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On Regular Fréchet-Lie Groups VI

Infinite Dimensional Lie Groups Which Appear in General Relativity

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It is well-known that Newtonian mechanics is an approximation for general relativity. However, mathematically, both mechanics can stand independently without any subordination by each other. That means, without experimental fact one cannot prove mathematically that general relativistic mechanics is more adequate than Newtonian mechanics.

Nevertheless, beside physics, we insist in this paper that under the group theoretical point of view, general relativistic mechanics is a more closed system than Newtonian mechanics.

Introduction

Hamiltonian mechanics is a beautifully organized mathematical expression of classical mechanics (cf. [1]) and is expressed as the triplet (M, Ω, H), where

(i) (M, Ω) is a symplectic manifold, called a *phase space*. M is even dimensional smooth manifold, and Ω a smooth symplectic structure on it.

(ii) H is a smooth function on M, called a *Hamiltonian*. Mechanical motions governed by the above Hamiltonian H are given by the integral curve of the Hamiltonian vector field X_H, where X_H is defined by

$$\Omega \lrcorner X_H = dH .$$

Let $\varphi_t(x)$ be the integral curve of X_H with an initial point $x \in M$, namely $x_t = \varphi_t(x)$ is the unique solution of the following equation:

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$$(1) \qquad \frac{d}{dt}x_t = X_H(x_t)\,, \quad x_0 = x\,.$$

Now, in an idealized, dynamically closed system, the mechanical state $\varphi_t(x)$ must exist for all $t \in \boldsymbol{R}$ and $x \in M$. Thus, it is natural to assume that X_H is a complete vector field on M, hence φ_t is a one parameter transformation group on M.

It is widely known that the Hamiltonian equation (1) is equivalent to the following *Poisson's equation*: Let f be a smooth function on M, called an *observable*, and set $f_t(x) = f(\varphi_t(x))$. Then, f_t expresses the time evolution of the observed quantities, which satisfies

$$(2) \qquad \frac{d}{dt}f_t = -\{H, f_t\}\,, \quad f_0 = f\,,$$

where $\{f, g\}$ is the Poisson bracket, defined by $\{f, g\} = \Omega(X_f, X_g)$. Obviously, (1) implies (2), and the converse is also true if every local coordinate function is an observable, or equivalently if sufficiently many functions are observables.

Now, remark that observables are, mathematically, non-defined concept, but it seems to be natural to assume that if f and g are observables, then all linear combinations $\lambda f + \mu g$, $\lambda, \mu \in \boldsymbol{R}$ are observables. We denote by $\mathcal{O}$ the linear space of all observables in a mechanical system (M, Ω). Obviously, all possible Hamiltonians should be observables. Moreover, every $f \in \mathcal{O}$ will be able to be a Hamiltonian. In such a situation, it is plausible to assume that every Hamiltonian vector field X_f is complete whenever $f \in \mathcal{O}$, for $f \in \mathcal{O}$ can be a Hamiltonian. In quantum mechanics, every observable A is assumed to be a self-adjoint operator, and hence by Stone's theorem it generates a unitary one parameter subgroup e^{itA}. This corresponds to the completeness of X_f, $f \in \mathcal{O}$.

It is also natural to assume that every time-evolution f_t of an observable f is an observable, because otherwise one can not recognize such unstable functions as observables. Since $(1/\delta)(f_{t+\delta} - f_t) \in \mathcal{O}$ for every $\delta > 0$, it is also natural to assume that $(d/dt)f_t \in \mathcal{O}$. Since every $f \in \mathcal{O}$ can be a Hamiltonian, Poisson's equation (2) shows that $\mathcal{O}$ must be closed under the Poisson bracket, that is, $\mathcal{O}$ is a Lie algebra.

Let $\mathfrak{A}_{\mathcal{O}}$ be the set of all Hamiltonian vector fields X_f, $f \in \mathcal{O}$. $\mathfrak{A}_{\mathcal{O}}$ is then a Lie algebra under the usual Lie bracket, because of the identity $d\{f, g\} = -\Omega \lrcorner [X_f, X_g]$. Let $\exp tX_f$ be the one parameter group generated by $X_f \in \mathfrak{A}_{\mathcal{O}}$. Since every time-evolution is an observable, we have $(\exp tX_f)^* \mathcal{O} = \mathcal{O}$ for every $f \in \mathcal{O}$, and hence

(3) $$\mathrm{Ad}\,(\exp tX_f)\mathfrak{A}_{\mathcal{O}}=\mathfrak{A}_{\mathcal{O}}\,, \qquad {}^{\forall}f\in\mathcal{O}\,,$$

where $\mathrm{Ad}\,(\varphi)u$ is defined by $(\mathrm{Ad}\,(\varphi)u)(x)=d\varphi u(\varphi^{-1}x)$.

By the above reasoning, we assume that $\mathcal{O}$ must have the following properties:

(A1) $\mathcal{O}$ is a Lie algebra under the Poisson bracket.
(A2) Each Hamiltonian vector field X_f for $f\in\mathcal{O}$ is complete.
(A3) $(\exp tX_f)^*\mathcal{O}=\mathcal{O}$ for every $f\in\mathcal{O}$ and for all $t\in\boldsymbol{R}$.

Note at first that (A1) and (A3) aren ot necessarily independent. If $\mathcal{O}$ is closed under a suitable topology (for instance, the C^∞ topology), then $d/dt|_{t=0}(\exp tX_f)^*g\in\mathcal{O}$ for every $f, g\in\mathcal{O}$. However,

$$\frac{d}{dt}\bigg|_{t=0}(\exp tX_f)^*g=X_fg=-\Omega(X_f, X_g)=\{g, f\}\,,$$

hence (A3) yields (A1). Although the converse is not necessarily true (cf. [2] §0 and §2) it holds in many natural situations.

If M is a compact symplectic manifold without boundary, then the space $C^\infty(M)$ of all smooth functions on M satisfies (A1)~(A3), and $\mathfrak{A}_{\mathcal{O}}(\mathcal{O}=C^\infty(M))$ is the Lie algebra of the group of all canonical transformations on M. In [4] p. 103, this group is known to be a strong ILB-Lie group, and hence it is a regular Fréchet-Lie group (cf. §6 in Th. 6.1 [7]).

However, if M is non-compact, the situation is quite different. For instance, let us consider Newtonian mechanics on a configuration space N. The cotangent bundle T^*N has a natural symplectic structure Ω such that $\Omega=-d\eta$, $\eta=p_idx^i$, where $(x^1, \cdots, x^n)$ is a local coordinate system on N and $(p_1, \cdots, p_n)$ a linear coordinate system corresponding to the basis $(dx^1, \cdots, dx^n)$. By (g_{ij}) we denote a smooth riemannian metric on N and by (g^{ij}) its inverse matrix. Now, Newtonian mechanics can be understood as a Hamiltonian system $\{M, \Omega, H\}$ with $M=T^*N$, Hamiltonian $H(x, p)=(1/2)g^{ij}p_ip_j+V$, where V is a smooth function on N, called a potential function, which is naturally regarded as a function on $M=T^*N$ through the projection π: $T^*N\to N$. In Newtonian mechanics, potential functions can be chosen almost arbitrarily, while, the riemannian metric (g_{ij}) is understood to be fixed. However, there exists the following quite curious fact:

THEOREM A. *Let* $\mathcal{O}$ *be a set of smooth functions on* T^*N *satisfying* (A1) *and* (A2), *and containing a smooth function* $(1/2)g^{ij}p_ip_j+V$ *for some* $V\in C^\infty(N)$. *If* $f\in C^\infty(N)\cap\mathcal{O}$ *satisfies* $j^2_xf=0$ *at a point* $x\in N$, *then*

$j^k_x f=0$ for all $k\geqq 2$, where $j^k_x f$ is the k-th jet of f at x.

PROOF. Assume there is $f\in C^\infty(N)\cap\mathcal{O}$ such that $j^2_x f=0$ but $j^k_x f\neq 0$ for some $k\geqq 3$. One may suppose that $j^{k-1}_x f=0$ without loss of generality.

Let $(x^1, \cdots, x^n)$ be a normal coordinate system at x with respect to the given riemannian metric (g_{ij}). We denote by $\mathrm{ad}(f)g$ the Poisson bracket $\{f, g\}$. Denoting $H=(1/2)g^{ij}p_i p_j+V$, we set

$$W=(-1)^{k-1}\,\mathrm{ad}\,(H)^{k-1}f\,.$$

Since $\{g, h\}=(\partial h/\partial p_i)(\partial g/\partial x^i)-(\partial h/\partial x^i)(\partial g/\partial p_i)$, we obtain

$$W=g^{i_1 j_1}\cdots g^{i_{k-1}j_{k-1}}p_{i_1}p_{i_2}\cdots p_{i_{k-1}}\frac{\partial^{k-1}f}{\partial_x^{j_1}\cdots\partial_x^{j_{k-1}}}+\Phi_{k-2}(f)\,,$$

where Φ is a linear combination of the terms containing the derivatives of f up to order $k-2$. Since $(x^1, \cdots, x^n)$ is a normal coordinate system at x, we have

$$dW\,|\,T^*_x N=\Gamma^{i_1,\cdots,i_k}p_{i_1}\cdots p_{i_{k-1}}dx^{i_k}$$

where $\Gamma^{i_1\cdots i_k}=(\partial^k f/\partial_x^{i_1}\cdots\partial_x^{i_k})(0)$ and $|T^*_x N$ means the restriction. Therefore the Hamiltonian vector field X_W satisfies

$$X_W\,|\,T^*_x N=\Gamma^{i_1\cdots i_k}p_{i_1}\cdots p_{i_{k-1}}\frac{\partial}{\partial p_{i_k}}\,,$$

hence $X_W|T^*_x N$ is a tangent vector field on T^*N. Let $(\bar{p}_1, \cdots, \bar{p}_n)$ be a point at which the homogenous polynomial $\Gamma^{i_1\cdots i_k}p_{i_1}\cdots p_{i_k}$ attains the maximum under the restriction $\sum p_i^2=1$. Since $(\bar{p}_1, \cdots, \bar{p}_n)$ is a stationary point, we get

$$\Gamma^{i_1\cdots i_k}\bar{p}_{i_1}\cdots\bar{p}_{i_{k-1}}dp_{i_k}=\frac{2}{k}\lambda\bar{p}_{i_k}dp_{i_k}\,,\qquad \lambda\in\boldsymbol{R}\,.$$

By a suitable linear change of coordinates one may assume that $(\bar{p}_1, \cdots, \bar{p}_n)=(1, \cdots, 0)$, hence we have

$$\frac{\partial^k f}{\partial(x^1)^k}(0)\neq 0\quad\text{and}\quad\frac{\partial^k f}{\partial(x^1)^{k-1}\partial x^i}(0)=0\,,\qquad i\geqq 2\,.$$

Therefore, $X_W|(p, 0, \cdots, 0)=cp^{k-1}(\partial/\partial p)$, $c\neq 0$. Thus X_W defines a tangent vector field on a one dimensional linear subspace $(p, 0, \cdots, 0)$, which is not complete because $k\geqq 3$. Thus, X_W is not complete contradicting the assumption (A2), because $W\in\mathcal{O}$. □

The above result shows that there are few potential functions in $\mathcal{O}$. Indeed, if N is a real analytic riemannian manifold, then

$$\dim C^{\omega}(N)\cap\mathcal{O}\leqq\frac{1}{2}(n+2)(n+1)\,,\qquad n=\dim N\,,$$

where $C^{\omega}(N)$ is the space of all real analytic functions on N.

The above curious fact is mainly caused by the assumption (A2). However, if we give up assuming (A2) then we have to struggle again to obtain reasonable criterions of observables and Hamiltonians without any group-theoretical stand point. To resolve the above difficulty, we have to consider relativistic Hamiltonians on (T^*N, Ω), which will be discussed in the next section.

§1. Relativistic Hamiltonians and the statement of Theorem.

In general relativity, the free motion of a particle is given by a timelike geodesic on an $(n+1)$-dimensional Lorentzian manifold N' regarded as the space-time or the universe. However, the universe N' is not an arbitrary Lorentzian manifold, but is under certain restrictions imposed by some cosmological reasons, such as the existence of a universal time t, where t is a C^∞ function on N' such that $\partial/\partial t$ is a timelike vector field.

Suppose the vector field $\operatorname{grad} t/|\operatorname{grad} t|^2$ is complete. Then, it is not hard to see that N' is C^∞ diffeomorphic to $\boldsymbol{R}\times N$, on which the given Lorentzian metric is written in the form

$$d\tau^2=c(t, x)^2dt^2-h_{ij}(t, x)dx^i dx^j\,, \tag{4}$$

where $c(t, x)$ is a C^∞ positive function and $h_{ij}(t, x)$ is a C^∞ riemannian metric on N for every fixed $t\in\boldsymbol{R}$.

In what follows, we impose the above restriction, and moreover we assume N is compact without boundary.

Consider a free motion of a particle. Since it is a timelike geodesic, it must be a solution of the variational equation $\delta\int d\tau=0$. Since every timelike curve can be parametrized by t, the above variational problem on $\boldsymbol{R}\times N$ can be regarded naturally as that on N. Take the Legendre transform and compute the Hamiltonian $H(t, x, p)$ ([1], [3]). Then, we get easily

$$H(t, x, p)=c(t, x)\sqrt{1+h^{ij}(t, x)p_ip_j}\,. \tag{5}$$

The motions of a particle are described by integral curves of X_H in T^*N, where X_H is a time-dependent vector field on T^*N. Namely, the general relativity in our situation is nothing but a Hamiltonian mechanics with H in (5) as a Hamiltonian. The above H will be called a *relativistic Hamiltonian.*

Note that in our situation, $c(t, x)$ and $h_{ij}(t, x)$ can be chosen almost arbitrarily just like potential functions in Newtonian mechanics. This is because the gravitational forces are involved in c and h_{ij} in general relativity.

Let g_{ij} be an arbitrarily fixed C^∞ riemannian metric on N and set $r=\sqrt{g^{ij}p_ip_j}$. We denote by S^*N the unit cosphere bundle in T^*N with respect to g^{ij}.

LEMMA 1.1. *Every relativistic Hamiltonian $H_t(x, p)$ has the following asymptotic expansion:*

$$H_t(x, p)\sim a_1r+a_0+a_{-1}r^{-1}+\cdots+a_{-m}r^{-m}+\cdots , \tag{6}$$

for $r\gg 0$, where a_j's are smooth functions on S^*N.

PROOF. Set $G_t(x, p)=h^{ij}p_ip_j/g^{ij}p_ip_j$ on $T^*N-\{0\}$. G is a positively homogeneous function of degree 0, hence naturally identified with a C^∞ function on S^*N. Since $G>0$ and

$$H_t(x, p)=rc(x, t)\sqrt{G}\sqrt{1+1/Gr^2} ,$$

H_t has an asymptotic expansion of type (6). □

Now, let Σ^1 be the space consisting of all smooth functions on T^*N which have asymptotic expansion of type (6) and let $\mathfrak{A}_{\Sigma^1}$ be the set of all Hamiltonian vector fields X_f of $f\in\Sigma^1$. Obviously, Σ^1 contains $C^\infty(N)$ and all relativistic Hamiltonians for every fixed t. The purpose of this paper is to show the following:

THEOREM B. *Σ^1 satisfies* (A1)$\sim$(A3). *Moreover, $\mathfrak{A}_{\Sigma^1}$ is a Lie algebra of a regular Fréchet-Lie group.*

Remark that the above result shows that general relativity is more closed system than Newtonian mechanics from the Lie group theoretical point of view.

§2. A compactification of T^*N.

For the proof of Theorem B, we have to compactify the cotangent bundle T^*N. Fix a C^∞ riemannian metric g on N and we denote by

$M=D^*N$ (resp. $\bar{M}=\bar{D}^*N$) the unit open disk bundle (resp. the unit closed disk bundle) in T^*N. Points in T^*N will be denoted by $(x;\xi)$, $x\in N$, $\xi\in T_x^*N$, or

$$(x;\xi)=(x;rz)\ ,\qquad r\in[0,\infty)\ ,\quad z\in S_x^*N\ ,$$

where S_x^*N is the fibre at x of the unit cosphere bundle $S^*N=\partial\bar{M}$. Let $\iota: S^*N\to T^*N$ be the inclusion mapping. Then, S^*N carries canonically a smooth contact structure

$$\omega=\iota^*\eta\ ,\qquad \eta=\sum \xi_i dx^i\ . \tag{7}$$

Note that $T^*N-\{0\}$ is diffeomorphic to $\boldsymbol{R}_+\times S^*N$, $\boldsymbol{R}_+=(0,\infty)$ and the symplectic form Ω on $T^*N-\{0\}$ can be written in the form

$$\Omega=-d(r\omega)=-(dr\wedge\omega+rd\omega)\ ,\qquad r=(g^{ij}\xi_i\xi_j)^{1/2}\ . \tag{8}$$

Let $\tau: M\to T^*N$ be the C^∞ diffeomorphism defined by

$$\tau(x;\theta z)=\left(x;\left(\tan\frac{\pi\theta}{2}\right)z\right)\ , \tag{9}$$

where $\theta\in[0,1)$, $z\in S_x^*N$ (cf. [6]). By using this diffeomorphism, everything on T^*N can be transferred to that on M, e.g., $\tilde{\Omega}=\tau^*\Omega$ is a C^∞ symplectic form on M, and we see

$$\begin{aligned}\tilde{\Omega}&=-d\left(\left(\tan\frac{\pi\theta}{2}\right)\omega\right)\\&=-\frac{\pi}{2}\frac{1}{\left(\cos\frac{\pi\theta}{2}\right)^2}d\theta\wedge\omega-\left(\tan\frac{\pi\theta}{2}\right)d\omega\end{aligned} \tag{10}$$

on $M-\{0\}$.

Let $C^\infty(\bar{M})$ be the space of all C^∞ functions on $\bar{M}=\bar{D}^*N$, where a function f on $\bar{M}$ is said to be C^∞ if f can be extended to a C^∞ function on a neighborhood of $\bar{M}$. Denote by $\phi(\theta)$ a non-decreasing C^∞ function on $[0,\infty)$ such that

$$\phi(\theta)=\begin{cases}0\ , & \theta\in[0,1/3)\\ 1\ , & \theta\in[2/3,\infty)\ .\end{cases} \tag{11}$$

Every $f\in C^\infty(S^*N)$ can be naturally regarded as a C^∞ function on $\bar{M}-\{0\}$, and hence $\phi f\in C^\infty(\bar{M})$.

Recall that each element h of Σ^1 has the following asymptotic expansion;

(12)
$$h(x;\xi)\sim a_1 r+a_0+a_{-1}r^{-1}+\cdots+a_{-m}r^{-m}+\cdots ,$$
$$r=(g^{ij}\xi_i\xi_j)^{1/2} \qquad \text{for} \quad r\gg 0 .$$

LEMMA 2.1. $\tau^*\Sigma^1=(\phi(\theta)\tan(\pi\theta/2))C^\infty(S^*N)\oplus C^\infty(\bar{M})$.

PROOF. For every $h\in\Sigma^1$, we see easily that $\lim_{r\to\infty}(1/r)h$ exists as C^∞ function a_1 on S^*N. Hence, setting $\tau^*k=\tau^*h-(\phi(\theta)\tan(\pi\theta/2))a_1$, k has the asymptotic expansion:

$$k\sim a_0+a_{-1}r^{-1}+\cdots+a_{-m}r^{-m}+\cdots \qquad (r\gg 0) .$$

Set $r=\tan\pi\theta/2$, and by the converse of the Taylor's theorem at $\theta=1$, we get that $\tau^*k\in C^\infty(\bar{M})$. Conversely, if $\tau^*k\in C^\infty(\bar{M})$, then Taylor's theorem shows that k has an above asymptotic expansion. This completes the proof. □

From now on we treat the unit disk bundle $\bar{M}=\bar{D}^*N$ and the symplectic structure $\tilde{\Omega}$.

Let $\mathscr{D}(\bar{M})$ be the group of all C^∞ diffeomorphisms of $\bar{M}$ onto itself. Then, $\mathscr{D}(\bar{M})$ is a strong ILH-Lie group with the Lie algebra $\Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the totality of C^∞ vector field u on $\bar{M}$ which is tangent to the boundary $\partial\bar{M}=S^*N$. (Cf. [4] §II, [5] p. 279). Let $\mathscr{D}_{\tilde{\Omega}}$ be the identity component of the group $\{\varphi\in\mathscr{D}(\bar{M});\ \varphi^*\tilde{\Omega}=\tilde{\Omega}$ on $M\}$. Then $\mathscr{D}_{\tilde{\Omega}}$ is a closed subgroup of $\mathscr{D}(\bar{M})$. We set

(13)
$$\Gamma_{\tilde{\Omega}}=\{u\in\Gamma(T\bar{M});\ \exp tu\in\mathscr{D}_{\tilde{\Omega}} \text{ for all } t\in \boldsymbol{R}\} .$$

LEMMA 2.2. *$\Gamma_{\tilde{\Omega}}$ has the following properties:*
(i) *$\Gamma_{\tilde{\Omega}}$ is a closed subalgebra of $\Gamma(T\bar{M})$.*
(ii) *Every $u\in\Gamma_{\tilde{\Omega}}$ is a complete vector field.*
(iii) $\mathrm{Ad}(\exp tu)\Gamma_{\tilde{\Omega}}=\Gamma_{\tilde{\Omega}}$ *for any $u\in\Gamma_{\tilde{\Omega}}$.*

PROOF. By 1.4.1 Theorem in [4] or Lemma 1.4 in [8], we see that $\Gamma_{\tilde{\Omega}}$ is a closed Lie algebra of $\Gamma(T\bar{M})$. Note that

$$\Gamma_{\tilde{\Omega}}=\{u\in\Gamma(T\bar{M});\ \mathscr{L}_u\tilde{\Omega}=0 \text{ on } M\} .$$

Then, we get easily (iii) (cf. [2]). □

Our first purpose is to get the following, which will be proved in §§4, 5 and 6.

THEOREM 2.3. *$\mathscr{D}_{\tilde{\Omega}}$ is a regular Fréchet-Lie group.* (For the definition of regular Fréchet-Lie group, see [7].)

Let $\mathfrak{A}_{\Sigma^1}$ be the set of all Hamiltonian vector fields X_f of $f\in\Sigma^1$, i.e., $\Omega\lrcorner X_f=df$, $f\in\Sigma^1$. We translate this Lie algebra to $d\tau^{-1}\mathfrak{A}_{\Sigma^1}$. For the proof of Theorem B, we shall show the following in §3:

THEOREM 2.4. *$d\tau^{-1}\mathfrak{A}_{\Sigma^1}$ is a closed ideal of $\Gamma_{\tilde{\Omega}}$ of codimension* $\dim H^1(N; \boldsymbol{R})+1$, *where $H^1(N, \boldsymbol{R})$ is the first cohomology group of N over $\boldsymbol{R}$.*

Before proving the above theorems, we shall show at first how to obtain Theorem B by Theorems 2.3 and 2.4.

PROOF OF THEOREM B. We shall prove Theorem B under Theorems 2.3 and 2.4. At first, we shall show that Σ^1 satisfies (A1)~(A3). Since $d\tau^{-1}\mathfrak{A}_{\Sigma^1}$ is a Lie algebra, so is $\mathfrak{A}_{\Sigma^1}$ and hence Σ^1 is a Lie algebra under the Poisson bracket. Since every $u\in\Gamma_{\tilde{\Omega}}$ is a complete vector field, we see that $d\tau u$ is also a complete vector field on T^*N.

Set $w(t)=\mathrm{Ad}(\exp tv)w$ for $v, w\in d\tau^{-1}\mathfrak{A}_{\Sigma^1}$. Then $w(t)$ satisfies $(d/dt)w(t)=[v, w(t)]$, and hence by Lemma 4.5 [7], we have $\mathrm{Ad}(\exp tv)d\tau^{-1}\mathfrak{A}_{\Sigma^1}=d\tau^{-1}\mathfrak{A}_{\Sigma^1}$. Since $d\tau(\mathrm{Ad}(\exp tv)w)=\mathrm{Ad}(\exp td\tau v)d\tau w$, the above equality shows that $\mathrm{Ad}(\exp tu)\mathfrak{A}_{\Sigma^1}=\mathfrak{A}_{\Sigma^1}$ for all $u\in\mathfrak{A}_{\Sigma^1}$. Now, remark that $\Omega\lrcorner\mathrm{Ad}(\exp tu)X_f=d(\exp -tu)^*f$. Since $\mathrm{Ad}(\exp tu)X_f\in\mathfrak{A}_{\Sigma^1}$ we have $(\exp tu)^*f+\text{const.}\in\Sigma^1$ and hence $(\exp tu)^*f\in\Sigma^1$. Thus Σ^1 satisfies (A1)~(A3).

As $\mathscr{D}_{\tilde{\Omega}}$ is a regular Fréchet-Lie group and $d\tau^{-1}\mathfrak{A}_{\Sigma^1}$ is a finite codimensional subalgebra of $\Gamma_{\tilde{\Omega}}$, Theorem 4.2 in [8] shows that there is a locally flat FL-subgroup $\tilde{G}_{\Sigma^1}$ of $\mathscr{D}_{\tilde{\Omega}}$ having $d\tau^{-1}\mathfrak{A}_{\Sigma^1}$ as its Lie algebra. By Corollary 2.4 in [8], $\tilde{G}_{\Sigma^1}$ is a regular Fréchet-Lie group.

Now, using the C^∞ diffeomorphism $\tau: M\to T^*N$, we put

$$\mathscr{D}_\Omega(\bar{T}^*N)_0=\tau\mathscr{D}_{\tilde{\Omega}}\tau^{-1}\ ,$$
$$\Gamma_\Omega(T\bar{T}^*N)=d\tau\Gamma_{\tilde{\Omega}}\ ,$$
$$G_{\Sigma^1}=\tau\tilde{G}_{\Sigma^1}\tau^{-1}\ .$$

Obviously $\mathscr{D}_\Omega(\bar{T}^*N)_0$ is a regular Fréchet-Lie group and G_{Σ^1} is an FL-subgroup. The Lie algebra of G_{Σ^1} is given by $\mathfrak{A}_{\Sigma^1}$. □

§3. Proof of Theorem 2.4.

First of all, we explain the structures on the closed unit disk bundle $\bar{M}=\bar{D}^*N$ which will be useful through this paper. Remark that M has two geometrical structures, that is, it is a symplectic manifold with symplectic structure $\tilde{\Omega}$ and also is a manifold with boundary $S^*N=\partial\bar{M}$

on which there is a canonical contact structure ω (cf. (7)).

In the following, we will omit $\bar{M}$ or $T\bar{M}$ in the notations for Lie subgroups or Lie subalgebras of $\mathscr{D}(\bar{M})$ or $\Gamma(T\bar{M})$, respectively, e.g., $\mathscr{D}_{\tilde{\Omega}}$ is a subgroup of $\mathscr{D}(\bar{M})$.

For the contact manifold (S^*N, ω), define the C^∞ vector field χ_ω on S^*N defined by

$$d_\omega \lrcorner \chi_\omega = 0\ , \qquad \omega \lrcorner \chi_\omega = 1\ . \tag{14}$$

χ_ω is called the *characteristic vector field of the contact form ω*. Denote by E_ω the distribution

$$E_\omega = \{X \in TS^*N;\ \omega \lrcorner X = 0\}\ ,$$

and by $E^*_\omega (\subset T^*S^*N)$ the anihilator of χ_ω. E_ω (resp. E^*_ω) is a smooth subbundle of the tangent (resp. the cotangent) bundle of S^*N of codimension one. It is obvious that

$$TS^*N = \boldsymbol{R}\chi_\omega \oplus E_\omega\ , \qquad T^*S^*N = \boldsymbol{R}\omega \oplus E^*_\omega\ .$$

The linear mapping $d\omega|_{E_\omega}$: $X \mapsto d\omega \lrcorner X$, defines an isomorphism of E_ω onto E^*_ω. So, its inverse will be denoted by $d\omega^{-1}$. The following fact is well-known in Hamiltonian mechanics (cf. [1] and [4] 8.3.1. Lemma):

LEMMA 3.1. *A vector field u on S^*N is a contact vector field if and only if there is a smooth function f on S^*N such that u can be written in the form $u = f\chi_\omega - \{f\}$, where $\{f\} = d\omega^{-1}(df - (\chi_\omega f)\omega)$.*

Since $\bar{M} - \{0\}$ is diffeomorphic to $(0, 1] \times S^*N$, χ_ω, E_ω and E^*_ω stated above can be naturally extended to $\bar{M} - \{0\}$. We use the same notations for the extended ones. It is again obvious that

$$\begin{cases} T\bar{M} - \{0\} = \boldsymbol{R}\dfrac{\partial}{\partial\theta} \oplus \boldsymbol{R}\chi_\omega \oplus E_\omega \\ T^*\bar{M} - \{0\} = \boldsymbol{R}d\theta \oplus \boldsymbol{R}\omega \oplus E^*_\omega\ . \end{cases} \tag{15}$$

For every vector field u on $\bar{M} = \bar{D}^*N$, we set

$$\tilde{\flat}(u) = \tilde{\Omega} \lrcorner u\ . \tag{16}$$

Also, for a smooth function f on $\bar{M}$, define $\tilde{\sharp} f \in \Gamma_{\tilde{\Omega}}$ by

$$\tilde{\sharp} f = \tilde{\flat}^{-1} df\ . \tag{17}$$

Let $\mathfrak{A}_{\Sigma^1}$ be the set of all Hamiltonian vector fields X_f, $f \in \Sigma^1$. By the diffeomorphism τ in (9), it is not hard to see that $d\tau^{-1}\mathfrak{A}_{\Sigma^1} = \tilde{\flat}^{-1} d(\tau^* \Sigma^1)$.

In accordance with the decomposition (15), we shall compute the components of $u \in d\tau^{-1}\mathfrak{A}_{\Sigma^1}$. Given $f \in \tau^*\Sigma^1$, there exists uniquely $a_1 \in C^\infty(S^*N)$ and $g \in C^\infty(\bar{M})$ such that

$$f=\left(\tan \frac{\pi\theta}{2}\right)\phi(\theta)a_1+g\ , \tag{18}$$

because of Lemma 2.1. Since each vector field u on $\bar{D}^*N-\{0\}$ can be written in the form

$$u=\lambda(x;\,\theta z)\frac{\partial}{\partial\theta}+\mu(x;\,\theta z)\chi_\omega+\hat{u}\ , \qquad \hat{u}\in\Gamma(E_\omega)\ , \tag{19}$$

if $df=\tilde{\Omega}\lrcorner u$, then we get by (10) and (19)

$$\begin{cases} \lambda(x;\,\theta z)=-\dfrac{1}{\pi}\phi(\theta)(\sin\pi\theta)\chi_\omega a_1-\dfrac{2}{\pi}\left(\cos\dfrac{\pi\theta}{2}\right)^2(\chi_\omega g) \\ \mu(x;\,\theta z)=\left(\phi+\dfrac{1}{\pi}\sin\pi\theta\dfrac{\partial\phi}{\partial\theta}\right)a_1+\dfrac{2}{\pi}\left(\cos\dfrac{\pi\theta}{2}\right)^2\dfrac{\partial g}{\partial\theta} \\ \hat{u}=-\phi(\theta)\{a_1\}-\left(\cot\dfrac{\pi\theta}{2}\right)\{g\}\ , \end{cases} \tag{20}$$

where $\{g\}=d\omega^{-1}(d'g-(\chi_\omega g)\omega)$ and $d'g$ is the derivative of g regarding θ as an arbitrarily fixed parameter.

LEMMA 3.2. *$\tilde{\sharp}(\tau^*\Sigma^1)$ is a closed subspace of $\Gamma_{\tilde{\Omega}}$.*

PROOF. Let f be an element of $\tau^*\Sigma^1$. Then $u=\tilde{\sharp}f$ can be expressed by (19) and (20). If $\theta=1$, then $\tilde{\sharp}f$ becomes $a_1\chi_\omega-\{a_1\}$. Hence, $\tilde{\sharp}f$ induces a smooth vector field on S^*N. Therefore, $\tilde{\sharp}f\in\Gamma(T\bar{M})$. Since $\mathscr{L}_u\tilde{\Omega}=d(\tilde{\Omega}\lrcorner u)=d\tilde{\flat}(u)$, we see easily that $\tilde{\sharp}f\in\Gamma_{\tilde{\Omega}}$.

Now, suppose that a sequence $\{\tilde{\sharp}f_m\}$, $f_m\in\tau^*\Sigma^1$, converges to an element $u\in\Gamma_{\tilde{\Omega}}$ in the C^∞ topology on $\bar{M}$. On $[1/3,1]\times S^*N$, the above convergence means those of each component of $\partial/\partial\theta$, χ_ω, and E_ω. Thus, using (20) we have that there are smooth functions $a_1'\in C^\infty(S^*N)$, $g'\in C^\infty([1/3,1]\times S^*N)$ such that

$$\lim_{m\to\infty}\tilde{\sharp}f_m=\tilde{\sharp}\left(\left(\tan\frac{\pi\theta}{2}\right)\phi(\theta)a_1'\right)+\tilde{\sharp}g'$$

on $[1/3,1]\times S^*N$. Choose an arbitrary extension of g' onto $\bar{M}$ and denote it by the same notation g'. Define $f'\in C^\infty(\bar{M})$ by

$$f'=\left(\tan\frac{\pi\theta}{2}\right)\phi(\theta)a_1'+g' .$$

Thus, we have $u=\tilde{\sharp}f'$ on $[1/3, 1]\times S^*N$, hence $\tilde{\sharp}(f_m-f')$ converges to 0 on $[1/3, 1]\times S^*N$. Therefore $\tilde{\sharp}(1-\phi)(f_m-f')$ converges in $\Gamma_{\tilde{\Omega}}$. Recall that $\tilde{\sharp}h=\tilde{\flat}^{-1}dh$, and remark that the mapping $\tilde{\flat}$ induces a linear homeomorphism of $\Gamma(T\bar{D}(2/3))$ onto $\Gamma(T^*\bar{D}(2/3))$, where $\bar{D}(2/3)$ is the closed 2/3-neighborhood of zero-section of $\bar{M}=\bar{D}^*N$. Therefore, there exists $h'\in C^\infty(\bar{M})$ such that $\operatorname{supp} h'\subset\bar{D}(2/3)$ and $\lim_{m\to\infty}\tilde{\sharp}(1-\phi)(f_m-f')=\tilde{\sharp}h'$. Thus, we have

$$\lim_{m\to\infty}\tilde{\sharp}f_m=\tilde{\sharp}f'+\lim_{m\to\infty}\tilde{\sharp}\phi(f_m-f')+\tilde{\sharp}h' .$$

Note that the second term of the right hand side converges to 0. Hence, we have $\lim_{m\to\infty}\tilde{\sharp}f_m=\tilde{\sharp}(f'+h')$. This implies the closedness of $\tilde{\sharp}(\tau^*\Sigma^1)$. □

LEMMA 3.3. *$\tilde{\sharp}(\tau^*\Sigma^1)=d\tau^{-1}\mathfrak{A}_{\Sigma^1}$ is an ideal of $\Gamma_{\tilde{\Omega}}$.*

PROOF. Since $\tilde{\flat}\mathrm{Ad}(\varphi)u=\varphi^{*-1}((\varphi^*\tilde{\Omega})\lrcorner u)$ for every $\varphi\in\mathscr{D}(\bar{M})$, we see

$$\mathrm{Ad}(\varphi^{-1})\tilde{\sharp}f=\tilde{\sharp}\varphi^*f , \qquad f\in\tau^*\Sigma^1 ,$$

for every $\varphi\in\mathscr{D}_{\tilde{\Omega}}$.

We shall show at first that $\varphi^*\tau^*\Sigma^1=\tau^*\Sigma^1$ for any $\varphi\in\mathscr{D}_{\tilde{\Omega}}$. To do this, it is enough to show that $\varphi^*(\tan\pi\theta/2)\in\tau^*\Sigma^1$, because of the expression in Lemma 2.1. Let $s=\cot\pi\theta/2$. Then $[0, \varepsilon)\times S^*N$ can be naturally identified with an ε-neighborhood of S^*N in $\bar{M}$. By this identification, φ can be written as

$$\begin{cases}\bar{s}=\bar{s}(s, z)\\ \bar{z}=\bar{z}(s, z)\end{cases}\qquad (s, z)\in[0, \varepsilon)\times S^*N . \tag{21}$$

Since $\varphi(S^*N)=S^*N$, we see $\bar{s}(0, z)\equiv 0$ and $\bar{s}=g(s, z)s$ on a sufficiently small neighborhood of S^*N, where $g(s, z)$ is a smooth positive function. Note that $\varphi^*(\tan\pi\theta/2)=\bar{s}^{-1}$, hence it is contained in $(\tan\pi\theta/2)C^\infty(\bar{M})$ on a sufficiently small neighborhood of S^*N, but this shows immediately $\varphi^*(\tan\pi\theta/2)\in\tau^*\Sigma^1$.

The above result shows especially

$$\mathrm{Ad}(\exp -tu)\tilde{\sharp}\tau^*\Sigma^1=\tilde{\flat}^{-1}d(\exp tu)^*\tau^*\Sigma^1=\tilde{\flat}^{-1}d(\tau^*\Sigma^1) ,$$

for every $u\in\Gamma_{\tilde{\Omega}}$. Since $\tilde{\sharp}\tau^*\Sigma^1$ is closed in $\Gamma_{\tilde{\Omega}}$, we see that $\tilde{\sharp}\tau^*\Sigma^1$ is an ideal of $\Gamma_{\tilde{\Omega}}$ by taking the derivative at $t=0$. □

Let $\Gamma_\omega(TS^*N)$ be the Lie algebra of all smooth contact vector fields on S^*N. By Lemma 3.1 and by (20), it is clear that $\Gamma_\omega(TS^*N)=\tilde{\sharp}(\tau^*\Sigma^1)|_{S^*N}$. However, we have in fact the following:

LEMMA 3.4. *Every $u\in\Gamma_{\tilde{\Omega}}$ induces a contact vector field on S^*N.*

PROOF. For every $u\in\Gamma_{\tilde{\Omega}}$ denote by φ_t the one parameter subgroup generated by u. For the proof of $u|_{S^*N}\in\Gamma_\omega(TS^*N)$, it is enough to show that $\varphi_t|_{S^*N}$ is a contact transformation. For an element $\varphi\in\mathscr{D}_{\tilde{\Omega}}$ we set $\hat{\varphi}=\varphi|_{S^*N}$. On a neighborhood of the boundary S^*N of $\bar{M}$, φ can be written in the form (21). Since $\varphi^*\tilde{\Omega}=\tilde{\Omega}$ and $\tilde{\Omega}=-d(s^{-1}\omega)$, we have

$$\bar{s}^{-2}d\bar{s}\wedge\hat{\varphi}^*\omega=s^{-2}ds\wedge\omega\ ,$$

hence

$$\hat{\varphi}^*\omega=\frac{\bar{s}^2}{s^2}\left(\frac{\partial\bar{s}}{\partial s}\right)^{-1}\Big|_{s=0}\omega\ .$$

Since $\bar{s}=g(s,z)s$, $g\neq 0$ because of $\bar{s}(0,z)\equiv 0$, we see that

$$\hat{\varphi}^*\omega=g(0,z)\omega\ ,$$

hence $\hat{\varphi}$ is a contact transformation. □

Now, let $p\colon\Gamma_{\tilde{\Omega}}\to\Gamma_\omega(TS^*N)$ be the restriction mapping $p(u)=u|_{S^*N}$. p is surjective, because $\Gamma_{\tilde{\Omega}}\supset\tilde{\sharp}(\tau^*\Sigma^1)$ and $\tilde{\sharp}(\tau^*\Sigma^1)|_{S^*N}=\Gamma_\omega(TS^*N)$. We denote by $\Gamma_{\tilde{\Omega},b}$ the kernel of p. By a standard diagram chasing, we see easily

$$\Gamma_{\tilde{\Omega}}/\tilde{\sharp}(\tau^*\Sigma^1)\cong\Gamma_{\tilde{\Omega},b}/\Gamma_{\tilde{\Omega},b}\cap\tilde{\sharp}(\tau^*\Sigma^1)\ .$$

On the other hand, by (20), we see $\Gamma_{\tilde{\Omega},b}\cap\tilde{\sharp}(\tau^*\Sigma^1)=\tilde{\sharp}C^\infty(\bar{M})$. Thus, we have

$$\Gamma_{\tilde{\Omega}}/d\tau^{-1}\mathfrak{A}_{\Sigma^1}\cong\Gamma_{\tilde{\Omega},b}/\Gamma_{\tilde{\Omega},b}\cap\tilde{\sharp}(\tau^*\Sigma^1)\cong\tilde{\flat}\,\Gamma_{\tilde{\Omega},b}/dC^\infty(\bar{M})\ .$$

Let $Z(\bar{M})$ be the space of all smooth closed 1-forms on $\bar{M}$. Then we have

LEMMA 3.5. $\tilde{\flat}(\Gamma_{\tilde{\Omega},b})=Z(\bar{D}^*N)+\boldsymbol{R}d(\phi\log\tan\pi\theta/2)$.

PROOF. For every $u\in\Gamma_{\tilde{\Omega},b}$, $\tilde{\flat}(u)$ is a closed 1-form on M. First, we show that $\tilde{\flat}(\Gamma_{\tilde{\Omega},b})$ is contained in $Z(\bar{M})\oplus\boldsymbol{R}d(\phi\log\tan\pi\theta/2)$. Given $u\in\Gamma_{\tilde{\Omega},b}$, put $u=\lambda(\partial/\partial\theta)+\mu X_\omega+\hat{u}$ in accordance with the decomposition (19). Since $u|_{S^*N}\equiv 0$, one may set

$$\lambda=(\cos\pi\theta/2)\tilde{\lambda}\ ,\qquad \mu=(\cos\pi\theta/2)\tilde{\mu}\ ,\qquad \hat{u}=(\cos\pi\theta/2)\tilde{u}\ ,$$

on $(2/3, 1]\times S^*N$, where $\tilde{\lambda}$, $\tilde{\mu}$ and $\tilde{u}$ are C^∞ functions and a C^∞ vector field on $(2/3, 1]\times S^*N$. Thus, on $(2/3, 1]\times S^*N$ we have

$$\begin{aligned}\tilde{\Omega}\lrcorner u &= -d((\tan \pi\theta/2)\omega)\lrcorner u \\ &= \frac{\pi}{2}\Big(\frac{\tilde{\mu}d\theta}{\cos \pi\theta/2}-\frac{\tilde{\lambda}\omega}{\cos \pi\theta/2}\Big)-(\sin \pi\theta/2)d\omega\lrcorner\tilde{u} \,.\end{aligned}$$

Set $a=\tilde{\lambda}|_{S^*N}$, $b=\tilde{\mu}|_{S^*N}$ and regard these as functions on $(2/3, 1]\times S^*N$. Then, $\tilde{\lambda}-a$ and $\tilde{\mu}-b$ can be devided by $\cos \pi\theta/2$, and hence

$$\begin{aligned}\tilde{\flat}(u) &= \tilde{\Omega}\lrcorner u \\ &= \frac{\pi}{2}\Big(\frac{b}{\cos \pi\theta/2}d\theta-\frac{a}{\cos \pi\theta/2}\omega\Big)+\alpha \,,\end{aligned}$$

where α is a smooth 1-form on $(2/3, 1]\times S^*N$.

Since $d(\tilde{\Omega}\lrcorner u)\equiv 0$, especially $d(\tilde{\Omega}\lrcorner u)$ has no singularity on $(2/3, 1]\times S^*N$, we see easily that $a\equiv 0$ and $b\equiv$const.. Thus,

$$\tilde{\flat}(u)=\frac{b}{\cos \pi\theta/2}d\theta+\alpha=Cd(\phi \log \tan \pi\theta/2)+\alpha' \,,$$

where C is a constant and α' is a smooth 1-form on $\bar{M}$.

Conversely, assume that

$$\tilde{\Omega}\lrcorner u=Cd(\phi \log \tan \pi\theta/2)+\beta \,, \qquad \beta\in \boldsymbol{Z}(\bar{M}) \,.$$

Then, $d(\tilde{\Omega}\lrcorner u)=0$ on $\bar{M}$, hence we only prove that $u|_{S^*N}=0$. On $(2/3, 1]\times S^*N$, the right hand of the above equality can be written as

$$\frac{C}{\cos \pi\theta/2}d\theta+\beta' \,, \qquad \beta'\in \boldsymbol{\Gamma}(T^*\bar{M}) \,.$$

Hence, by using (20), we see easily that $u|_{S^*N}=0$. □

PROOF OF THEOREM 2.4.

By Lemma 3.2 and Lemma 3.3, $d\tau^{-1}\mathfrak{A}_{\Sigma^1}$ is a closed ideal $\boldsymbol{\Gamma}_{\tilde{\Omega}}$. Since $\boldsymbol{\Gamma}_{\tilde{\Omega}}/d\tau^{-1}\mathfrak{A}_{\Sigma^1}\cong\tilde{\flat}(\boldsymbol{\Gamma}_{\tilde{\Omega},\flat})/dC^\infty(\bar{M})$, Lemma 3.5 shows that it is isomorphic to $H^1(\bar{M}; \boldsymbol{R})\oplus\boldsymbol{R}d(\phi \log \tan \pi\theta/2)$. Since $H^1(N; \boldsymbol{R})\cong H^1(D^*N; \boldsymbol{R})$, we have the desired results. □

§4. A reduction of Theorem 2.3.

To complete the proof of Theorem B, it remains proving only that $\mathscr{D}_{\tilde{\Omega}}$ is a regular Fréchet-Lie group (cf. Theorem 2.3 in §2). In this section, we shall show how the proof will be done. Put

(22) $$\mathscr{D}_{\tilde{\Omega},b}=\{\varphi\in\mathscr{D}_{\tilde{\Omega}};\ \varphi|_{S^*N}=\text{identity}\}\ .$$

We start with the following remark.

LEMMA 4.1. *$\mathscr{D}_{\tilde{\Omega},b}$ is a closed normal subgroup of $\mathscr{D}_{\tilde{\Omega}}$ and $\mathscr{D}_{\tilde{\Omega}}/\mathscr{D}_{\tilde{\Omega},b}$ is naturally isomorphic to the identity component of $\mathscr{D}_\omega(S^*N)$, the group of the C^∞ contact transformations.*

PROOF. Obviously, $\mathscr{D}_{\tilde{\Omega},b}$ is a normal subgroup of $\mathscr{D}_{\tilde{\Omega}}$. By the same proof as in Lemma 3.4, we see that $\mathscr{D}_{\tilde{\Omega}}/\mathscr{D}_{\tilde{\Omega},b}\subset\mathscr{D}_\omega(S^*N)$. To prove that $\mathscr{D}_{\tilde{\Omega}}/\mathscr{D}_{\tilde{\Omega},b}$ is isomorphic to the identity component of $\mathscr{D}_\omega(S^*N)$, we remark at first that $\mathscr{D}_\omega(S^*N)$ is a strong ILH-Lie group (cf. [4], 8.3.6 Theorem). Hence, for every element φ in that identity component there is a smooth curve φ_t joining between the identity and $\varphi=\varphi_1$. Define u_t by $(d\varphi_t/dt)\cdot\varphi_t^{-1}$. Then $u_t\in\Gamma_\omega(TS^*N)$ and by Lemma 3.1, $u_t=g_t\chi_\omega-\{g_t\}$, $g_t\in C^\infty(S^*N)$.

Now, $\phi(\theta)$ being as in §2, (9), $f_t=(\tan\pi\theta/2)\phi(\theta)g_t$ is an element of $\tau^*\Sigma^1$, hence $\tilde{\sharp}f_t\in\Gamma_{\tilde{\Omega}}$ and $\tilde{\sharp}f_t|_{S^*N}=u_t$. Setting $X_{f_t}=\tilde{\sharp}f_t$, we solve the equation

$$\frac{d}{dt}\psi_t=X_{f_t}\cdot\psi_t\ ,\qquad \psi_0=\text{id.}\ .$$

Then, it is easy to see $\psi_t\in\mathscr{D}_{\tilde{\Omega}}$, for $\mathscr{D}_{\tilde{\Omega}}$ is a closed subgroup of $\mathscr{D}(\bar{M})$. Since $X_{f_t}|_{S^*N}=u_t$ (cf. (20)), $\psi_t|_{S^*N}$ satisfies the same equation as φ_t. Hence by the uniqueness, we have $\psi_t|_{S^*N}=\varphi_t$. □

For the proof of Theorem 2.3, we shall show the following theorem in §6.

THEOREM 4.2. *$\mathscr{D}_{\tilde{\Omega},b}$ is an FL-subgroup of $\mathscr{D}(\bar{M})$.*

In this section, we shall show the above theorem yields the desired result.

PROOF OF THEOREM 2.3. Using Theorem 4.2, we will prove that $\mathscr{D}_{\tilde{\Omega}}$ is a regular Fréchet-Lie group. Since $\mathscr{D}(\bar{M})$ is a regular Fréchet-Lie group, the above theorem and Proposition 2.4 [8] show that $\mathscr{D}_{\tilde{\Omega},b}$ is a regular Fréchet-Lie group.

Let $\xi\colon U\to\mathscr{D}_\omega(S^*N)$ be a C^∞ local coordinate system at the identity such that $\xi(0)=e$ (the identity), where U is an open convex neighborhood of 0 in $\Gamma_\omega(TS^*N)$. For each $u\in U$, we set $\kappa(u,t)=(d\xi(tu)/dt)\cdot\xi(tu)^{-1}$. Then, $\kappa(u,t)\in\Gamma_\omega(TS^*N)\subset\Gamma(TS^*N)$ and $\kappa\colon U\times[0,1]\to\Gamma_\omega(TS^*N)$ is a C^∞ mapping. By Lemma 3.1, $\kappa(u,t)$ can be written in the form

$$\kappa(u, t)=\sigma(u, t)\chi_\omega-\{\sigma(u, t)\}\ ,$$

where σ is a C^∞ mapping of $U\times[0, 1]$ into $C^\infty(S^*N)$.

Let $X_{(u,t)}$ be the Hamiltonian vector field defined by the C^∞ function $(\tan \pi\theta/2)\phi(\theta)\sigma(u, t)$ on $\bar{M}$ i.e.,

$$\tilde{\Omega} \lrcorner X_{(u,t)}=d((\tan \pi\theta/2)\phi(\theta)\sigma(u, t))\ .$$

We see easily that the mapping $(u, t)\mapsto X_{(u,t)}$ is a C^∞ mapping of $U\times[0, 1]$ into $d\tau^{-1}\mathfrak{A}_{\Sigma^1}(\subset\Gamma(T\bar{M}))$. Set

$$\Phi(u, t)=\prod_0^t (1+X_{(u,s)})ds\ .$$

Then, by the second fundamental theorem of regular Fréchet-Lie groups (cf. [7]), we see that Φ: $U\times[0, 1]\to\mathscr{D}(\bar{M})$ is a C^∞ mapping. Since $X_{(u,t)}|_{S^*N}=\kappa(u, t)$, we see $\Phi(u, t)|_{S^*N}=\xi(tu)$ by the same reasoning as in the proof of the above lemma.

Set $\tilde{U}=\xi(U)$, and let $\tilde{V}$ be an open neighborhood of e in $\mathscr{D}_\omega(S^*N)$ such that $\tilde{V}^{-1}=\tilde{V}$ and $\tilde{V}^2\subset\tilde{U}$. For every $g\in\tilde{U}$, we define $\gamma(g)$ by

$$\gamma(g)=\Phi(\xi^{-1}(g), 1)\ .$$

Obviously, γ is a C^∞ mapping of $\tilde{U}$ into $\mathscr{D}(\bar{M})$ such that $\gamma(g)|_{S^*N}=g$. Moreover, since $X_{(u,t)}\in\Gamma_{\tilde{\Omega}}$, we have $\gamma(\tilde{U})\subset\mathscr{D}_{\tilde{\Omega}}$ because $\mathscr{D}_{\tilde{\Omega}}$ is a closed subgroup of $\mathscr{D}(\bar{M})$ (cf. §2). Define

$$r_\gamma(g, h)=\gamma(gh)^{-1}\gamma(g)\gamma(h) \quad \text{for} \quad g, h\in\tilde{V}\ .$$

Then $r_\gamma(g, h)\in\mathscr{D}_{\tilde{\Omega},b}$ and r_γ: $\tilde{V}\times\tilde{V}\to\mathscr{D}(\bar{M})$ is a C^∞ mapping. Thus, by the assumed theorem we see that r_γ: $\tilde{V}\times\tilde{V}\to\mathscr{D}_{\tilde{\Omega},b}$ is a C^∞ mapping (cf. Lemma 2.3 (ii) [8]).

For $n\in\mathscr{D}_{\tilde{\Omega},b}$ we set

$$\alpha_\gamma(g)n=\gamma(g)^{-1}n\gamma(g)\ , \qquad g\in\tilde{V}\ .$$

It is obvious that $\alpha_\gamma(g)n\in\mathscr{D}_{\tilde{\Omega},b}$ because $\mathscr{D}_{\tilde{\Omega},b}$ is a normal subgroup of $\mathscr{D}_{\tilde{\Omega}}$ by Lemma 4.1. Also, the map α_γ: $\tilde{V}\times\mathscr{D}_{\tilde{\Omega},b}\to\mathscr{D}_{\tilde{\Omega},b}$ is a C^∞ mapping by the same reasoning as above. Consider the following exact sequence:

$$(23) \qquad 1\longrightarrow\mathscr{D}_{\tilde{\Omega},b}\longrightarrow\mathscr{D}_{\tilde{\Omega}}\longrightarrow\mathscr{D}_\omega(S^*N)\longrightarrow 1\ .$$

(23) has a local section γ which satisfies (Ext. 1)~(Ext. 3) in [8] §6, that is, (23) is an FL-extension. Hence, by Proposition 5.2 and Theorem 5.3 in [8], we have that $\mathscr{D}_{\tilde{\Omega}}$ is a regular Fréchet-Lie group. Taking the derivative of a smooth family $\varphi_t\in\mathscr{D}_{\tilde{\Omega}}$ at the identity, we get

$\mathscr{L}_u\tilde{\Omega}=0$, where $u=(d\varphi_t/dt)_{t=0}$. Therefore, we see that the Lie algebra of $\mathscr{D}_{\tilde{\Omega}}$ is $\Gamma_{\tilde{\Omega}}$. □

§5. Subgroups $\mathscr{D}_b$ and $\mathscr{D}_\Delta$ of $\mathscr{D}(\bar{M})$.

Let $\bar{M}$, $\tilde{\Omega}$ be the closed unit disk bundle over N and the symplectic structure on M defined by (10). Recall that $\mathscr{D}(\bar{M})$ is a strong ILH-Lie group with the Lie algebra $\Gamma(T\bar{M})$ (cf. §2), hence it is a regular Fréchet-Lie group (cf. [7] §6). Set

(24) $$\mathscr{D}_b=\{\varphi\in\mathscr{D}(\bar{M});\ \varphi|_{\partial M}=\mathrm{id.}\}\ .$$

By recalling how a local coordinate system on $\mathscr{D}(\bar{M})$ was given in §II. 4 in [4], we easily see that $\mathscr{D}_b$ is a locally flat FL-subgroup of $\mathscr{D}(\bar{M})$. Indeed, $\mathscr{D}_b$ has a structure, which one may call a strong ILH-Lie subgroup (cf. [4] I. 4 or [5]). $\mathscr{D}_{\tilde{\Omega},b}$ (cf. (22)) is by definition a subgroup of $\mathscr{D}_b$. However, by the certain reason which will be mentioned in this section, it is convenient to regard this as a subgroup of the following subgroup $\mathscr{D}_\Delta$:

(25) $$\mathscr{D}_\Delta=\{\varphi\in\mathscr{D}_b;\ \varphi^*\tilde{\Omega}-\tilde{\Omega}\ \text{can be extended to a smooth 2-form on}\ \bar{M}\}\ .$$

To prove Theorem 4.2, we shall show at first the following, which is indeed the goal of this section.

THEOREM 5.1. *$\mathscr{D}_\Delta$ is a closed* FL-*subgroup of $\mathscr{D}_b$.*

REMARK. The proof which will be given in this section shows also that $\mathscr{D}_\Delta$ has the structure which one may call a strong ILH-Lie subgroup of $\mathscr{D}_b$.

To prove Theorem 5.1 we shall have some steps as below. First, we get the following lemma.

LEMMA 5.2. *$\mathscr{D}_\Delta$ is a closed subgroup of $\mathscr{D}_b$.*

PROOF. We use the notations in the proof of Lemma 3.3. Near the boundary $S^*N=\partial\bar{M}$ every $\varphi\in\mathscr{D}(\bar{M})$ can be written in the form (21). We write $\varphi\in\mathscr{D}_\Delta$ in the form

(26) $$\begin{cases}\bar{s}=\bar{s}(s,z)\\ \bar{z}=\bar{z}(s,z)\ ,\end{cases}\qquad s=\cot\pi\theta/2\ ,\qquad (s,z)\in[0,\varepsilon)\times S^*N\ ,$$

where $\bar{s}(0,z)=0$ and $\bar{z}(0,z)=z$. Put

$$(27) \qquad \varphi^*\omega=\bar{z}^*\omega+a_\varphi ds ,$$

where a_φ is a smooth function on $[0, \varepsilon)\times S^*N$. Recall that $\tilde{\Omega}=-d(\omega/s)$ and also remark by (26) and (27),

$$(28) \qquad \varphi^*\left(\frac{\omega}{s}\right)-\frac{\omega}{s}=\frac{s}{\bar{s}}\left(\frac{\bar{z}^*\omega-\omega}{s}+\frac{a_\varphi}{s}ds+\frac{1-(\bar{s}/s)}{s}\omega\right) .$$

Since $\varphi\in\mathscr{D}(\bar{M})$, $s/\bar{s}$ is smooth on $[0, \varepsilon)\times S^*N$ and never vanishes. As $\varphi|_{S^*N}=\mathrm{id.}$, we see $(z^*\omega-\omega)|_{S^*N}=0$. Hence $(1/s)(\bar{z}^*\omega-\omega)$ is a C^∞ 1-form on $[0, \varepsilon)\times S^*N$. Also, the third term of the right hand side of (28) is smooth on $[0, \varepsilon)\times S^*N$ because $\bar{s}/s\equiv 1$ on S^*N. Compute $d(\varphi^*(\omega/s)-(\omega/s))$. Thus, $(1/s)da_\varphi\wedge ds$ must be smooth on $[0, \varepsilon)\times S^*N$, hence we see that $a_\varphi|_{S^*N}=\mathrm{const.}$. It is now not to hard to see that $\varphi\in\mathscr{D}_b$ is an element of $\mathscr{D}_\Delta$ if and only if $\bar{s}/s\equiv 1$ on S^*N and $a_\varphi=\omega\lrcorner d\varphi(\partial/\partial s)$ is constant on S^*N. Obviously, these are closed conditions and hence $\mathscr{D}_\Delta$ is a closed subgroup of $\mathscr{D}_b$. □

Set

$$\Gamma_\Delta=\{u\in\Gamma(T\bar{M});\ \exp tu\in\mathscr{D}_\Delta\} .$$

By Lemma 1.4 [8] we see that Γ_Δ is a closed Lie algebra of $\Gamma(T\bar{M})$.

LEMMA 5.3. *An element u of $\Gamma(T\bar{M})$ is contained in Γ_Δ if and only if it satisfies the following*:

(i) $u|_{S^*N}=0$.

(ii) *$d(\tilde{\Omega}\lrcorner u)$ can be extended to a smooth 2-form on $\bar{M}$.*

PROOF. For every $u\in\Gamma_\Delta$, $(\exp tu)^*\tilde{\Omega}-\tilde{\Omega}$ is a smooth 2-form on $\bar{M}$. Take the derivative with respect to t at $t=0$ and use $\mathscr{L}_u\tilde{\Omega}=d(\tilde{\Omega}\lrcorner u)$, where $\mathscr{L}_u$ is the Lie derivation. Then, we see $d(\tilde{\Omega}\lrcorner u)$ can be extended to a smooth 2-form on $\bar{M}$. Obviously, $u|_{S^*N}=0$.

To get the converse, remark that

$$(\exp tu)^*\tilde{\Omega}-\tilde{\Omega}=\int_0^t(\exp su)^*\mathscr{L}_u\tilde{\Omega}ds .$$

Since $\mathscr{L}_u\tilde{\Omega}=d(\tilde{\Omega}\lrcorner u)$ is extended to a smooth 2-form on $\bar{M}$ and $\exp tu\in\mathscr{D}_b$, we have $(\exp tu)^*\tilde{\Omega}-\tilde{\Omega}$ is a smooth 2-form on $\bar{M}$. □

Now, recall the notation $\tilde{\flat}: \Gamma(T\bar{M})\to\Gamma(T^*\bar{M})$ defined by $\tilde{\flat}(u)=\tilde{\Omega}\lrcorner u$ (cf. §2) and also use the notation (19) and (20) for $u\in\Gamma(T\bar{M})$.

LEMMA 5.4. $\tilde{\flat}(\Gamma_\Delta)=\Gamma(T^*\bar{M})+\boldsymbol{R}d(\phi\log\tan\pi\theta/2)$. *Moreover, $u\in\Gamma_\Delta$ if and only if near the boundary $S^*N=\partial\bar{M}$, u can be written in the form*

$$u=(\cos\pi\theta/2)^2 a'\frac{\partial}{\partial\theta}+(c\cos\pi\theta/2+(\cos\pi\theta/2)^2 g')X_\omega+(\cos\pi\theta/2)\hat{u}' ,$$

where a', g' *are smooth functions on* $\bar{M}$, c *a real constant and* $\hat{u}'$ *is a smooth section of* E_ω *regarded naturally as a subbundle of* $T(\bar{M}-\{0\})$.

PROOF. Use (19) and compute $\tilde{\Omega}\lrcorner(\lambda(\partial/\partial\theta)+\mu X_\omega+\hat{u})$, $\tilde{\Omega}=-d((\tan\pi\theta/2)\omega)$ by putting $\lambda=\lambda_0+\lambda_1\cos\pi\theta/2+\lambda_2(\cos\pi\theta/2)^2+\cdots$, etc.. The first statement follows immediately from the second one, and second one is obtained by the same proof as in Lemma 3.5. □

Now, recall that $\tilde{\mathfrak{b}}(\boldsymbol{\Gamma}_{\tilde{\Omega},b})=\boldsymbol{Z}(\bar{D}^*N)+\boldsymbol{R}d(\phi\log\tan\pi\theta/2)$ and that it is naturally contained in $\tilde{\mathfrak{b}}(\boldsymbol{\Gamma}_\Delta)$.

To prove Theorem 5.1, we shall recall how we define a structure of a strong ILH-Lie group (cf. [4], [5]). Put a smooth riemannian metric on $\bar{M}$ such that the boundary $S^*N=\partial\bar{M}$ is a totally geodesic. We denote by Exp the exponential mapping with respect to the metric on $\bar{M}$. For every $u\in\boldsymbol{\Gamma}(T\bar{M})$, put $\xi(u)(x)=\mathrm{Exp}_x u(x)$. If u is in a sufficiently small neighborhood U of 0 in the C^1-topology, then $\xi(u)$ is a smooth diffeomorphism on $\bar{M}$. The mapping $\xi\colon U\cap\boldsymbol{\Gamma}(T\bar{M})\to\mathscr{D}(\bar{M})$ gives a local coordinate system at the identity which satisfies the axioms (N. 1)~(N. 7) in §6 of [7] (cf. [4] §1 and [5] chap. 3). Set

$$\boldsymbol{\Gamma}_b=\{u\in\boldsymbol{\Gamma}(T\bar{M});\ u|_{\partial\bar{M}}\equiv 0\} .$$

Then, it is obvious that

$$\xi(U\cap\boldsymbol{\Gamma}_b)=\xi(U)\cap\mathscr{D}_b(\bar{M}) ,$$

hence $\xi\colon U\cap\boldsymbol{\Gamma}_b\to\mathscr{D}_b(\bar{M})$ gives a local coordinate system of $\mathscr{D}_b$ which also satisfies (N. 1)~(N. 7) in §6 of [7]. Since $U\cap\boldsymbol{\Gamma}_b$ is linearly imbedded in $\boldsymbol{\Gamma}(T\bar{M})$, one may call $\mathscr{D}_b$ a strong ILH-Lie subgroup of $\mathscr{D}(\bar{M})$.

REMARK. We have not necessary to use an exponential mapping. By an inverse mapping theorem in [4] §III or [5] Chap. 1, it may be replaced by any smooth mapping $\mathscr{E}$ of a neighborhood of zero-section of $T\bar{M}$ into $\bar{M}\times\bar{M}$ satisfying the following:

(*1) For $y\in\bar{M}$, $Y\in T_y\bar{M}$, $\mathscr{E}(y, Y)=(y, \mathscr{E}_y(Y))$ and $\mathscr{E}_y$ is a smooth diffeomorphism of a neighborhood of 0 on $T_y\bar{M}$ onto a neighborhood of y in T^*N.

(*2) If $y\in S^*N$ and $Y\in T_yS^*N$, then $\mathscr{E}_y(Y)\in S^*N$.

Recall that $\bar{M}-\{0\}$ is diffeomorphic to $(0, 1]\times S^*N$. We fix a smooth riemannian metric on $\bar{M}$ such that on $(1/2, 1]\times S^*N$, it is a direct product of $(1/2, 1]$ and S^*N. Then for $1/2<\varepsilon\leqq 1$, $\{\varepsilon\}\times S^*N$ is a totally geodesic submanifold. Let Exp be the exponential mapping with respect to the above riemannian metric. Note that there is a δ-neighborhood W_y of 0 in $T_y\bar{M}$ for every $y \in \bar{M}$ such that Exp_y is a smooth diffeomorphism of W_y onto a δ-neighborhood of y in T^*N. If y is near to S^*N, y is written as (θ, z), $z \in S^*N$, and a tangent vector $Y \in T_y\bar{M}$ is written as

$$(29) \qquad Y=\bar{\theta}\frac{\partial}{\partial\theta}+Y' , \qquad Y' \in T_zS^*N .$$

By this expression, if y is near to S^*N, we see

$$(30) \qquad \mathrm{Exp}_y Y=(\bar{\theta}+\theta, \mathrm{Exp}_z Y') , \qquad y=(\theta, z) ,$$

for small $Y \in T_y\bar{M}$.

Let E_ω is the extended distribution on $T^*N-\{0\}$ defined by ω (cf. (14), (15)). For an arbitrarily fixed $y \in \bar{M}-\{0\}$, we define a smooth non-involutive distribution E'_ω on the above W_y by

$$E'_\omega(Y)=(d\,\mathrm{Exp}_y^{-1})^*_Y E_\omega(\mathrm{Exp}_y Y) , \qquad Y \in W_y ,$$

where $E_\omega(y)$ (resp. $E'_\omega(Y)$) is the fiber of E_ω (resp. E'_ω) of y (resp. Y). Since $T_y\bar{M}=\boldsymbol{R}(\partial/\partial\theta)+\boldsymbol{R}\chi_\omega+E_\omega(y)$ (cf. (15)), we may use $(\bar{\theta}, \bar{\lambda}, \bar{Y})$, $\bar{Y} \in E_{\omega,y}$, as coordinate system on $T_y\bar{M}$. Suppose δ is sufficiently small and y is sufficiently near to S^*N. Then the fiber $E'_\omega(\bar{\theta}, \bar{\lambda}, \bar{Y})$ of E'_ω at $Y=(\bar{\theta}, \bar{\lambda}, \bar{Y}) \in W_y$ does not depend on $\bar{\theta}$ and there is a smooth mapping $\Lambda(\bar{\lambda}, \bar{Y})$ of W_y into the dual space $E^*_\omega(y)$ of $E_\omega(y)$ such that

$$E'_\omega(\bar{\theta}, \bar{\lambda}, \bar{Y})=\{(\Lambda(\bar{\lambda}, \bar{Y})Z)\chi_\omega(y)+Z, \forall Z \in E_\omega(y)\}$$

because of (30).

Now, solve the equation $d\bar{\lambda}(t)/dt=\Lambda(\bar{\lambda}(t), t\bar{Y})\bar{Y}$, $\bar{\lambda}(0)=0$, for an arbitrarily fixed small $\bar{Y} \in E_\omega(y)$. The solution will be denoted by $\bar{\lambda}(t, \bar{Y})$. $\bar{\lambda}(t, \bar{Y})$ is smooth with respect to $(t, \bar{Y})$, and by definition $d(\bar{\lambda}(t, \bar{Y})\chi_\omega(y)+t\bar{Y})/dt=\Lambda(\bar{\lambda}(t, \bar{Y}), t\bar{Y})\bar{Y}\chi_\omega(y)+\bar{Y}$ is an element of $E'_\omega(\bar{\theta}, \bar{\lambda}(t, \bar{Y}), t\bar{Y})$. Set $e_y(\bar{Y})=\mathrm{Exp}_y(\bar{\lambda}(1, \bar{Y})\chi_\omega(y)+\bar{Y})$. Then, e_y is a diffeomorphism of $W_y \cap E_\omega(y)$ into T^*N such that $de_y(t\bar{Y})/dt \in E_\omega(e_y(t\bar{Y}))$ for every $0 \leq t \leq 1$, where y is assumed to be near to the boundary S^*N.

Let $\mathscr{T}_\omega(t)(z)$ be the integral curve of χ_ω with the initial point $z \in \bar{M}-\{0\}$, i.e.,

$$\frac{d}{dt}\mathscr{T}_\omega(t)(z)=\chi_\omega(\mathscr{T}_\omega(t)(z)) , \qquad \mathscr{T}_\omega(0)(z)=z .$$

Since $\mathscr{L}_{\chi_\omega}\omega=0$, we see that $d\mathscr{T}_\omega(t)E_\omega=E_\omega$. Especially, if y is sufficiently close to S^*N, then $d\mathscr{T}_\omega(t)(de_y(t\bar{Y})/dt) \in E_\omega(\mathscr{T}_\omega(t)(e_y(t\bar{Y})))$. For

$Z=\bar{\theta}(\partial/\partial\theta)+\bar{\lambda}\chi_\omega(y)+\bar{Y}\in T_y\bar{M}$, we set

$$\mathscr{E}'_y(Z)=\mathrm{Exp}_{\mathscr{I}_\omega(\bar{\lambda})e_y(\bar{Y})}\bar{\theta}\frac{\partial}{\partial\theta} . \tag{31}$$

If we use a coordinate system $y=(\theta, z)$, $z\in S^*N$, then

$$\mathscr{E}'_{(\theta,z)}(Z)=(\theta+\bar{\theta}, \mathscr{I}_\omega(\bar{\lambda})(e_y(\bar{Y}))) . \tag{32}$$

Therefore, $\mathscr{E}'_y$ is a diffeomorphism of a neighborhood of 0 in $T_y\bar{M}$ onto a neighborhood of y in T^*N such that $\mathscr{E}_y(Z)\in S^*N$ if $y\in S^*N$, $Z\in W_y\cap T_yS^*N$ and $\mathscr{E}_y(Z)=y$ if $Z=0$.

Let ϕ be the smooth function on $[0, \infty)$ defined by (9). Since $\bar{M}-\{0\}$ is diffeomorphic to $(0, 1]\times S^*N$, ϕ can be regarded as a smooth function on $\bar{M}$. Thus, we set

$$\mathscr{E}_y(X)=\mathscr{E}'_z(\phi(y)X) , \qquad z=\mathrm{Exp}_y (1-\phi(y))X . \tag{33}$$

Then, $\mathscr{E}_y$ is a diffeomorphism of a neighborhood of 0 in $T_y\bar{M}$ onto a neighborhood of y in T^*N. If y is sufficiently near to S^*N, then $\mathscr{E}_y(X)=\mathscr{E}'_y(X)$. Set $\mathscr{E}(y, X)=(y, \mathscr{E}_y(X))$. Then, $\mathscr{E}$ is a smooth mapping satisfying (*1) and (*2) mentioned as above.

LEMMA 5.5. *Set $\xi'(u)(y)=\mathscr{E}'_y(u(y))$. Then, there is a neighborhood U of 0 of $\Gamma(T\bar{M})$ in the C^1 topology such that $\xi'\colon U\cap\Gamma(T\bar{M})\to\mathscr{D}(\bar{M})$ defines a local coordinate system such that if $u\in U\cap\Gamma_\Delta$ (resp. $U\cap\Gamma_b$), then $\xi'(u)\in\mathscr{E}_\Delta$ (resp. $\mathscr{D}_b$).*

PROOF. By the inverse mapping theorem [4] §III, we see that ξ' gives a local coordinate system of a strong ILH-Lie group $\mathscr{D}(\bar{M})$. For $u\in\Gamma_b$ we set $v_t=(d\xi'(tu)/dt)\xi'(tu)^{-1}$. We shall prove at first that $v_t\in\Gamma_b$ for $t\in[0, 1]$. For the proof, we have only to consider a sufficiently small neighborhood of the boundary. Using the diffeomorphism $\bar{M}-\{0\}\approx(0, 1]\times S^*N$, we may assume $y\in(1/2, 1]\times S^*N$.

Set $y=(\theta, z)$, $z\in S^*N$ and

$$u(\theta, z)=\bar{\theta}(\theta, z)\frac{\partial}{\partial\theta}+\bar{\lambda}(\theta, z)\chi_\omega(\theta, z)+\hat{u}(\theta, z) ,$$

where $\hat{u}(\theta, z)\in E_\omega((\theta, z))$. Therefore, using the direct product structure on $(1/2, 1]\times S^*N$, we set

$$\frac{d}{dt}\mathscr{E}'_{(\theta,z)}(tu(\theta, z))=\Big(\bar{\theta}(\theta, z), \bar{\lambda}(\theta, z)\chi_\omega(\mathscr{E}'_{(\theta,z)}(tu(\theta, z)))$$

$$+d\mathscr{J}_{\omega}(t\bar{\lambda}(\theta, z))\frac{d}{dt}e_{(\theta,z)}(t\hat{u}(\theta, z))\Big) .$$

Remark that $d\mathscr{J}_{\omega}(t\bar{\lambda}(\theta, z))(d/dt)e_{(\theta,z)}(t\hat{u}(\theta, z)) \in E_{\omega}(\mathscr{E}'_{(\theta,z)}(t\hat{u}(\theta, z)))$ and this quantity is zero if $\theta=1$ because $u|_{S^*N}=0$. Since $\bar{\theta}(1, z)=\bar{\lambda}(1, z)=0$, we have easily $v_t|_{S^*N}\equiv 0$, hence $\xi'(u) \in \mathscr{D}_b$.

If $u \in \Gamma_{\Delta}$, then using the above equality, we see $v_t(\theta, z)=(\bar{\theta}(\bar{y}), \bar{\lambda}(\bar{y})\chi_{\omega}(\theta, z)+d\mathscr{J}_{\omega}(t\bar{\lambda}(\bar{y}))(d/dt)e_{\bar{y}}(t\hat{u}(\bar{y})))$, where $\bar{y}=\xi'(tu)^{-1}(\theta, z)$. On the other hand, by Lemma 5.4, we see easily that $u \in \Gamma_{\Delta}$ if and only if

$$\begin{cases} \bar{\theta}(1, z)\equiv 0 , & \dfrac{\partial\bar{\theta}}{\partial\theta}(1, z)\equiv 0 , \qquad \hat{u}(1, z)=0 , \\ \bar{\lambda}(1, z)\equiv 0 , & \dfrac{\partial\bar{\lambda}}{\partial\theta}(1, z)\equiv \text{const.} . \end{cases}$$

Hence, remarking $\xi'(tu) \in \mathscr{D}_b$, we get $v_t \in \Gamma_{\Delta}$ from the fact $u \in \Gamma_{\Delta}$. Since $(d/dt)\xi'(tu)=v_t\xi'(tu)$, we have

$$\xi'(u)^*\tilde{\Omega}-\tilde{\Omega}=\int_0^1 \xi'(tu)^*\mathscr{L}_{v_t}\tilde{\Omega}dt \in d(\Gamma(T^*\bar{M})) ,$$

hence $\xi'(u) \in \mathscr{D}_{\Delta}$. □

PROOF OF THEOREM 5.1. Notations being as above, $\xi': U\cap\Gamma(T\bar{M})\rightarrow\mathscr{D}(\bar{M})$ is a local coordinate system at the identity such that $\xi'(U\cap\Gamma_b)\subset\mathscr{D}_b$ and $\xi'(U\cap\Gamma_{\Delta})\subset\mathscr{D}_{\Delta}$. Thus, it is enough to show that $\xi'(U\cap\Gamma_b(T\bar{M}))=\xi'(U)\cap\mathscr{D}_b$ and $\xi'(U\cap\Gamma_{\Delta})=\xi'(U)\cap\mathscr{D}_{\Delta}$. The first equality is trivial, because $\mathscr{E}_y$ is a local diffeomorphism. Thus, we have only to show the second one. Remark that "$\subset$" is given by Lemma 5.5. We concern only "$\supset$".

Let $(z_1, \cdots, z_m)$, $m=\dim M-1$, be a local coordinate system of S^*N at z. Then, putting $\tau=1-\theta$, $(\tau, z_1, \cdots, z_m)$ is a local coordinate system of $\bar{M}$ at z. For $u=\hat{\tau}(\partial/\partial\tau)+\sum\hat{z}_i(\partial/\partial z_i)$, $\mathscr{E}_{(\tau,z_1,\cdots,z_m)}(u)$ is expressed in the form

$$\begin{cases} \bar{\tau}=\tau+\hat{\tau} \\ \bar{z}_i=\bar{z}_i(\hat{\tau}, \hat{z}_1, \cdots, \hat{z}_m; \tau, z_1, \cdots, z_m) . \end{cases}$$

Then, we see easily that $(\bar{\tau}, \bar{z}_1, \cdots, \bar{z}_m)=(\tau, z_1, \cdots, z_m)$ if and only if $(\hat{\tau}, \hat{z}_1, \cdots, \hat{z}_m)=(0, 0, \cdots, 0)$. Moreover, it is not hard to see by the definition of $\mathscr{E}_{(\tau,z_1,\cdots,z_m)}$ that

$$\left(\begin{array}{c|c} (\partial\tau/\partial\hat{\tau}) & 0 \\ \hline (\partial\bar{z}_i/\partial\hat{\tau}) & (\partial\bar{z}_i/\partial\hat{z}_j) \end{array}\right)_{(\hat{\tau},\hat{z}_1,\cdots,\hat{z}_m)=(0,\cdots,0)} = \text{identity} .$$

Now, let $\varphi \in \xi'(U) \cap \mathscr{D}_\Delta$. Then, φ can be written in the form

$$\begin{cases} \bar{\tau} = \varphi_0(\tau, z_1, \cdots, z_m) \\ \bar{z}_i = \varphi_i(\tau, z_1, \cdots, z_m) \qquad i = 1 \sim m , \end{cases}$$

where if $\tau = 0$, then $\bar{\tau} = 0$ and $\bar{z}_i = z_i$ because $\varphi \in \mathscr{D}_b$. By the proof of Lemma 5.2, we get $\varphi \in \mathscr{D}_\Delta$ if and only if $\varphi \in \mathscr{D}_b$ and $(\partial\bar{\tau}/\partial\tau)|_{\tau=0} \equiv 1$ and $\sum \omega_i(z_1, \cdots, z_m)(\partial\bar{z}_i/\partial\tau) = \text{const.}$ on S^*N, where ω_i are the components of ω, i.e., $\omega = \sum \omega_i dz_i$.

Since $\bar{\tau} = \tau + \hat{\tau}$ and $\bar{z}_i = \bar{z}_i(\hat{\tau}, \hat{z}_1, \cdots, \hat{z}_m; \tau, z_1, \cdots, z_m)$, we obtain $(\partial\hat{\tau}/\partial\tau)|_{\tau=0} = 0$ and $\sum \omega_i(\partial\bar{z}_i/\partial\tau) = \sum \omega_i((\partial\bar{z}_i/\partial\hat{\tau})(\partial\hat{\tau}/\partial\tau) + (\partial\bar{z}_i/\partial\hat{z}_j)(\partial\hat{z}_j/\partial\tau))$ is constant on S^*N. As $\varphi|S^*N = \text{identity}$, we have $(\partial\bar{z}_i/\partial\hat{z}_j)|_{\tau=0} = \text{identity}$, and hence $\sum \omega_i(\partial\hat{z}_i/\partial\tau) = \text{const.}$ on S^*N. Hence by Lemma 5.4, we get that $u = \hat{\tau}(\partial/\partial\tau) + \sum \hat{z}_i(\partial/\partial z_i)$ is contained in $\boldsymbol{\Gamma}_\Delta$. Thus, we see $\xi'(U \cap \boldsymbol{\Gamma}_\Delta) \supset \xi'(U) \cap \mathscr{D}_\Delta$, hence the desired equality. □

REMARK. It is not hard to see that ξ': $U \cap \boldsymbol{\Gamma}_\Delta \to \mathscr{D}_\Delta$ satisfies (N. 1~7) in [7] §6. Therefore $\mathscr{D}_\Delta$ has a structure of a strong ILH-Lie group. See also the next section for the modeled Sobolev chain.

§6. An implicit function theorem and $\mathscr{D}_{\tilde{\Omega}}$.

Remark that $\mathscr{D}_\Delta$ acts naturally on the space of smooth 2-forms on $\bar{M}$. We denote by $\boldsymbol{E}_{\tilde{\Omega}}$ the affine space $\tilde{\Omega} + d\boldsymbol{\Gamma}(T^*\bar{M})$.

LEMMA 6.1. *Suppose φ is an element of the identity component of $\mathscr{D}_\Delta$. Then $\varphi^*\boldsymbol{E}_{\tilde{\Omega}} = \boldsymbol{E}_{\tilde{\Omega}}$.*

PROOF. Let φ_t be a smooth curve in $\mathscr{D}_\Delta$ such that $\varphi_0 = e$ and $\varphi_1 = \varphi$. Set $v_t = (d\varphi_t/dt) \cdot \varphi_t^{-1}$. Then $v_t \in \boldsymbol{\Gamma}_\Delta$. Therefore

$$\varphi^*\tilde{\Omega} - \tilde{\Omega} = \int_0^1 \varphi_t^* \mathscr{L}_{v_t}\tilde{\Omega}dt \in d\boldsymbol{\Gamma}(T^*\bar{M}) , \quad \text{i.e.,} \quad \varphi^*\tilde{\Omega} \in \boldsymbol{E}_{\tilde{\Omega}} .$$

It follows immediately $\varphi^*\boldsymbol{E}_{\tilde{\Omega}} = \boldsymbol{E}_{\tilde{\Omega}}$. □

Let $\mathscr{D}'_\Delta$ be the totality of $\varphi \in \mathscr{D}_\Delta$ such that $\varphi^*\boldsymbol{E}_{\tilde{\Omega}} = \boldsymbol{E}_{\tilde{\Omega}}$. By the above lemma, $\mathscr{D}'_\Delta$ is an open subgroup of $\mathscr{D}_\Delta$ and $\mathscr{D}_{\tilde{\Omega}} \subset \mathscr{D}'_\Delta$. Set $\Phi_{\tilde{\Omega}}(\varphi) = \varphi^*\tilde{\Omega}$. Then by identifying $\boldsymbol{E}_{\tilde{\Omega}}$ with the Fréchet space $d\boldsymbol{\Gamma}(T^*\bar{M})$, $\Phi_{\tilde{\Omega}}$: $\mathscr{D}'_\Delta \to \boldsymbol{E}_{\tilde{\Omega}}$ is a smooth mapping. The derivative $d\Phi_{\tilde{\Omega}}$ at the identity is given by $(d\Phi_{\tilde{\Omega}})u = d(\tilde{\Omega} \lrcorner u) \in d\boldsymbol{\Gamma}(T^*\bar{M})$. Remark that Lemma 5.4 shows also $d\boldsymbol{\Gamma}(T^*\bar{M}) = \{d(\tilde{\Omega} \lrcorner u); u \in \boldsymbol{\Gamma}_\Delta\}$. Therefore

$$d\Phi_{\tilde{\Omega}}: \boldsymbol{\Gamma}_\Delta \longrightarrow d\boldsymbol{\Gamma}(T^*\bar{M})$$

is a surjection, and the kernel of $d\Phi_{\tilde{\Omega}}$ is given by $\Gamma_{\tilde{\Omega}}$. We are going to apply the implicit function theorem in [4] to the mapping $\Phi_{\tilde{\Omega}}$. If it were done without any obstruction, then $\mathscr{D}_{\tilde{\Omega}}=\Phi_{\tilde{\Omega}}^{-1}(\tilde{\Omega})$ would be a strong ILH-Lie subgroup of $\mathscr{D}'_{\Delta}$. However there is an obstruction, and hence we can get only a weaker conclusion, which is stated as follows and indeed this is the goal of this section:

Theorem 6.2. *$\mathscr{D}_{\tilde{\Omega}}$ is an FL-subgroup of $\mathscr{D}'_{\Delta}$.*

Remark. Since $\mathscr{D}'_{\Delta}$ is an FL-subgroup of $\mathscr{D}_b$ and $\mathscr{D}_b$ is also an FL-subgroup of $\mathscr{D}(\bar{M})$, the above theorem gives the proof of Theorem 4.2, and hence Theorem B.

Let $\langle\ ,\ \rangle_k$, $k\geqq 0$, be an inner product on $\Gamma(T\bar{M})$ defined by

$$\langle u, v\rangle_k=\sum_{s=0}^{k}\int_{\bar{M}}\langle \nabla^s u, \nabla^s v\rangle(y)dV(y)\ ,$$

where $\nabla^s u$ is the s-times covariant derivative of u with respect to the riemannian connection on $\bar{M}$ given in the previous section, $\langle\ ,\ \rangle$ inside the integration sign is the fiberwise inner product on $\otimes^s T^*\bar{M}\otimes T\bar{M}$ and dV is the volume element on $\bar{M}$. Let $\|u\|_k$ be the norm given by $\sqrt{\langle u, u\rangle_k}$. We denote by $\Gamma^k(T\bar{M})$ the completion of $\Gamma(T\bar{M})$ with respect to the norm $\|\ \ \|_k$. We define also a norm $\|\ \ \|_k$ on the space $\Gamma(\bigwedge^p\bar{M})$ of smooth p-forms on $\bar{M}$ by the similar manner. Let $\Gamma^k(\bigwedge^p\bar{M})$ be the completion of $\Gamma(\bigwedge^p\bar{M})$. By Sobolev's imbedding theorem, if $k\geqq k_0=[(1/2)\dim\bar{M}]+1$, then every $u\in\Gamma^k(T\bar{M})$ is a C^{k-k_0} vector field on $\bar{M}$. Let Γ^k_{Δ} be the closure of Γ_{Δ} in $\Gamma^k(T\bar{M})$. Then, $\mathscr{D}_{\Delta}$ is in fact a strong ILH-Lie group modeled on an ILH-chain $\{\Gamma_{\Delta}, \Gamma^k, k\geqq\dim\bar{M}+5\}$ (cf. [4] §II).

Let $\tilde{\flat}$: $\Gamma_{\Delta}\to\Gamma(T^*\bar{M})\oplus\boldsymbol{R}d(\phi\log\tan\pi\theta/2)$ be the isomorphism defined by $\tilde{\flat}(u)=\tilde{\Omega}\lrcorner u$. The inverse mapping $\tilde{\flat}^{-1}$ is sometimes regarded as a mapping of $\Gamma(T^*\bar{M})$ into Γ_{Δ} by restriction.

Lemma 6.3. *$\tilde{\flat}^{-1}$: $\Gamma(T^*\bar{M})\to\Gamma_{\Delta}$ satisfies the following inequalities: $C_k\|\alpha\|_{k-2}\leqq\|\tilde{\flat}^{-1}(\alpha)\|_k\leqq C\|\alpha\|_k+D_k\|\alpha\|_{k-1}$, $k\geqq 2$, where C_k, C, D_k are positive constants such that C does not depend on k.*

Remark. There is no inequality such as $C'\|\alpha\|_k-D'_k\|\alpha\|_{k-1}\leqq\|\tilde{\flat}^{-1}(\alpha)\|_k$. Therefore, $\tilde{\flat}$ gives only an isomorphism of Fréchet spaces. It can not be extended to an isomorphism of ILH-chains.

Proof of Lemma 6.3. Note that for every $y\in M$, the mapping $X\mapsto\tilde{\Omega}\lrcorner X$ is a linear isomorphism of T_yM onto T_y^*M. The inverse mapping will be denoted by $\tilde{\Omega}^{-1}$. Let ϕ be a smooth function on [0, 1] such that $\phi\equiv 1$ on $[2/3, 1]$, $\equiv 0$ on $[0, 1/3]$ and $\phi\geqq 0$ on $[0, 1]$. For every $\alpha\in\Gamma(T^*\bar{M})$, set $\alpha=\phi\alpha+(1-\phi)\alpha$. We see easily

$$\tilde{\flat}^{-1}((1-\phi)\alpha)=(1-\phi)\tilde{\flat}^{-1}(\alpha)=\tilde{\Omega}^{-1}((1-\phi)\alpha) .$$

Since $1-\phi\equiv 0$ on a neighborhood of S^*N, we have

$$\|\tilde{\flat}^{-1}((1-\phi)\alpha)\|_k\leqq C'\|\alpha\|_k+D'_k\|\alpha\|_{k-1} , \qquad k\geqq 1 ,$$

where C', D'_k are positive constants and C' does not depend on k.

On $\bar{M}-\{0\}$, every α can be written in the form

$$\alpha=fd\theta+g\omega+\hat{\alpha} , \qquad \hat{\alpha}(y)\in E^*_\omega(y) ,$$

hence on $\bar{M}-\{0\}$

$$\tilde{\flat}^{-1}(\alpha)=\frac{2}{\pi}\Big(\cos\frac{\pi\theta}{2}\Big)^2 g\frac{\partial}{\partial\theta}-\frac{2}{\pi}\Big(\cos\frac{\pi\theta}{2}\Big)^2 f\chi_\omega+\Big(\cot\frac{\pi\theta}{2}\Big)d\omega^{-1}\hat{\alpha} .$$

Note that $\tilde{\flat}^{-1}(\phi\alpha)=\phi\tilde{\flat}^{-1}(\alpha)$ and that $\phi\equiv 0$ on $[0, 1/3]$. Hence we get

$$\|\tilde{\flat}^{-1}(\phi\alpha)\|_k\leqq C''\{\|\phi g\|_k+\|\phi f\|_k+\|\phi\hat{\alpha}\|_k\}+D''_k\{\|\phi g\|_{k-1}+\|\phi f\|_{k-1}+\|\phi\hat{\alpha}\|_{k-1}\} , \qquad k\geqq 1 .$$

Since $d\theta$, ω, E^*_ω are linearly independent at every $y\in\bar{M}-\{0\}$, we have

$$\|\phi g\|_k+\|\phi f\|_k+\|\phi\hat{\alpha}\|_k\leqq\bar{C}\|\phi\alpha\|_k+\bar{D}_k\|\phi\alpha\|_{k-1} , \qquad k\geqq 1$$

and hence

$$\|\tilde{\flat}^{-1}(\phi\alpha)\|_k\leqq\tilde{C}\|\phi\alpha\|_k+\tilde{D}_k\|\phi\alpha\|_{k-1} , \qquad k\geqq 1 .$$

Thus, by using $\|\tilde{\flat}^{-1}(\alpha)\|_k\leqq\|\tilde{\flat}^{-1}(\phi\alpha)\|_k+\|\tilde{\flat}^{-1}((1-\phi)\alpha)\|_k$ we obtain the second inequality.

Now, we shall prove the first inequality. If $\tilde{\flat}^{-1}(\alpha)=u$, then $\tilde{\Omega}\lrcorner(1-\phi)u=(1-\phi)\alpha$. Since $1-\phi\equiv 0$ on an neighborhood of S^*N, we see that there is a positive constant C'_k such that

$$\|\tilde{\Omega}\lrcorner(1-\phi)u\|_k\leqq C'_k\|(1-\phi)u\|_k , \qquad k\geqq 0 .$$

Especially

$$\|\tilde{\flat}^{-1}((1-\phi)\alpha)\|_k\geqq C''_k\|(1-\phi)\alpha\|_{k-2} , \qquad k\geqq 2 .$$

Since $\partial/\partial\theta$, χ_ω, E_ω are linearly independent on $\bar{M}-\{0\}$, there is a positive constant $\bar{C}_k$ such that

$$\|\tilde{\flat}^{-1}(\phi\alpha)\|_k \geqq \bar{C}_k\left\{\left\|\left(\cos\frac{\pi\theta}{2}\right)^2\phi g\right\|_k + \left\|\left(\cos\frac{\pi\theta}{2}\right)^2\phi f\right\|_k + \left\|\left(\cot\frac{\pi\theta}{2}\right)\phi\hat{\alpha}\right\|_k\right\} .$$

Remark that $(\cos\pi\theta/2)/(1-\theta)$ is a smooth function on $(0, 1]$. Hence we may set

$$\begin{aligned}(\cos\pi\theta/2)^2\phi g &= (1-\theta)^2\hat{\phi}g\\ (\cos\pi\theta/2)^2\phi f &= (1-\theta)^2\hat{\phi}f\\ (\cot\pi\theta/2)\phi\hat{\alpha} &= (1-\theta)\bar{\phi}\hat{\alpha}\ ,\end{aligned}$$

where $\hat{\phi}$, $\bar{\phi}$ are smooth functions in θ such that $\equiv 0$ on a neighborhood of zero. Set $(1-\theta)\bar{\phi}\hat{\alpha}=\hat{\beta}$ and denote by (θ, z) a point in $\bar{M}-\{0\}$, where $z\in S^*N$. Then it is easy to see that

$$\bar{\phi}\hat{\alpha}(\theta, z) = -\int_0^1 \frac{\partial\hat{\beta}}{\partial\theta}(\theta+(1-\theta)t, z)dt .$$

Therefore

$$C_k\|\bar{\phi}\hat{\alpha}\|_{k-1} \leqq \|\hat{\beta}\|_k = \|(\cot\pi\theta/2)\phi\hat{\alpha}\|_k .$$

Similarly, we get

$$\begin{aligned}C'_k\|\hat{\phi}g\|_{k-2} &\leqq \|(\cos\pi\theta/2)^2\phi g\|_k\\ C''_k\|\hat{\phi}f\|_{k-2} &\leqq \|(\cos\pi\theta/2)^2\phi f\|_k\ , \qquad k\geqq 2 .\end{aligned}$$

By the above three inequalities, there are positive constants, $\tilde{C}_k$, $\hat{C}_k$ such that

$$\begin{aligned}\|\tilde{\flat}^{-1}(\phi\alpha)\|_k &\geqq \tilde{C}_k\{\|\hat{\phi}g\|_{k-2} + \|\hat{\phi}f\|_{k-2} + \|\bar{\phi}\hat{\alpha}\|_{k-1}\}\\ &\geqq \hat{C}_k\{\|\phi g\|_{k-2} + \|\phi f\|_{k-2} + \|\phi\hat{\alpha}\|_{k-1}\} .\end{aligned}$$

The reason of the last inequality is in the fact that $\hat{\phi}/\phi$ and $\bar{\phi}/\phi$ are positive on $[1/3, 1]$. Note that there is $C_k>0$ such that

$$\|\phi g\|_{k-2} + \|\phi f\|_{k-2} + \|\phi\hat{\alpha}\|_{k-2} \geqq C_k\|\phi\alpha\|_{k-2} .$$

Thus, using $\|\phi\hat{\alpha}\|_{k-2} \leqq \|\phi\hat{\alpha}\|_{k-1}$ we get

$$\|\tilde{\flat}^{-1}(\phi\alpha)\|_k \geqq C'_k\|\phi\alpha\|_{k-2} .$$

Note that there is a constant $C''_k>0$ such that

$$\|\tilde{\flat}^{-1}(\alpha)\|_k \geqq C''_k\{\|\tilde{\flat}^{-1}((1-\phi)\alpha)\|_k + \|\tilde{\flat}^{-1}(\phi\alpha)\|_k\} .$$

Then, we get

$$\|\tilde{\flat}^{-1}(\alpha)\|_k \geqq \bar{C}_k\{\|(1-\phi)\alpha\|_{k-2}+\|\phi\alpha\|_{k-2}\}$$
$$\geqq C_k\|\alpha\|_{k-2}\,. \qquad \square$$

Now, we define a new series of norms $\{\|\cdot\|_k\}$ on $\boldsymbol{\Gamma}_\Delta$ by putting $\|u\|_k = \|\tilde{\flat}^{-1}(u)\|_k$, where we put $\|d(\phi \log \tan \pi\theta/2)\|_k \equiv 1$ for all k, and $\boldsymbol{\Gamma}(T^*\bar{M})$ and $\boldsymbol{R}d(\phi \log \tan \pi\theta/2)$ are assumed to be orthogonal. Denote by $\hat{\Gamma}^k_\Delta$ the completion of $\boldsymbol{\Gamma}_\Delta$ with respect to $\|\cdot\|_k$. $\{\boldsymbol{\Gamma}_\Delta, \hat{\Gamma}^k_\Delta, k \geqq \dim \bar{M}+7\}$ is then an ILH-chain.

REMARK. $\mathscr{D}_\Delta$ is in fact a strong ILH-Lie group modeled on an ILH-chain $\{\boldsymbol{\Gamma}_\Delta, \Gamma^k_\Delta; k \geqq \tilde{k}_0\}$ for every $\tilde{k}_0 \geqq \dim \bar{M}+5$. However, it may not be modeled on $\{\boldsymbol{\Gamma}_\Delta, \hat{\Gamma}^k_\Delta, k \geqq \dim \bar{M}+7\}$. In fact, a group can have many strong ILH-Lie group structures modeled on various ILH-chains. It is very likely that $\mathscr{D}_\Delta$ is a strong ILH-Lie group modeled on $\{\boldsymbol{\Gamma}_\Delta, \hat{\Gamma}^k_\Delta, k \geqq \dim \bar{M}+7\}$. However, at this moment this is only a conjecture. This is the reason why we could not conclude that $\mathscr{D}_{\tilde{\Omega}}$ is a strong ILH-Lie subgroup. To apply an implicit function theorem in [4] §III, we have to use the ILH-chain $\{\boldsymbol{\Gamma}_\Delta, \hat{\Gamma}^k_\Delta, k \geqq \dim \bar{M}+7\}$.

Now, let U be an open neighborhood of 0 in $\Gamma^{k_0}_\Delta$, $k_0 = \dim \bar{M}+5$, such that $\xi'\colon U \cap \boldsymbol{\Gamma}_\Delta \to \mathscr{D}_\Delta$ defined in Lemma 5.5 gives a local coordinate system of $\mathscr{D}_\Delta$ at the identity e. Since

$$\tilde{\flat}^{-1}\colon \Gamma^{k_0+2}(T^*\bar{M}) \oplus \boldsymbol{R}d(\phi \log \tan \pi\theta/2) \longrightarrow \Gamma^{k_0}_\Delta$$

is bounded there is an open neighborhood U' of 0 in the source space such that $\tilde{\flat}^{-1}(U') \subset U$. Define $\Psi_{\tilde{\Omega}}\colon U' \cap \boldsymbol{\Gamma}(T^*\bar{M}) \to \boldsymbol{E}_{\tilde{\Omega}}$ by

$$\Psi_{\tilde{\Omega}}(\alpha) = \Phi_{\tilde{\Omega}}(\xi'(\tilde{\flat}^{-1}(\alpha)))\,.$$

Then, we have

LEMMA 6.4. *$\Psi_{\tilde{\Omega}}\colon U' \cap \boldsymbol{\Gamma}(T^*\bar{M}) \to \boldsymbol{E}_{\tilde{\Omega}}$ is a C^∞ ILB C^2-normal mapping* (cf. [4] §III), *namely*

(1) *$\Psi_{\tilde{\Omega}}$ can be extended to a smooth mapping of $U' \cap \Gamma^k(T\bar{M})$ into E^{k-1} for every $k \geqq \dim \bar{M}+8$.*

(2) *There are a positve constant C independent of k and a polynomial $P_k(t)$ with positive coefficients depending on k such that for every $k \geqq k_0+1$,*

$$\|(d\Psi_{\tilde{\Omega}})_\alpha\beta\|_{k-1} \leqq C\{\|\alpha\|_k\|\beta\|_{k_0}+\|\beta\|_k\}+P_k(\|\alpha\|_{k-1})\|\beta\|_{k-1}$$

$$\|(d^2\Psi_{\tilde{\Omega}})_\alpha(\beta_1, \beta_2)\|_{k-1} \leqq C\{\|\alpha\|_k\|\beta_1\|_{k_0}\|\beta_2\|_{k_0}+\|\beta_1\|_k\|\beta_2\|_{k_0}$$
$$+\|\beta_1\|_{k_0}\|\beta_2\|_k\}+P_k(\|\alpha\|_{k-1})\|\beta_1\|_{k-1}\|\beta_2\|_{k-1}\,.$$

PROOF. It is known that $\Psi'(u)=\Phi_{\tilde{\Omega}}(\xi'(u))$ is a C^∞ ILB C^2-normal mapping of $U\cap\Gamma_\Delta$ into $E_{\tilde{\Omega}}=\tilde{\Omega}\oplus d\Gamma(T^*\bar{M})$ (cf. [4] Lemma 2.5.3). By the second inequality in Lemma 6.3, $\tilde{\flat}^{-1}$: $U'\cap\Gamma(T^*\bar{M})\to U\cap\Gamma(T\bar{M})$ is a C^∞ ILB C^2-normal mapping. Hence so is the composition $\Psi_{\tilde{\Omega}}=\Psi'\cdot\tilde{\flat}^{-1}$ (cf. [5] Chap. I). □

For simplicity of notations, we set

$$F=\Gamma(T^*\bar{M})\oplus Rd(\phi\log\tan\pi\theta/2)$$
$$F^k=\Gamma^k(T^*\bar{M})\oplus Rd(\phi\log\tan\pi\theta/2)\ .$$

Note that F^k is isomorphic to $\hat{\Gamma}^k_\Delta$, and that $(d\Psi_{\tilde{\Omega}})_0$: $F\to d\Gamma(T^*\bar{M})$ is given by $(d\Psi_{\tilde{\Omega}})_0\beta=d\beta$ for $\beta\in\Gamma(T^*\bar{M})$ and $(d\Psi_{\tilde{\Omega}})_0 d(\phi\log\tan\pi\theta/2)=0$. Thus, to apply the implicit function theorem (cf. Theorem 3.3.1 in [4] or [5] §6 in Chap. I), we have only to see the following:

LEMMA 6.5. *For every $k\geqq 1$, $d\Gamma^k(T^*\bar{M})$ is a closed subspace of $\Gamma^{k-1}(\wedge^2\bar{M})$, and there is a right inverse B of d: $\Gamma(T^*\bar{M})\to d\Gamma(T^*\bar{M})$ such that for $k\geqq 2$,*

$$\|B\beta\|_k\leqq C\|\beta\|_{k-1}+D_k\|\beta\|_{k-2}\ ,$$

where C, D_k are positive constants, and C does not depend on k.

PROOF. The above fact is an immediate consequence of Hodge theory on a compact manifold with boundary, and rather widely known. Roughly speaking it is proved by using Stokes' theorem and a regularity theorem in [5] combined with Gårding's inequality in [5]. □

PROOF OF THEOREM 6.2. U' was an open neighborhood of 0 in F^{k_1}, $k_1=k_0+2$, and $\Psi_{\tilde{\Omega}}$ was a C^∞ ILB C^2-normal mapping of $U'\cap F$ into $\tilde{\Omega}\oplus d\Gamma(T^*\bar{M})$. By the above lemma $(d\Psi_{\tilde{\Omega}})_0$: $F\to d\Gamma(T^*\bar{M})$ has a right inverse B satisfying the inequality stated in the above lemma. Let

$$\hat{Z}=Z(\bar{M})+Rd(\phi\log\tan\pi\theta/2)\ .$$

Using B, we have the following splitting:

$$F=\hat{Z}\oplus Bd\Gamma(T^*\bar{M})\ ,\qquad \alpha=(\alpha-Bd\alpha)+Bd\alpha\ .$$

Let $\hat{Z}^k$ be the closure of $\hat{Z}$ in F^k. For $\alpha\in\hat{Z}$, $\beta\in Bd\Gamma(T^*\bar{M})$, we consider a mapping $\tilde{\Psi}(\alpha+\beta)=(\alpha,\Psi_{\tilde{\Omega}}(\alpha+\beta))$ of $U'\cap F$ into $\hat{Z}\oplus E_{\tilde{\Omega}}$. By Lemma 6.4, $\tilde{\Psi}$ is a C^∞ ILB C^2-normal mapping, and by Lemma 6.5, $(d\tilde{\Psi})_0$: $F\to\hat{Z}\oplus$

$d\Gamma(T^*\bar{M})$ has the inverse A satisfying

$$\|A(\alpha+d\beta)\|_k \leqq C\{\|\alpha\|_k+\|d\beta\|_{k-1}\}+D_k\{\|\alpha\|_{k-1}+\|d\beta\|_{k-2}\}\ , \qquad k\geqq 2\ .$$

Thus, rewriting the suffix $k-1$ by k in the ILH-chain $\{d\Gamma(T^*\bar{M}), d\Gamma^k(T^*\bar{M}), k\geqq k_1\}$, one can apply the inverse mapping theorem of [4] §III. Hence, there are open neighborhoods W_1, W_2 of zeros in $\hat{Z}^{k_1}$, $d\Gamma^{k_1}(T^*\bar{M})$, respectively and a C^∞ ILB C^2-normal mapping $\lambda=\lambda_1+\lambda_2$ of $W_1\cap \boldsymbol{Z}\oplus W_2\cap d\Gamma(T^*\bar{M})$ into $\boldsymbol{F}=\hat{\boldsymbol{Z}}\oplus Bd\Gamma(T^*\bar{M})$ such that

$$\tilde{\Psi}(\lambda_1(\alpha, \omega)+\lambda_2(\alpha, \omega))=(\alpha, \tilde{\Omega}+\omega)$$

i.e., $\lambda_1(\alpha, \omega)\equiv\alpha$, $\Psi_{\tilde{\Omega}}(\alpha+\lambda_2(\alpha, \omega))\equiv\tilde{\Omega}+\omega$. Now, for $\alpha\in\hat{\boldsymbol{Z}}$, $\beta\in Bd\Gamma(T^*\bar{M})$, we set $\nu(\alpha+\beta)=\alpha+(\lambda_2(\alpha, 0)+\beta)$. Then, ν is a C^∞ ILB C^2-normal diffeomorphism at the origin, namely ν and ν^{-1} are C^∞ ILB C^2-normal mappings.

Note that $\tilde{\flat}^{-1}\hat{\boldsymbol{Z}}=\Gamma_{\tilde{\Omega}}$, $\tilde{\flat}\boldsymbol{F}=\Gamma_\Delta$, and that $\xi': U\cap\Gamma_\Delta\rightarrow\mathscr{D}_\Delta$ can be regarded as a local coordinate system at e of regular Fréchet-Lie group $\mathscr{D}_\Delta$. Through the linear mapping $\tilde{\flat}$, ν defines a smooth change of coordinates. Namely, $\nu'=\tilde{\flat}^{-1}\nu\tilde{\flat}$ is a smooth diffeomorphism of a neighborhood W of 0 in Γ_Δ onto another neighborhood of 0. Remark that $\xi''=\xi'\nu'$: $W\rightarrow\mathscr{D}_\Delta$ is a local coordinate system of $\mathscr{D}_\Delta$ at e. Since the above implicit function theorem shows that

$$\xi''(W)\cap\mathscr{D}_{\tilde{\Omega}}=\xi''(W\cap\Gamma_{\tilde{\Omega}})\ ,$$

$\xi''|W\cap\Gamma_{\tilde{\Omega}}$ gives an FL-subgroup structure on $\mathscr{D}_{\tilde{\Omega}}$. This completes the proof of Theorem 6.2. □

References

[1] R. ABRAHAM and J. MARSDEN, Foundation of Mechanics, Benjamin N.Y., 1967.
[2] A. KORIYAMA, Y. MAEDA and H. OMORI, On Lie algebras of vector fields, Trans. Amer. Math. Soc., **226** (1977), 89-117.
[3] L. D. LANDAU and E. M. LIFSHITZ, The Classical Theorey of Fields, Pergman Press, 1975.
[4] H. OMORI, Infinite Dimensional Lie Transformation Groups, Lecture Notes in Math., **427**, Springer, 1974.
[5] H. OMORI, Theory of Infinite Dimensional Lie Groups, Kinokuniya, 1978 (in Japanese).
[6] H. OMORI, Y. MAEDA and A. YOSHIOKA, On regular Fréchet-Lie groups, I, Some differential geometrical expressions of Fourier integral operators on a Riemannian manifold, Tokyo J. Math., **3** (1980), 353-390.
[7] H. OMORI, Y. MAEDA, A. YOSHIOKA and O. KOBAYASHI, Onreg ular Fréchet-Lie groups IV, Definition and fundamental theorems, Tokyo J. Math., **5** (1982), 365-398.
[8] H. OMORI, Y. MAEDA, A. YOSHIOKA and O. KOBAYASHI, On regular Fréchet-Lie groups

V, Several basic properties, Tokyo J. Math., **6** (1983), 39–64.

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