK. By Lemma 2.5, $\prod_a^t(h, d\tau)$ depends only on $u(t)=(\partial h/\partial s)(0, t)$, hence $\prod_a^t(h, d\tau)$ is denoted often by $\prod_a^t(1+u(\tau))d\tau$.

Now, let Δ_n: $a<a+(1/n)(b-a)<a+(2/n)(b-a)<\cdots<a+(n/n)(b-a)=b$, be a division of $J=[a, b]$. By $(\sigma_n(h), \Delta_n)$ we denote the step function defined by (10) (see also Remark after Lemma 3.7) for an arbitrarily fixed $h\in H^1_{\varepsilon,J}(G)$.

Recall that $\prod_a^t(h, d\tau)$ is continuous in t by Lemma 3.7 and that

$\prod_a^t(\sigma_n(h), \Delta_n)$ converges uniformly to $\prod_a^t(h, d\tau)$, $t \in J$. Then, we easily obtain the following:

LEMMA 4.2. *For any open neighborhood $\tilde{U}$ of e in G, there are $\delta > 0$ and n_0 such that if $n \geqq n_0$ then $\prod_a^t(\sigma_n(h), \Delta_n)$ and $\prod_a^t(h, d\tau)$ are both contained in $\tilde{U}$ for every $t \in J_\delta$, where $J_\delta = [a, a+\delta]$.*

Now suppose $\xi: U \to \tilde{U}$ is a local coordinate system at e in G. By the above lemma one may assume that $\prod_a^t(\sigma_n(h), \Delta_n) \in \tilde{U}$, $\prod_a^t(h, d\tau) \in \tilde{U}$, for large n and $t \in J_0 = [a, b']$, if we choose b' such that $b' - a$ is small.

Let $\eta(u, v) = \xi^{-1}(\xi(u)\xi(v))$. This map η is defined for all pairs u, v in U such that $\xi(u)\xi(v) \in \tilde{U}$. For a small $\varepsilon > 0$ and *a* C^1-hair $h \in H^1_{\varepsilon, J}(G)$, set

$$\lambda(s, t) = \xi^{-1} \circ h(s, t) .$$

We see easily that $\lambda(0, t) \equiv 0$, $(\partial\lambda/\partial s)(0, t) = u(t)$, where $u(t) = (\partial h/\partial s)(0, t)$.

LEMMA 4.3. *Given $u(a) \in \mathfrak{g}$ and any convex neighborhood V of 0 in $\mathfrak{g}$, there is a convex neighborhood V_1 of 0 which is contained in a coordinate neighborhood U such that*

$$((d\eta_w)_v - \mathrm{Id})u' \in V , \quad \textit{for} \quad w, v \in V_1,\ u' \in u(a) + V_1 ,$$

where we set $\eta_w(v) = \eta(v, w)$, $u(a) + V_1 = \{u(a) + v_1;\ v_1 \in V_1\}$.

Proof is easy, because $(d\eta_0)_0 = \mathrm{Id}$ and $(d\eta_w)_v u'$ is continuous in (w, v, u'). □

By Lemma 4.2, there are $\delta > 0$ and n_0 such that $\xi^{-1} \circ \prod_a^t(h, d\tau) \in V_1$ for $t \in J_\delta$, $J_\delta = [a, a+\delta]$ and that $\xi^{-1} \circ \prod_a^t(\sigma_n(h), \Delta_n) \in V_1$ for $n \geqq n_0$, $t \in J_\delta$. Moreover, since $\lambda(s, t)$ is continuous and $\lambda(0, t) \equiv 0$, one may assume that $\lambda(s, t) \in V_1$ for $t \in J_\delta$, $s \in [0, |\Delta_{n_0}|]$, $|\Delta_{n_0}| = (b-a)/n_0$.

Now, set $w_n(t) = \xi^{-1} \circ \prod_a^t(\sigma_n(h), \Delta_n)$, $w(t) = \xi^{-1} \circ \prod_a^t(h, d\tau)$. For simplicity we set also $t_i = (i(b-a)/n) + a$. Then we have

$$\tag{12} \begin{cases} w_n(t) = \eta(\lambda(t - t_k, t_k), w_n(t_k)) , & t \in [t_k, t_{k+1}) , \\ w_n(t_{i+1}) = \eta(\lambda(|\Delta_n|, t_i), w_n(t_i)) , & 0 \leqq i \leqq k-1 , \end{cases}$$

where $|\Delta_n| = (b-a)/n$. Hence we get

$$\begin{cases} w_n(t) = \displaystyle\int_0^1 (d\eta_{w_n(t_k)})_{s\lambda(t-t_k, t_k)} \lambda(t - t_k, t_k) ds + w_n(t_k) , & t \in [t_k, t_{k+1}) , \\ w_n(t_{i+1}) = \displaystyle\int_0^1 (d\eta_{w_n(t_i)})_{s\lambda(|\Delta_n|, t_i)} \lambda(|\Delta_n|, t_i) ds + w_n(t_i) , & 0 \leqq i \leqq k-1 . \end{cases}$$

Therefore, for $t \in [t_k, t_{k+1}]$,

$$
\begin{aligned}
(13)\qquad w_n(t) &= \int_0^1 (d\eta_{w_n(t_k)})_{s\lambda(t-t_k,t_k)}\lambda(t-t_k,\, t_k)ds \\
&\quad + \sum_{i=0}^{k-1}\int_0^1 (d\eta_{w_n(t_i)})_{s\lambda(|\Delta_n|,t_i)}\lambda(|\Delta_n|,\, t_i)ds \\
&= \lambda(t-t_k,\, t_k) + \sum_{i=0}^{k-1}\lambda(|\Delta_n|,\, t_i) \\
&\quad + \int_0^1 [(d\eta_{w_n(t_k)})_{s\lambda(t-t_k,t_k)} - \mathrm{Id}]\lambda(t-t_k,\, t_k)ds \\
&\quad + \sum_{i=0}^{k-1}\int_0^1 [(d\eta_{w_n(t_i)})_{s\lambda(|\Delta_n|,t_i)} - \mathrm{Id}]\lambda(|\Delta_n|,\, t_i)ds\ .
\end{aligned}
$$

LEMMA 4.4. *For a sufficiently large n and for a sufficiently small $\delta_0>0$, if $t\in J_{\delta_0}$, then*

$$
\begin{aligned}
&\frac{1}{t-t_k}\lambda(t-t_k,\, t_k)\in u(a)+V_1\ , \quad t\in[t_k,\, t_{k+1})\ , \\
&\frac{1}{|\Delta_n|}\lambda(|\Delta_n|,\, t_i)\in u(a)+V_1\ , \quad 0\leqq i\leqq k-1\ ,
\end{aligned}
$$

where V_1 is the convex neighborhood given in Lemma 4.3.

PROOF. Note that $\lim_{s\to 0}(1/s)\lambda(s,\, t)=u(t)$. Using the mean value theorem (1), we see easily that the above convergence is uniform in $t\in J_\delta$. Hence, there is $\delta'>0$ such that $(1/s)\lambda(s,\, t)\in u(t)+(1/2)V_1$ for $s\leqq\delta'$, where $aV_1=\{au';\, u'\in V_1\}$. As $u(t)$ is continuous, we get the desired result. □

Recall that $w_n(t)=\xi^{-1}\circ\prod_a^t(\sigma_n(h),\, \Delta_n)$ for $t\in J_\delta$, $n\geqq n_0$, and $\lambda(s,\, t)\in V_1$ for $(s,\, t)\in[0,\, |\Delta_{n_0}|]\times J_\delta$. It follows from Lemma 4.3 and Lemma 4.4 that

$$
\begin{aligned}
&[(d\eta_{w_n(t_k)})_{s\lambda(t-t_k,t_k)} - \mathrm{Id}]\lambda(t-t_k,\, t_k)\in(t-t_k)V\subset|\Delta_n|V\ , \\
&[(d\eta_{w_n(t_i)})_{s\lambda(|\Delta_n|,t_i)} - \mathrm{Id}]\lambda(|\Delta_n|,\, t_i)\in|\Delta_n|V\ .
\end{aligned}
$$

Since V is convex, we get from (13) that for any $t\in J_{\delta_0}$,

$$
(14)\qquad w_n(t)-\Big\{\lambda(t-t_k,\, t_k)+\sum_{i=0}^{k-1}\lambda(|\Delta_n|,\, t_i)\Big\}\in(k+1)|\Delta_n|V\ ,
$$

where $t\in[t_k,\, t_{k+1})$.

Keep in mind that V is an arbitrary convex neighborhood of 0 in $\mathfrak{g}$.

LEMMA 4.5. $\lambda(t-t_k,\, t_k)+\sum_{i=0}^{k-1}\lambda(|\Delta_n|,\, t_i)\in(t-a)u(a)+(k+1)|\Delta_n|V_1$, $t\in[t_k,\, t_{k+1})$.

PROOF. By Lemma 4.4, we see for $t\in[t_k,\, t_{k+1})$,

$$\lambda(t-t_k, t_k) \in (t-t_k)(u(a)+V_1) \subset (t-t_k)u(a)+|\Delta_n| V_1 ,$$
$$\lambda(|\Delta_n|, t_i) \in |\Delta_n|(u(a)+V_1)$$

Thus we get the desired one by using the convexity of V_1. □

Using Lemma 4.5 and (14), we obtain that

$$w_n(t) \in (t-a)u(a)+(k+1)|\Delta_n| V_1+(k+1)|\Delta_n| V,\ t \in J_{\delta_0} .$$

Note that $t \in [t_k, t_{k+1})$, hence $k|\Delta_n| \leqq t-a<(k+1)|\Delta_n|$. Thus,

$$w_n(t) \in (t-a)(u(a)+V_1+V)+|\Delta_n|(V_1+V) .$$

Hence recalling $w(t)=\lim_{n\to\infty} w_n(t)$, we get

$$\lim_{t\to a+} \frac{1}{t-a} w(t) \in u(a)+(V_1+V)^- ,$$

where $(V_1+V)^-$ is the closure of V_1+V. Remark that V and V_1 can be chosen arbitrarily small. Therefore, the above result shows that

$$\lim_{t\to a+} \frac{1}{t-a} w(t)=u(a) .$$

Namely,

$$\lim_{t\to a+} \frac{1}{t-a} \xi^{-1} \circ \prod_a^t (h, d\tau)=\frac{\partial(\xi^{-1}\circ h)}{\partial s}(0, a)=\frac{\partial h}{\partial s}(0, a) .$$

This means that $w(t)$ is differentiable at a from the right hand side. Since $\prod_t^s(h, d\tau)\cdot\prod_a^t(h, d\tau)=\prod_a^s(h, d\tau)$, the above result shows also that $w(t)=\xi^{-1}\circ\prod_a^t(h, d\tau)$ is differentiable from the right hand side at every $t \in [a, a+\delta_0]$ and the derivative $D^+w(t)$ is given by

$$D^+w(t)=(d\eta_{w(t)})_0 u(t) .$$

As $w(t)$ and $u(t)$ are continuous in $t \in [a, a+\delta_0]$, so is $(d\eta_{w(t)})_0 u(t)$. Set $\bar{w}(t)=\int_0^t (d\eta_{w(s)})_0 u(s) ds$. Then $\bar{w}(t)$ is C^1 and $D^+(w(t)-\bar{w}(t)) \equiv 0$, $w(a)-\bar{w}(a)=0$.

LEMMA 4.6. *Let $v(t)$ be a continuous mapping from $[a, a+\delta_0]$ into $\mathfrak{g}$ such that $v(t)$ is differentiable from the right hand side and $D^+v \equiv 0$ on $[a, a+\delta_0]$. Suppose $v(a)=0$. Then $v \equiv 0$.*

PROOF. This fact is well-known for $\mathbf{R}$-valued functions. Let $\kappa: \mathfrak{g} \to \mathbf{R}$ be an arbitrary continuous linear mapping. Then we have $\kappa v(a)=0$ and $D^+\kappa v(t)=\kappa\circ D^+v(t) \equiv 0$. Thus, $\kappa v \equiv 0$ for every κ. It follows $v \equiv 0$ because $\mathfrak{g}$ is assumed to be locally convex. □

PROOF OF THEOREM 4.1. The above lemma shows that $w(t)=\xi^{-1}\circ\prod_a^t(h, d\tau)$ is C^1 for $t\in[a, a+\delta]$, where δ is sufficiently small, and $(d/dt)w(t)=(d\eta_{w(t)})_0 u(t)$. This implies that

$$\frac{d}{dt}\prod_a^t(h, d\tau)=u(t)\cdot\prod_a^t(h, d\tau)\ , \quad t\in[a, a+\delta]\ .$$

Recall again that $\prod_s^t(h, d\tau)\cdot\prod_a^s(h, d\tau)=\prod_a^t(h, d\tau)$. This relation gives us that $\prod_a^t(h, d\tau)$ is C^1 for all $t\in[a, b]$ and satisfies the above equation. Theorem 4.1 is thereby proved. □

§ 5. The second fundamental theorem.

Suppose G is a regular Fréchet-Lie group with the Lie algebra $\mathfrak{g}$. As G is a C^∞ Fréchet manifold, there is a C^∞ local coordinate system $\xi: U\to\tilde{U}$ at e such that $\xi(0)=e$. Let u be a continuous mapping of the closed interval $I=[0, 1]$ into $\mathfrak{g}$. If we set $h(s, t)=\xi(su(t))$ for a sufficiently small s, then it is clear that $h\in H^1_{\varepsilon, I}(G)$ for some $\varepsilon>0$. By Theorem 4.1 $g(t)=\prod_0^t(h, d\tau)$ satisfies the equation

$$\frac{d}{dt}g(t)=u(t)\cdot g(t)\ , \quad g(0)=e\ .$$

Hence by the uniqueness theorem (cf. Lemma 2.5) we denote this product integral by $\prod_a^t(1+u(\tau))d\tau$, for it depends only on $u(t)$.

Let $C^0(I, \mathfrak{g})$ be the linear space of all C^0 mappings of I into $\mathfrak{g}$. $C^0(I, \mathfrak{g})$ is a Fréchet-Lie group (Lemma 3.9). We denote by $C_e^1(I, G)$ the totality of C^1 mappings $c: I\to G$ such that $c(0)=e$. $C_e^1(I, G)$ is a group under the pointwise group-operations, and a topological group under the C^1 uniform topology. Moreover, by Lemma 1.2 $C_e^1(I, G)$ is a C^∞ Fréchet manifold. The goal of this section is as follows:

THEOREM 5.1 (Second fundamental theorem). *Notations and assumptions being as above $C_e^1(I, G)$ is an FL-group, and the mapping $\mathscr{F}: C^0(I, \mathfrak{g})\to C_e^1(I, G)$ defined by $\mathscr{F}(u)(t)=\prod_0^t(1+u(t))d\tau$ is a C^∞-diffeomorphism.*

First of all, we shall remark the following:

LEMMA 5.2. *Suppose G is an FL-group. Then $C_e^1(I, G)$ is an FL-group with the Lie algebra $C_0^1(I, \mathfrak{g})$ which is the totality of C^1 mappings $u: I\to\mathfrak{g}$ such that $u(0)=0$.*

PROOF. It is easy to see that $C_0^1(I, \mathfrak{g})$ is a Fréchet space under the C^1 uniform topology (cf. Lemma 1.2). Let $\xi: U\to\tilde{U}$ be a local coordinate system at e of G. We set

$$\Sigma=\{u \in C_0^1(I, \mathfrak{g});\ u(t) \in U \quad \text{for all} \quad t \in I\}$$
$$\tilde{\Sigma}=\{c \in C_e^1(I, G);\ c(t) \in \tilde{U} \quad \text{for all} \quad t \in I\} .$$

Define $\hat{\xi}\colon \Sigma \to \tilde{\Sigma}$ by $\hat{\xi}(u)(t)=\xi(u(t))$. We set $\hat{\eta}(u, v)=\hat{\xi}^{-1}(\hat{\xi}(u)\hat{\xi}(v))$, $\hat{\iota}(u)=\hat{\xi}^{-1}(\hat{\xi}(u)^{-1})$. These are well-defined if u, v are contained in a small neighborhood of 0, and these are C^∞ because $\hat{\eta}(u, v)(t)=\eta(u(t), v(t))$, $\hat{\iota}(u)(t)=\iota(u(t))$ and η, ι are C^∞ (cf. the proof of Lemma 1.2.). For every $g \in C_e^1(I, G)$, we set

$$\hat{A}_g(u)=\hat{\xi}^{-1}(g\hat{\xi}(u)g^{-1}) .$$

Then, $\hat{A}_g(u)(t)=\xi^{-1}(g(t)\xi(u(t))g(t)^{-1})$, and hence it is C^∞ on a neighborhood of 0 in $C_0^1(I, \mathfrak{g})$. Thus, $C_e^1(I, \mathfrak{g})$ is an FL-group. □

REMARK. It will be proved in the next paper that if G is a regular Fréchet-Lie group, then $C_e^1(I, G)$ is a regular Fréchet-Lie group.

First we remark that the mapping $\mathscr{I}\colon C^0(I, \mathfrak{g}) \to C_e^1(I, G)$ is bijective. In fact, $\mathscr{I}$ is injective by the uniqueness theorem (Lemma 2.5). Also, $\mathscr{I}$ is invertible since the inverse mapping $\mathscr{I}^{-1}$ can be written as

$$\mathscr{I}^{-1}(g)(t)=\frac{dg}{dt}\cdot g(t)^{-1} , \quad g \in C_e^1(I, G) . \tag{15}$$

LEMMA 5.3. *Let G be a regular Fréchet-Lie group. Then the mapping $\mathscr{I}\colon C^0(I, \mathfrak{g}) \to C_e^1(I, G)$ is continuous.*

PROOF. For every $u \in C^0(I, \mathfrak{g})$, define $\sigma(u)(s, t)=\xi(su(t))$ for small s. Then, $\sigma(u) \in H_{\varepsilon,I}^1(G)$ for a sufficiently small $\varepsilon>0$. Note that there is a neighborhood W of u such that $\sigma(u') \in H_{\varepsilon,I}^1(G)$ for any $u' \in W$. It is obvious that $\sigma\colon W \to H_{\varepsilon,I}^1(G)$ is continuous, and hence by Lemma 3.7, $\mathscr{I}\colon C^0(I, \mathfrak{g}) \to C_e^0(I, G)$ is continuous.

Remark that $\mathscr{I}(u)(t)$ is C^1 in t and $(d/dt)\mathscr{I}(u)(t)=u(t)\cdot\mathscr{I}(u(t))$, hence $(d/dt)\mathscr{I}(u)$ depends continuously on u. Thus, $\mathscr{I}\colon C^0(I, \mathfrak{g}) \to C_e^1(I, G)$ is continuous. □

Now, let G be an FL-group. Then the tangent bundle T_G is a C^∞ Fréchet manifold modeled on $\mathfrak{g}\oplus\mathfrak{g}$. By Lemma 2.2, T_G can be regarded as an FL-group $\mathfrak{g}*G$. Let $C^0(I, T_G)$ be the space of all continuous mappings of I into T_G. By Lemma 1.2, $C^0(I, T_G)$ is a C^∞ Fréchet manifold. Let $\hat{d}\colon C_e^1(I, G) \to C^0(I, T_G)$ be a mapping defined by $\hat{d}(g)(t)=(dg/dt)(t) \in T_G$, for $g \in C_e^1(I, G)$. It is not hard to see that $\hat{d}$ is a C^∞-mapping.

LEMMA 5.4. *Suppose G is a regular Fréchet-Lie group. Then, the mapping* $\mathscr{I}^{-1}: C_e^1(I, G) \to C^0(I, \mathfrak{g})$ *is* C^∞.

PROOF. Let $\pi: T_G \to G$ be the projection. For every $w \in C^0(I, T_G)$, there is a C^0 mapping $v: I \to \mathfrak{G}$ such that $w(t)=v(t)\cdot \pi w(t)$. By this manner $C^0(I, T_G)$ is C^∞-diffeomorphic to $C^0(I, \mathfrak{g}*G)$. We denote this diffeomorphism by ψ. Note that $\mathfrak{g}*G=\mathfrak{g}\times G$ as Fréchet manifolds. Hence $C^0(I, \mathfrak{g}*G)= C^0(I, \mathfrak{g})\times C^0(I, G)$. Let ν be the projection of $C^0(I, \mathfrak{g})\times C^0(I, G)$ to the first component. Remark that $\mathscr{I}^{-1}(g)(t)=(dg/dt)(t)\cdot g(t)^{-1}$ and hence $\mathscr{I}^{-1}= \nu\circ\psi\circ\hat{d}$. Since ν, ψ, $\hat{d}$ are C^∞ we get the desired result. □

Combining Lemma 5.3 and Lemma 5.4, we get

COROLLARY 5.5. *Let G be a regular Fréchet-Lie group. Then the mapping* $\mathscr{I}: C^0(I, \mathfrak{g}) \to C_e^1(I, G)$ *is a homeomorphism.*

To prove Theorem 5.1 we must study about the differentiability of $\mathscr{I}$. Let $g \in C_e^1(I, G)$ and ω an element of the tangent space $T_g C_e^1(I, G)$ of $C_e^1(I, G)$ at g. ω is a C^1 mapping of I into T_G such that $\pi\omega(t)=g(t)$ and $\omega(0)=0$. Put $v(t)=\omega(t)\cdot g(t)^{-1}$. Then, v is a C^1 mapping of I into $\mathfrak{g}$ such that $v(0)=0$.

Lemma 5.6. *Notations and assumptions being as above. The derivative* $(d\mathscr{I}^{-1})_g\omega$, $\omega \in T_g C_e^1(I, G)$, *is given by*

$$((d\mathscr{I}^{-1})_g\omega)(t)=\frac{d}{dt}v(t)-[u(t), v(t)]\ ,$$

where $v(t)=\omega(t)\cdot g(t)^{-1}$ *and* $u(t)$ *is defined by* $(dg/dt)(t)=u(t)g(t)$.

PROOF. Let $\xi: U \to \tilde{U}$ be a local coordinate system of G at e. Then, we see that

$$\begin{aligned}((d\mathscr{I}^{-1})_g\omega)(t)&=\frac{\partial}{\partial s}\Big|_{s=0}\Big(\frac{\partial}{\partial t}\xi(sv(t))g(t)\Big)g(t)^{-1}\xi(sv(t))^{-1}\\ &=\frac{\partial}{\partial s}\Big|_{s=0}\Big(\frac{\partial}{\partial t}\xi(sv(t))g(t)\Big)g(t)^{-1}+\frac{\partial}{\partial s}\Big|_{s=0}u(t)\cdot\xi(sv(t))^{-1}\\ &=\frac{\partial}{\partial s}\Big|_{s=0}(d\xi)_{sv(t)}s\dot{v}(t)+\frac{\partial^2}{\partial s\partial\delta}\Big|_{\substack{s=0\\ \delta=0}}\xi(sv(t))\cdot\xi(\delta u(t))-\partial_{v(t)}u^*(t)\\ &=\dot{v}(t)+\partial_{u(t)}v^*(t)-\partial_{v(t)}u^*(t)\end{aligned}$$

where $u^*(t)$, $v^*(t)$ are right-invariant vector field on G such that $u^*(t)(g)=$

$u(t)\cdot g$, $v^*(t)(g)=v(t)\cdot g$ respectively. Remark that $(\partial_{u(t)\cdot g(t)}v^*(t))\cdot g(t)^{-1}=\partial_{u(t)}v^*(t)$. Hence by (8) in §2 we have the desired equality. □

Now, we consider the differential equation

$$\frac{d}{dt}v(t)-[u(t), v(t)]=w(t)\ , \quad v(0)=0\ , \tag{16}$$

for an arbitrarily given $w\in C^0(I, \mathfrak{g})$. By Lemma 2.6, we see that (16) has the unique solution

$$v(t)=\mathrm{Ad}(g(t))\int_0^t \mathrm{Ad}(g(s)^{-1})w(s)ds\ .$$

LEMMA 5.7. *Notations and assumptions being as above, the derivative* $(d\mathscr{I}^{-1})_g$: $T_gC_e^1(I, G)\to C^0(I, \mathfrak{g})$ *is a continuous linear isomorphism and* $(d\mathscr{I}^{-1})^{-1}$: $C_e^1(I, G)\times C^0(I, \mathfrak{g})\to T_{C_e^1(I,G)}$ *(the tangent bundle) defined by* $(d\mathscr{I}^{-1})^{-1}(g, w)=(d\mathscr{I}^{-1})_g^{-1}w$ *is continuous with respect to* (g, w).

PROOF. Using Lemma 5.6 and the uniqueness of the solution (16), we have easily the first statement. Remark that $\mathrm{Ad}(g(t))\int_0^t \mathrm{Ad}(g(s)^{-1})w(s)ds$ and its derivative in t is continuous with respect to (g, w). This implies the desired continuity. □

We globalize Lemma 1.1 and easily obtain the following from Lemma 5.4 and Corollary 5.5.

LEMMA 5.8. *Suppose for a while that* $\mathscr{I}: C^0(I, \mathfrak{g})\to C_e^1(I, G)$ *is differentiable at every point. Then,* $\mathscr{I}$ *is* C^∞ *diffeomorphism.*

By the above lemma, we have only to show the differentiability of $\mathscr{I}$ for the proof of Theorem 5.1.

For a sufficiently small $\varepsilon>0$, let $\lambda\in H_{\varepsilon,I}^1(\mathfrak{g})$. Then, $h(s, t)=\xi\circ\lambda(s, t)\in H_{\varepsilon,I}^1(G)$. What we shall prove at first is the following:

PROPOSITION 5.9. *Notations being as above, set* $h_r(s, t)=h(rs, t)$ $(0<r<1)$. *Then the function* $G(r, \lambda)$ *defined by*

$$G(r, \lambda)(t)=\begin{cases}\dfrac{1}{r}\xi^{-1}\circ\prod_0^t(h_r, d\tau)-\displaystyle\int_0^t\frac{\partial\lambda}{\partial s}(0, \tau)d\tau\ , & r\neq 0\\ 0\ , & r=0\ ,\end{cases}$$

is continuous on a neighborhood of $(0, \lambda)\in \boldsymbol{R}^+\times H_{\varepsilon,I}^1(\mathfrak{g})$.

PROOF. Remark at first that $\xi^{-1}\circ\prod_0^t(h_0, d\tau)=0$. Hence for every fixed λ, there is a neighborhood $\hat{W}$ of λ in $H_{\varepsilon,I}^1(\mathfrak{g})$ such that $\xi\circ\mu\in H_{\varepsilon,I}^1(G)$ for all $\mu\in\hat{W}$, and there is $\delta>0$ such that $\xi^{-1}\circ\prod_0^t(\xi\circ\mu_r, d\tau)$, $\mu_r(s, t)=\mu(rs, t)$ (cf. Corollary 3.8), is well-defined for all $\mu\in\hat{W}$, $r\in[0, \delta]$ and $t\in I$.

Let $\{\Delta_n\}$ be a sequence of division I such that $\lim_{n\to\infty}|\Delta_n|=0$ and $|\Delta_n|<\varepsilon$ for all n. Recall that $\prod_0^t(\xi\circ\mu_r, d\tau)=\lim_{n\to\infty}\prod_0^t(\sigma_n(\xi\circ\mu_r), \Delta_n)$. By the same computation as in (12) and (13), we have the following, by setting $w_{n,r}(t)=\xi^{-1}\circ\prod_0^t(\sigma_n(\xi\circ\mu_r), \Delta_n)$:

$$\begin{aligned}w_{n,r}(t)=&\mu((t-t_k)r, t_k)+\sum_{i=0}^{k-1}\mu((\Delta t_{i+1})r, t_i)\\&+\int_0^1[(d\eta_{w_{n,r}(t_k)})_{s\mu((t-t_k)r,t_k)}-\mathrm{Id}]\mu((t-t_k)r, t_k)ds\\&+\sum_{i=0}^{k-1}\int_0^1[(d\eta_{w_{n,r}(t_i)})_{s\mu((\Delta t_{i+1})r,t_i)}-\mathrm{Id}]\mu((\Delta t_{i+1})r, t_i)ds\ ,\end{aligned}$$

where $t\in[t_k, t_{k+1})$, $\Delta t_{i+1}=t_{i+1}-t_i$. Remark that

$$((d\eta_w)_v-\mathrm{Id})u=(d\eta_w)_vu-(d\eta_0)_vu=\int_0^1(d_2d_1\eta)_{(v,\tau w)}(u, w)d\tau\ .$$

We have the following by setting

$$\mu_{r,k}=\mu((t-t_k)r, t_k)\ ,\qquad \mu_{r,i}=\mu((\Delta t_{i+1})r, t_i)\ ,$$

$$\begin{aligned}w_{n,r}(t)=&\int_0^1\frac{\partial\mu}{\partial s}(sr(t-t_k), t_k)ds\cdot(t-t_k)r\\&+\sum_{i=0}^{k-1}\int_0^1\frac{\partial\mu}{\partial s}(sr(\Delta t_{i+1}), t_i)ds\cdot(\Delta t_{i+1})r\\&+\int_0^1\int_0^1(d_2d_1\eta)_{(s\mu_{r,k},\nu w_{n,r}(t_k))}(\mu_{r,k}, w_{n,r}(t_k))d\nu ds\\&+\sum_{i=0}^{k-1}\int_0^1\int_0^1(d_2d_1\eta)_{(s\mu_{r,i},\nu w_{n,r}(t_i))}(\mu_{r,i}, w_{n,r}(t_i))d\nu ds\ .\end{aligned}$$

Thus, putting $w_r(t)=\lim_{n\to\infty}w_{n,r}(t)$, we have that

$$\frac{1}{r}w_r(t)=\int_0^t\frac{\partial\mu}{\partial s}(0, \tau)d\tau+\int_0^t\int_0^1(d_2d_1\eta)_{(0,\nu w_r(t))}\Big(\frac{\partial\mu}{\partial s}(0, \tau), w(\tau)\Big)d\nu d\tau\ .$$

Therefore, if $r\neq0$, then

$$G(r, \mu)(t)=\int_0^t\int_0^1(d_2d_1\eta)_{(0,\nu w_r(\tau))}\Big(\frac{\partial\mu}{\partial s}(0, \tau), w_r(\tau)\Big)d\nu d\tau\ .$$

Recall that $w_r(t)=\xi^{-1}\circ\prod_0^t(h_r, d\tau)$, and that $\lim_{r\to0}w_r(t)=0$ by Corollary 3.8. Thus, we get that $G(r, \mu)$ is continuous on a neighborhood of $(0, \lambda)$. □

COROLLARY 5.10. *$\mathscr{I}: C^0(I, \mathfrak{g}) \to C^1_e(I, G)$ is differentiable at* 0.

PROOF. Apply the above result to $\mu(s, t)=su(t)$. Then, we see that

$$G(r, \mu)(t)=\begin{cases} \dfrac{1}{r}\xi^{-1}\circ\prod_0^t(1+ru(\tau))d\tau-\int_0^t u(\tau)d\tau\,, & r\neq 0\,, \\ 0\,, & r=0\,, \end{cases}$$

is continuous on a neighborhood of (0, 0) in $\mathbf{R}^+\times C^0(I, \mathfrak{g})$, where $\mathbf{R}^+=\{r\geqq 0\}$. This implies that $\mathscr{I}$ is differentiable at $u=0$ and $((d\mathscr{I})_0 u)(t)=\int_0^t u(\tau)d\tau$. □

To prove the differentiability at $u\in C^0(I, \mathfrak{g})$, we set

$$\xi(s(u(t)+v(t)))=\xi(\hat{v}(s, t))\xi(su(t))\,,$$

for a sufficiently small s, say $0\leqq s\leqq \varepsilon$. Denote

$$\hat{h}(s, t)=\xi\circ\hat{v}(s, t),\ h(s, t)=\xi(su(t)),\ \hat{h}h(s, t)=\xi(\hat{v}(s, t))\xi(su(t))\,.$$

Let $\{\Delta_n\}$ be a sequence of division of I such that $|\Delta_n|<\varepsilon$ for all n. Then putting $A(g, h)=ghg^{-1}$, we have for $t\in[t_k, t_{k+1})$,

$$\begin{aligned}(17)\quad &\prod_0^t(\sigma_n(\hat{h}h), \Delta_n)\\ &=\hat{h}(t-t_k, t_k)h(t-t_k, t_k)\hat{h}(\Delta t_k, t_{k-1})h(\Delta t_k, t_{k-1})\cdots\hat{h}(\Delta t_1, t_0)h(\Delta t_1, t_0)\\ &=\hat{h}(t-t_k, t_k)A\Big(\prod_{t_k}^t(\sigma_n(h), \Delta_n), \hat{h}(\Delta t_k, t_{k-1})\Big)\times\cdots\\ &\quad\cdots\times A\Big(\prod_{t_i}^t(\sigma_n(h), \Delta_n), \hat{h}(\Delta_{t_i}, t_{i-1})\Big)\times\cdots\\ &\quad\cdots\times A\Big(\prod_{t_1}^t(\sigma_n(h), \Delta_n), \hat{h}(\Delta t_1, t_0)\Big)\times\prod_0^t(\sigma_n(h), \Delta_n)\,.\end{aligned}$$

Consider a step function $\tilde{h}_n$ by

$$\tilde{h}_n(s, t')=\begin{cases}\hat{h}(s, t_k)\,, & t'\in[t_k, t_{k+1})\\ A\Big(\prod_{t_i}^t(\sigma_n(h), \Delta_n), \hat{h}(s, t_{i-1})\Big)\,, & t'\in[t_{i-1}, t_i),\ 1\leqq i\leqq k\end{cases}$$

defined on $[0, \varepsilon]\times[0, t]$. Then, obviously,

$$\prod_0^t(\sigma_n(\hat{h}h), \Delta_n)=\prod_0^t(\tilde{h}_n, \Delta_n)\prod_0^t(\sigma_n(h), \Delta_n)\,.$$

The following lemma is easy to prove.

LEMMA 5.11. *$\{\tilde{h}_n\}$ converges to a C^1-hair $\tilde{h} \in H^1_{\varepsilon,[0,t]}(G)$ with respect to $\tilde{\rho}$, where*

$$\tilde{h}(s, t') = A\Big(\prod_{t'}^{t}(h, d\tau), \hat{h}(s, t')\Big) = A\Big(\prod_{0}^{t}(h, d\tau), A\Big(\prod_{0}^{t'}(h, d\tau)^{-1}, \hat{h}(s, t')\Big)\Big)$$

PROOF OF THEOREM 5.1. Suppose s is sufficiently small. Then $\tilde{\mu}(s, t) = \xi^{-1}\circ\tilde{h}(s, t)$ is well-defined, and

$$\frac{\partial\tilde{\mu}}{\partial s}(0, t') = \mathrm{Ad}\Big(\prod_0^t(h, d\tau)\Big)\mathrm{Ad}\Big(\prod_0^{t'}(h, d\tau)^{-1}\Big)\frac{\partial\hat{v}}{\partial s}(0, t') \qquad (18)$$
$$= \mathrm{Ad}(g(t))\mathrm{Ad}(g(t')^{-1})v(t') \,,$$

where $g(t) = \prod_0^t(h, d\tau) = \prod_0^t(1+u(\tau))d\tau$. Remark that

$$\prod_0^t(1+u(\tau)+v(\tau))\,d\tau\Big\{\prod_0^t(1+u(\tau))d\tau\Big\}^{-1} = \prod_0^t(\tilde{h}, d\tau)\,. \qquad (19)$$

So, set

$$\tilde{h}_r(s, \tau) = A_{g(t)}A_{g(\tau)^{-1}}\hat{h}(rs, \tau)(= A_{g(t)}A_{g(\tau)^{-1}}\xi(\hat{v}(rs, t)))\,,$$

and define $G(r, v)$ by

$$G(r, v)(t) = \begin{cases} \dfrac{1}{r}\xi^{-1}\circ\prod_0^t(\tilde{h}_r, d\tau) - \mathrm{Ad}(g(t))\displaystyle\int_0^t \mathrm{Ad}(g(\tau)^{-1})v(\tau)d\tau\,, & r\neq 0\,, \\ 0\,, & r = 0\,. \end{cases}$$

By (18) and Proposition 5.9, we see that $G(r, v)$ is continuous on a neighborhood of (0, 0). Using (19) we see that $\mathscr{I}: C^0(I, \mathfrak{g}) \to C^1_e(I, G)$ is differentiable at $u \in C^0(I, \mathfrak{g})$ and $((d\mathscr{I})_u v)(t) = \mathrm{Ad}(g(t))\int_0^t \mathrm{Ad}(g(t)^{-1})v(t)dt$. By Lemma 5.8, we obtain that $\mathscr{I}$ is a C^∞ diffeomorphism. □

Now, let G be a regular Fréchet-Lie group with the Lie algebra $\mathfrak{g}$. The following is an immediate conclusion from Theorem 5.1:

COROLLARY 5.12. *There is a C^∞ mapping* $\exp: \mathfrak{g} \to G$ *such that for each $u \in \mathfrak{g}$ $\{\exp tu\}_{t\in\mathbf{R}}$ is a C^∞ one parameter subgroup of G.*

PROOF. For $u \in \mathfrak{g}$, we define $\exp tu$ by $\prod_0^t(1+u)d\tau$. Then, $g(t) = \exp tu$ satisfies the equation $(dg/dt)(t) = u\cdot g(t)$, $g(0) = e$. Since $\exp tu\cdot\exp su$, $\exp(t+s)u$ satisfy the same differential equation, we see that $\exp tu\cdot\exp su = \exp(t+s)u$ by using Lemma 2.5. Hence $\exp tu$ is a one parameter subgroup of G. As $((d/dt)\exp tu)(\exp -tu) \equiv u$, we see that $\exp tu$ is C^∞ with respect to t. By the second fundamental theorem, $\exp u$ depends smoothly on u. □

REMARK. In the next paper, we shall prove that regular Fréchet-Lie group have the properties (L1)~(L3) mentioned in the introduction.

§6. Strong *ILB*-Lie groups are regular Fréchet-Lie groups.

In this section, we shall show that strong *ILB*-Lie groups defined in [8] or [9] are regular Fréchet-Lie groups. By this result, we have a lot of concrete examples of regular Fréchet-Lie groups. Roughly speaking, a strong *ILB*-Lie group is a *Lie* group, which is modeled on a system $\{\mathbf{E}, E^k, k\in N(d)\}$ called an *ILB*-chain instead of a single vector space.

DEFINITION 6.1. A system $\{\mathbf{E}, E^k, k\in N(d)\}$ is called an *ILB-chain* if the following are satisfied:

(i) $N(d)$ is the set of integers such that $k\geqq d$.

(ii) E^k is a Banach space such that $E^{k+1}\subset E^k$. The inclusion is continuous and has a dense image.

(iii) $\mathbf{E}=\cap E^k$ with the inverse limit topology.

If all E^k's are Hilbert spaces, then we call $\{\mathbf{E}, E^k, k\in N(d)\}$ an *ILH-chain*.

DEFINITION 6.2. A group G will be called a *strong ILB-Lie group* modeled on an *ILB*-chain $\{\mathbf{E}, E^k, k\in N(d)\}$, if the following conditions (N, 1)~(N, 7) are satisfied:

(N, 1) There are an open convex neighborhood U of 0 in E^d and a bijective mapping ξ of $U\cap\mathbf{E}$ onto a subset $\tilde{U}$ of G such that $\xi(0)=e$.

(N, 2) There is an open convex neighborhood V of 0 in E^d such that

$$\xi(V\cap\mathbf{E})^2\subset\xi(U\cap\mathbf{E})\,,\quad \xi(V\cap\mathbf{E})^{-1}\subset\xi(U\cap\mathbf{E})\,.$$

(N, 3) Set $\eta(u, v)=\xi^{-1}(\xi(u)\xi(v))$ for $u, v\in V\cap\mathbf{E}$. Then $\eta\colon V\cap\mathbf{E}\times V\cap\mathbf{E}\to U\cap\mathbf{E}$ can be extended to a continuous mapping of $V\cap E^k\times V\cap E^k$ into $U\cap E^k$ for every $k\in N(d)$. (The extended mapping will be denoted by the same notation.)

(N, 4) Set $\eta_v(u)=\eta(u, v)$ for $v\in V\cap E^k$. Then $\eta_v\colon V\cap E^k\to U\cap E^k$ is a C^∞ mapping.

(N, 5) Set $\theta(w, u, v)=(d\eta_v)_u w$. For every integer $l\geqq 0$, $k\in N(d)$, θ can be extended to a C^l-mapping of $E^{k+l}\times(V\cap E^{k+l})\times(V\cap E^k)$ into E^k.

(N, 6) Define $\iota\colon V\cap\mathbf{E}\to U\cap\mathbf{E}$ by $\iota(u)=\xi^{-1}(\xi(u)^{-1})$. Then ι can be extended to a continuous mapping of $V\cap E^k$ into $U\cap E^k$ for every $k\in N(d)$.

(N, 7) For every $g\in G$, there is a neighborhood W of 0 in E^d such that $g^{-1}\xi(W\cap \mathbf{E})g\subset\xi(U\cap \mathbf{E})$, and the mapping $u\to\xi^{-1}(g^{-1}\xi(u)g)$ can be extended to a C^∞ mapping of $W\cap E^k$ into $U\cap E^k$ for every $k\in N(d)$.

REMARK 1. The above conditions are little weaker than those in [8] in the statement. However, it is not hard to see that those are in fact the same conditions. (See [9], pp. 52–59.)

REMARK 2. If the model space $\{\mathbf{E}, E^k, k\in N(d)\}$ is an *ILH*-chain, we call G a *strong ILH-Lie group*. If $\mathbf{E}=\cdots=E^k=\cdots=E^d$, and E^d is a Banach (resp. Hilbert) space, then G will be called a Banach (resp. Hilbert) Lie group. Of course, if $\dim E^d<\infty$, then G is a usual Lie group.

Now, we summarize several properties of strong *ILB*-Lie groups which will be used later.

LEMMA 6.3. *Let $\mathfrak{N}^k$ be a basis of neighborhoods of* 0 *in E^k such that $W\subset V\cap E^k$ for every $W\in\mathfrak{N}^k$. Then $\{\xi(W\cap \mathbf{E}); W\in\mathfrak{N}^k\}$ satisfies the axioms of neighborhoods of the identity e of a topological group. Hence G turns out to be a topological group under this topology, which will be denoted by* $(G, \mathfrak{N}^k)$. (See [9] 1.2 Proposition, or [8] 1.2.4 Lemma.)

Let G^k be the completion of $(G, \mathfrak{N}^k)$ by the right-uniform structure. In general, G^k is only a topological semi-group. However, in our case the property (N, 6) ensures that G^k is a topological group.

THEOREM 6.4. *Notations being as above, $\{G^k, k\in N(d)\}$ has the following properties*:

(G, 1) *Each G^k is a C^∞ Banach manifold modeled on E^k.*

(G, 2) *$G^{k+1}\subset G^k$. The inclusion map is a C^∞ homomorphism having a dense image.*

(G, 3) *$G=\cap G^k$. (Thus, G is a topological group under the inverse limit topology.)*

(G, 4) *The product $G\times G\to G$, $(g, h)\to g\cdot h$, can be extended to a C^l-mapping of $G^{k+l}\times G^k$ into G^k for any $l\geqq 0$, $k\in N(d)$.*

(G, 5) *The inversion $G\to G$, $g\to g^{-1}$, can be extended to a C^l-mapping of G^{k+l} into G^k for any $l\geqq 0$, $k\in N(d)$.*

(G, 6) *For any $g\in G^k$, the right-translation R_g: $G^k\to G^k$ is C^∞.*

(G, 7) *Define $dR(u, g)=dR_g u$. Then dR: $T_{G^{k+l}}\times G^k\to T_{G^k}$ is a C^l-mapping for every $l\geqq 0$, $k\in N(d)$.*

(G, 8) *$\xi: V\cap \mathbf{E}\to G$ can be extended to a C^∞ diffeomorphism of $V\cap E^k$ onto a neighborhood $\tilde{V}\cap G^k$ of e in G^k, where $\tilde{V}=\xi(V\cap E^d)$.*

REMARK. We call a system $\{G, G^k, k\in N(d)\}$ of topological groups an *ILB*-Lie group, if it satifies above (G, 1)$\sim$(G, 7). It is not hard to see that if an *ILB*-Lie group $\{G, G^k, k\in N(d)\}$ satisfies (G, 8), then G is a strong *ILB*-group.

By the above definition and the above theorem, we see that every strong *ILB*-Lie group is an *FL*-group.

Suppose we have a strong *ILB*-Lie group G modeled on an *ILB*-chain $\{\mathbf{E}, E^k, k\in N(d)\}$. Notations being as in Theorem 6.4, let $\mathfrak{g}^k$ be the tangent space of G^k at the identity e, and let $\mathfrak{g}=\cap\mathfrak{g}^k\cdot\mathfrak{g}$ can be naturally identified with the model space $\mathbf{E}$. Note that

$$\eta: V\cap E^k\times V\cap E^{k-1}\longrightarrow U\cap E^{k-1}$$

is a C^1 mapping by virtue of (G, 4) in Theorem 6.4. Set $\eta'_u(v)=\eta(u, v)$. Then $(d\eta'_u)_v\colon E^{k-1}\to E^{k-1}$ is a bounded linear mapping, which is continuous with respect to u, v such that $u\in V\cap E^k$, $v\in V\cap E^{k-1}$.

LEMMA 6.5. *For an arbitrarily fixed $k\in N(d)$, there are a δ-neighborhood W^k of 0 in $\mathfrak{g}^k(0<\delta<1)$, and a constant C_k satisfying the following:*

(a) $\bar{W}^k\subset V\cap E^k$.

(b) $\|(d\eta'_u)_v w\|_{k-1}\leqq C_k\|w\|_{k-1}$ *for every* $u, v\in W^k$.

(c) $\|\theta(w, u, v)\|_{k-j}\leqq C_k\|w\|_{k-j}(j=0, 1)$ *for every* $u, v\in W^k$, $w\in\mathfrak{g}^{k-1}$.

(d) $\|(d_3\theta)_{(w,u,v)}(v')\|_{k-1}\leqq C_k\|v'\|_{k-1}\|w\|_k$ *for every* $u, v\in W^k$, $v'\in\mathfrak{g}^{k-1}$ *and* $w\in\mathfrak{g}^k$, *where $d_3\theta$ is the partial derivative with respect to the third variable.*

PROOF. Note that $(d\eta'_0)_0 0=0$, $\theta(0, 0, 0)=0$, (a)$\sim$(c) follow immediately by the continuity of $d\eta'_u$ and θ. Remark that

$$\theta\colon \mathfrak{g}^k\times V\cap\mathfrak{g}^k\times V\cap\mathfrak{g}^{k-1}\longrightarrow\mathfrak{g}^{k-1}$$

is C^1(cf. (N, 5) in Definition 6.2). Hence $d_3\theta$ makes sense, and $(d_3\theta)_{(w,u,v)}(v')$ defines a continuous bilinear mapping of $\mathfrak{g}^k\times\mathfrak{g}^{k-1}$ into $\mathfrak{g}^{k-1}$ for every fixed u, v. Note that $(d_3\theta)_{(0,0,0)}(0)=0$. Then (d) follows from the continuity of $d_3\theta$. □

Let $\{(h_n, \Delta_n)\}$ be a sequence of step functions in G defined on $[0, \varepsilon]\times J$, $J=[a, b]$, such that $\lim_{n\to\infty}|\Delta_n|=0$, and $\{h_n\}$ converges to a C^1-hair $h\in H^1_{\varepsilon,J}(G)$ with respect to $\tilde{\rho}$ (cf. (6)). For an arbitrarily fixed $k\in N(d)$, we choose W^k as in Lemma 6.5. Since $h(0, t)\equiv e$, we see that if s is sufficiently close to 0, say $s\leqq\varepsilon'$, then $h(s, t)\in\xi(W^k\cap\mathfrak{g})$ for every $t\in J$. Thus, one may assume without loss of generality that $\varepsilon'=\varepsilon$. Moreover, as $\{h_n\}$ converges uniformly to h, one may assume that $h_n(s, t)\in\xi(W^k\cap\mathfrak{g})$ for all n and $(s, t)\in[0, \varepsilon]\times J$.

Now, we set $\lambda_n(s, t)=\xi^{-1}\circ h_n(s, t)$, $\lambda(s, t)=\xi^{-1}\circ h(s, t)$. Obviously, $\{\lambda_n\}$ converges uniformly to λ with their partial derivatives $\{\partial\lambda_n/\partial s\}$.

LEMMA 6.6. *Notations and assumptions being as above there is a constant* K_k *such that* $\|\lambda_n(s, t)\|_k \leqq K_k s$ *for all* n *and* $(s, t) \in [0, \varepsilon] \times J$.

PROOF. Note that $\lambda_n(s, t) = \int_0^s (\partial\lambda_n/\partial s)(\sigma, t)d\sigma$. Since $\{\partial\lambda_n/\partial s\}$ converges uniformly to $\partial\lambda/\partial s$, there is K_k such that $\|(\partial\lambda_n/\partial s)(s, t)\|_k \leqq K_k$ for all n, and $(s, t) \in [0, \varepsilon] \times J$. Thus, using $0<\varepsilon<1$, we get the desired one. □

LEMMA 6.7. *Notations and assumptions being as above, if* $t-a<\delta/C_k K_k$, *then* $\prod_a^t(h_n, \Delta_n) \in \xi(W^k \cap \mathfrak{g})$ *for a sufficiently large* n. *More precisely,*

$$\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)\right\|_k \leqq C_k K_k(t-a) .$$

PROOF. Let $\Delta_n = \{t_0, t_1, \cdots, t_{m_n}\}$, and let l be the integer such that $t \in [t_l, t_{l+1})$. Note that one may assume $C_k \geqq 1$ without loss of generality. If $t \in (t_0, t_1]$, then $\xi^{-1}\circ\prod_a^t(h_n, \Delta_n) = \lambda_n(t-t_0, t_0)$, and hence $\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)\|_k \leqq C_k K_k(t-a)$ by the above lemma. Suppose that desired inequality holds for $t \in (t_0, t_l]$ and suppose $t \in (t_l, t_{l+1}]$. Then

$$\xi^{-1}\circ\left(\prod_a^t(h_n, \Delta_n)\right) = \eta\left(\lambda_n(t-t_l, t_l), \xi^{-1}\circ\prod_a^{t_l}(h_n, \Delta_n)\right) .$$

We get therefore

$$\begin{aligned}\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)\right\|_k \leqq \int_0^1 \left\|\theta\left(\lambda_n(t-t_l, t_l), \sigma\lambda_n(t-t_l, t_l), \xi^{-1}\circ\prod_a^{t_l}(h_n, \Delta_n)\right)\right\|_k d\sigma \\ +C_k K_k(t_l-a) .\end{aligned}$$

Apply inequality (c) in Lemma 6.5, and use Lemma 6.6. Then,

$$\begin{aligned}\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)\right\|_k &\leqq C_k\|\lambda_n(t-t_l, t_l)\|_k + C_k K_k(t_l-a) \\ &\leqq C_k K_k(t-a) < \delta .\end{aligned}$$

□

LEMMA 6.8. *Notations being as above, let* Δ_n' *be a subdivision of* Δ_n. *Then* $(h_n, \Delta_n') \in \mathscr{S}_{\varepsilon,J}(G)$ *and*

$$\lim_{n\to\infty}\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n) - \xi^{-1}\circ\prod_a^t(h_n, \Delta_n')\right\|_{k-1} = 0$$

uniformly on the interval $a \leqq t \leqq a+\delta/(C_k K_k)$.

PROOF. Let $\Delta_n = \{t_0, t_1, \cdots, t_{m_n}\}$, and let l be the integer such that $t \in [t_l, t_{l+1})$. By the same proof as in Lemma 6.7, we see that $\omega_n(i) =$

$\xi^{-1}\circ\prod_{t_l}^{t_{l+1}}(h_n, \Delta'_n)$, $\omega_n(l)=\xi^{-1}\circ\prod_{t_l}^{t}(h_n, \Delta'_n)$ are well-defined, and $\prod_a^t(h_n, \Delta'_n)=\xi(\omega_n(l))\xi(\omega_n(l-1))\cdots\xi(\omega_n(0))$ is contained in $\xi(W^k\cap\mathfrak{g})$ for every t such that $t-a\leqq\delta/C_kK_k$. We set

$$\alpha_n(i)=\xi^{-1}\circ\prod_a^{t_i}(h_n, \Delta_n)\ ,\quad \beta_n(i)=\xi^{-1}\circ\prod_{t_i}^{t}(h_n, \Delta'_n)\ .$$

Then using the telescope equality

$$a_1a_2\cdots a_m-b_1b_2\cdots b_m=\sum_{j=1}^{m}b_1\cdots b_{j-1}(a_j-b_j)a_{j+1}\cdots a_m\ ,$$

we obtain that

$$\begin{aligned}&\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)-\xi^{-1}\circ\prod_a^t(h_n, \Delta'_n)\\&\quad=\sum_{j=0}^{l}[\eta(\beta_n(j+1), \eta(\lambda_n(t_{j+1}-t_j, t_j), \alpha_n(j)))-\eta(\beta_n(j+1), \eta(\omega_n(j), \alpha_n(j)))]\end{aligned}$$

where we use the convention $t=t_{l+1}$. Since $\beta_n(j), \alpha_n(j)\in W^k$ by Lemma 6.7, we can apply the following inequality

$$\begin{aligned}\|\eta(\beta, \eta(v_1, \alpha))-\eta(\beta, \eta(v_2, \alpha))\|_{k-1}&=\|\eta'_\beta(\eta(v_1, \alpha))-\eta'_\beta(\eta(v_2, \alpha))\|_{k-1}\\&\leqq C_k\|\eta(v_1, \alpha)-\eta(v_2, \alpha)\|_{k-1}\leqq C_k^2\|v_1-v_2\|_{k-1}\ .\end{aligned}$$

Therefore, we have

$$\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)-\xi^{-1}\circ\prod_a^t(h_n, \Delta'_n)\right\|_{k-1}\leqq C_k^2\sum_{j=0}^{l}\|\lambda_n(t_{j+1}-t_j, t_j)-\omega_n(j)\|_{k-1}\ .$$

Set $\lambda_{n,j}(s)=\lambda_n(s, t_j)$, and let $t_j=\tau_0<\tau_1<\cdots<\tau_p=t_{j+1}$ be the points in Δ'_n contained in $[t_j, t_{j+1}]$. Then,

$$\begin{aligned}\omega_n(j)&=\eta(\lambda_{n,j}(\tau_p-\tau_{p-1}), \xi^{-1}\circ\prod_{\tau_0}^{\tau_{p-1}}(h_n, \Delta'_n))\\&=\int_0^1\theta(\lambda_{n,j}(\tau_p-\tau_{p-1}), \sigma\lambda_{n,j}(\tau_p-\tau_{p-1}), \xi^{-1}\circ\prod_{\tau_0}^{\tau_{p-1}}(h_n, \Delta'_n))d\sigma+\xi^{-1}\circ\prod_{\tau_0}^{\tau_{p-1}}(h_n, \Delta'_n)\\&=\sum_{i=0}^{p-1}\int_0^1\theta(\lambda_{n,j}(\tau_{i+1}-\tau_i), \sigma\lambda_{n,j}(\tau_{i+1}-\tau_i), \xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n, \Delta'_n))d\sigma\\&=\sum_{i=0}^{p-1}\int_0^1\int_0^{\tau_{i+1}-\tau_i}\theta(v_n(s, t_j), \sigma\lambda_{n,j}(\tau_{i+1}-\tau_i), \xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n, \Delta'_n))dsd\sigma\ ,\end{aligned}$$

where $v_n(s, t_j)=(\partial\lambda_n/\partial s)(s, t_j)$. Thus, we get

$$\lambda_n(t_{j+1}-t_j,\ t_j)-\omega_n(j)$$
$$=\sum_{i=0}^{p-1}\int_0^1\int_0^{\tau_{i+1}-\tau_i}\Big[v_n(s+\tau'_1,\ t_j)-\theta(v_n(s,\ t_j),\ \sigma\lambda_{n,j}(\tau_{i+1}-\tau_i),\ \xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n,\ \Delta'_n))\Big]dsd\sigma\ ,$$

where $\tau'_i=\tau_i-t_j$. Note that

$$\theta(v,\ u,\ \alpha)=v+\int_0^1(d_3\theta)_{(v,u,\lambda\alpha)}\alpha d\lambda\ .$$

Using this, we have

$$\lambda_n(t_{j+1}-t_j,\ t_j)-\omega_n(j)$$
$$=\sum_{i=1}^{p-1}\int_0^{\tau_{i+1}-\tau_i}[v_n(s+\tau'_i,\ t_j)-v_n(s,\ t_j)]ds$$
$$+\sum_{i=0}^{p-1}\int_0^1\int_0^{\tau_{i+1}-\tau_i}\int_0^1(d_3\theta)_{(*)}\xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n,\ \Delta'_n)d\lambda ds\, d\sigma\ ,$$

where $(*)=(v_n(s,\ t_j),\ \sigma u_{n,j}(\tau_{i+1}-\tau_i),\ \lambda\xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n,\ \Delta'_n))$. Remark that $\lambda\xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n,\ \Delta'_n)\in W^k\cap\mathfrak{g}$ for every $\lambda\in[0,1]$. Thus, applying inequality (d) in Lemma 6.5, we obtain

$$\Big\|\xi^{-1}\circ\prod_a^t(h_n,\ \Delta_n)-\xi^{-1}\circ\prod_a^t(h_n,\ \Delta'_n)\Big\|_{k-1}$$
$$\leqq C_k^2\sum_{j=0}^{l}\Big[\sum_{i=0}^{p-1}\int_0^{\tau_{i+1}-\tau_i}\|v_n(s+\tau'_i,\ t_j)-v_n(s,\ t_j)\|_{k-1}ds$$
$$+\sum_{i=0}^{p-1}\int_0^1\int_0^{\tau_{i+1}-\tau_i}\int_0^1C_k\Big\|\xi^{-1}\circ\prod_{\tau_0}^{\tau_i}(h_n,\ \Delta'_n)\Big\|_{k-1}\|v_n(s,\ t_j)\|_k d\lambda ds\, d\sigma\Big]\ .$$

Note that $\|v_n(s,\ t_j)\|_k\leqq K_k$, and note that there is a constant C'_k such that $\|z\|_{k-1}\leqq C'_k\|z\|_k$ for every $z\in\mathfrak{g}^k$. Hence, using Lemma 6.7, we get that the above quantity is not larger than

$$C_k^2\sum_{j=0}^{l}\sum_{i=0}^{p-1}\int_0^{\tau_{i+1}-\tau_i}\|v_n(s+\tau'_i,\ t_j)-v_n(s,\ t_j)\|_{k-1}ds$$
$$+C_k^2\sum_{j=0}^{l}\sum_{i=0}^{p-1}(K_kC_k)^2C'_k(\tau_{i+1}-\tau_i)|\Delta_n|\ .$$

Remark $\{v_n\}$ is equi-continuous on $[0,\varepsilon]\times J$, that is, for every $\delta_1>0$, there is $\delta_2>0$ such that if $|s-s'|+|t-t'|<\delta_2$, then $\|v_n(s,\ t)-v_n(s',\ t')\|_{k-1}<\delta_1$. For sufficiently large n, we have $|\Delta_n|<\delta_2$ and hence $\tau'_i<\delta_2$. Therefore

$$\|v_n(s+\tau'_i,\ t_j)-v_n(s,\ t_j)\|_{k-1}<\delta_1$$

and hence for a sufficiently large n

$$\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)-\xi^{-1}\circ\prod_a^t(h_n, \Delta'_n)\right\|_{k-1} \leqq C_k^2(t-a)\delta_1+K_k^2C_k^4C'_k(t-a)|\Delta_n| .$$

Thus, we get the desired result. □

THEOREM 6.9. *Every strong ILB-Lie group is a regular Fréchet-Lie group.*

PROOF. Notations and assumptions being as above, we shall show at first that $\{\prod_a^t(h_n, \Delta_n)\}$ converges in G^{k-1} uniformly on $[a, a+(\delta/C_kK_k)]$. To prove this, we have only to show that $\{\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)\}$ is a uniform Cauchy sequence in W^k for t, $a\leqq t\leqq a+(\delta/C_kK_k)$. Thus, we consider $\|\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)-\xi^{-1}\circ\prod_a^t(h_m, \Delta_m)\|_{k-1}$. Assume $n\geqq m$ and let Δ'_m be a common subdivision of Δ_n, Δ_m. By the above lemma we have only to show that

$$\lim_{m\to\infty}\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta'_m)-\xi^{-1}\circ\prod_a^t(h_m, \Delta'_m)\right\|_{k-1}=0$$

uniformly in t. Let $\Delta'_m=\{t_0, t_1, t_2, \cdots, t_{n_m}\}$, and set

$$\alpha_n(i)=\xi^{-1}\circ\prod_a^{t_i}(h_n, \Delta'_m) , \quad \beta_m(i)=\prod_{t_i}^t(h_m, \Delta'_m) \quad (\in W^k\cap\mathfrak{g}) .$$

Using telescope equality

$$\begin{aligned}
&\left\|\xi^{-1}\circ\prod_a^t(h_n, \Delta'_m)-\xi^{-1}\circ\prod_a^t(h_m, \Delta'_m)\right\|_{k-1}\\
&\leqq\sum_{j=0}^{l}\|\eta(\beta_m(j+1), \eta(\lambda_n(t_{j+1}-t_j, t_j), \alpha_n(j)))\\
&\qquad-\eta(\beta_m(j+1), \eta(\lambda_m(t_{j+1}-t_j, t_j), \alpha_n(j)))\|_{k-1}\\
&\leqq C_k^2\sum_{j=0}^{l}\|\lambda_n(t_{j+1}-t_j, t_j)-\lambda_m(t_{j+1}-t_j, t_j)\|_{k-1}\\
&\leqq C_k^2\sum_{j=0}^{l}\int_0^{t_{j+1}-t_j}\|v_n(s, t_j)-v_m(s, t_j)\|_{k-1}ds ,
\end{aligned}$$

where $v_n=\partial\lambda_n/\partial s$, $v_m=\partial\lambda_m/\partial s$. Since $\{v_n\}$ converges uniformly, for every $\delta_1>0$, there is n_0 such that if $n\geqq m\geqq n_0$ then $\|v_n(s, t)-v_m(s, t)\|_{k-1}<\delta_1$. Therefore, if $n\geqq m\geqq n_0$ then the above quantity is less than

$$C_k^2\sum_{j=0}^{l}(t_{j+1}-t_j)\delta_1=C_k^2(t-a)\delta_1 .$$

Thus, we have that $\{\xi^{-1}\circ\prod_a^t(h_n, \Delta_n)\}$ converges in $\mathfrak{g}^{k-1}$ uniformly on $[a, a+\delta/C_kK_k]$. It follows immediately that $\{\prod_a^t(h_n, \Delta_n)\}$ converges uniformly in G^{k-1} on the same interval.

Remark that C_k, K_k depends only on W^k and $h\in H^1_{t,J}(G)$. Hence the above argument shows also that if $a'\in J$, then $\{\prod_{a'}^t(h_n, \Delta_n)\}$ converges uniformly in G^{k-1} on the interval $[a', a'+\delta/C_kK_k]$. Hence by using $\prod_a^\gamma(h_n, \Delta_n)=\prod_\beta^\gamma(h_n, \Delta_n)\cdot\prod_a^\beta(h_n, \Delta_n)$, we see that $\{\prod_a^t(h_n, \Delta_n)\}$ converges in G^{k-1} uniformly on the interval $J=[a, b]$. As k is arbitrary, the above result shows that $\{\prod_a^t(h_n, \Delta_n)\}$ converges in G uniformly on J. This completes the proof of Theorem 6.9. □

References

[1] P. DE LA HARP, Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space, Lecture Notes in Math., **285**, Springer, 1972.

[2] H. H. KELLER, Differential calculus in locally convex spaces, Lecture Notes in Math., **417**, Springer, 1974.

[3] J. LESLIE, On a differential structures of groups of diffeomorphisms, Topology, **6** (1967).

[4] J. LESLIE, Some Frobenius theorems in global analysis, J. Differential Geometry, **3** (1968), 279–298.

[5] B. MAISSEN, Lie-Gruppen mit Banachraümen als Parameterräume, Acta Math., **108** (1962), 229–270.

[6] D. MONTGOMERY and L. ZIPPIN, Topological Transformation Groups, Interscience, N. Y., 1955.

[7] E. NELSON, Topics in dynamics I. flows, Math. Note, Princeton University Press, Princeton, 1969.

[8] H. OMORI, Infinite Dimensional Lie Transformation Groups, Lecture Notes in Math., **427**, Springer, 1974.

[9] H. OMORI, Theory of Infinite Dimensional Lie Groups, Kinokuniya, Tokyo, 1978 (in Japanese).

[10] H. OMORI, Y. MAEDA and A. YOSHIOKA, On regular Fréchet-Lie groups, I, Some differential geometrical expressions of Fourier-integral operators on a Riemannian manifold, Tokyo J. Math., **3** (1980), 353–390.

[11] H. OMORI, Y. MAEDA and A. YOSHIOKA, On regular Fréchet-Lie groups II, Composition rules of Fourier-integral operators on a Riemannian manifold, Tokyo J. Math., **4** (1981). 221–253.

[12] H. OMORI, Y. MAEDA, A. YOSHIOKA and O. KOBAYASHI, On regular Fréchet-Lie groups III, A second cohomology class related to the Lie algebra of pseudo-differential operators of order one, Tokyo J. Math., **4** (1981), 255–277.

[13] S. YAMAMURO, A theory of differentiation in locally convex spaces, Memoirs of the Amer. Math. Soc., **212**, 1979.

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