On regular polynomial matrices II

The associated matrix pencils; simple matrix pencils

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1. Introduction

Our paper [1] was devoted to regular polynomial matrices giving also a new criterion for a polynomial matrix to be simple.

The present paper is composed of two parts. In part one we will show some relations existing between a regular polynomial matrix $D_l(\lambda, \mu)$ and the associated matrix pencil. The second part is a supplement to [1], since in the special case of simple matrix pencils some statements of [1] can be formulated sharper.

The following terms and notations will be used. A regular polynomial matrix $D_l(\lambda, \mu)$ of degree l will always denote its homogeneous form

$$(1.1)\qquad D_l(\lambda, \mu) = A_0\lambda^l + A_1\lambda^{l-1}\mu + \ldots + A_l\mu^l; \quad A_0 \neq 0; \quad |D_l(\lambda, \mu)| \not\equiv 0,$$

where all A_i are n-square complex matrices. The characteristic polynomial of $D_l(\lambda, \mu)$ is

$$(1.2)\qquad \Delta(\lambda, \mu) \doteq |D_l(\lambda, \mu)| = c\mu^{\alpha_0} \prod_{k=1}^{s} (\lambda - \lambda_k\mu)^{\alpha_k}$$

$$c \neq 0; \quad \lambda_k \neq \lambda_i \Leftrightarrow k \neq i; \quad \sum_{k=0}^{s} \alpha_k = nl.$$

The reduced characteristic polynomial

$$(1.3)\qquad \delta(\lambda, \mu) \doteq \frac{\Delta(\lambda, \mu)}{\Delta_{n-1}(\lambda, \mu)} = c\mu^{\beta_0} \prod_{k=1}^{s} (\lambda - \lambda_k\mu)^{\beta_k}$$

$\lambda_k \neq \lambda_i \Leftrightarrow k \neq i; \quad 1 \leqq \beta_k \leqq \alpha_k \quad (k = 1, \ldots, s), \quad 1 \leqq \beta_0 \leqq \alpha_0$ if $\alpha_0 \neq 0$.*)
and the reduced adjoint of $D_l(\lambda, \mu)$

$$(1.4)\qquad F(\lambda, \mu) \doteq \frac{\operatorname{adj} D_l(\lambda, \mu)}{\Delta_{n-1}(\lambda, \mu)}$$

play an important role in our discussions. Here $\Delta_{n-1}(\lambda, \mu)$ denotes the gcd of all $(n-1)$th order subdeterminants of $D_l(\lambda, \mu)$.

*) The inequality $1 \leqq \beta_k \leqq \alpha_k$ is proved in [1].

The roots of an arbitrary polynomial $f(\lambda, \mu)$ will always mean the — finite and infinite — roots of the polynomial equation $f(\lambda, \mu)=0$. For any homogeneous polynomial $f(\lambda, \mu)$ we will use the abbreviation

$$(1.5)\qquad f(\lambda_k) = \begin{cases} f(\lambda, \mu)|_{\lambda=\lambda_k,\, \mu=1} & k = 1, \dots, s \\ f(\lambda, \mu)|_{\lambda=1,\, \mu=0} & k = 0 \end{cases}$$

$$f^{(\nu)}(\lambda_k) = \begin{cases} \left.\dfrac{\partial^\nu f(\lambda, \mu)}{\partial \lambda^\nu}\right|_{\lambda=\lambda_k,\, \mu=1} & k = 1, \dots, s \\ \left.\dfrac{\partial^\nu f(\lambda, \mu)}{\partial \mu^\nu}\right|_{\lambda=1,\, \mu=0} & k = 0, \end{cases}$$

where λ_k $(k=0, 1, \dots, s)$ are the roots of $\delta(\lambda, \mu)$ in (1.3).

The asterisk on a matrix (or column vector) indicates the conjugate transpose; E_r is the r-square unit matrix; $\langle d_0, d_1, \dots, d_s \rangle$ denotes the diagonal matrix with d_k in the kth position of the main diagonal; $\varrho(X)$ is the rank of the matrix X.

A regular polynomial matrix will always mean a polynomial matrix represented by (1.1)—(1.3). A regular polynomial matrix $D_l(\lambda, \mu)$ is called *simple* if $\varrho[D_l(\lambda_k)]= =n-\alpha_k$ $(k=0, 1, \dots, s)$ (cf. [1]). A regular matrix pencil is a linear binomial matrix

$$(1.6)\qquad \lambda B+\mu A; \quad |\lambda B+\mu A| \not\equiv 0.$$

2. The associated matrix pencil

To a regular polynomial matrix $D_l(\lambda, \mu)$ there can be associated a regular matrix pencil is several ways (cf. [2], [3], [4]). We associate to $D_l(\lambda, \mu)$ the matrix pencil $\lambda\mathscr{B}+\mu\mathscr{A}$, where $\mathscr{B}$ and $\mathscr{A}$ are ln-square matrices in the partitioned form

$$(2.1)\qquad \mathscr{B} = \begin{bmatrix} A_0 & & & \\ & E & & \\ & & \ddots & \\ & & & E \end{bmatrix}; \quad \mathscr{A} = \begin{bmatrix} A_1 & A_2 & \dots & A_{l-1} & A_l \\ -E & & & & \\ & & \ddots & & \\ & & & -E & 0 \end{bmatrix}.$$

In the following some relations between $D_l(\lambda, \mu)$ and the associated encil $\lambda\mathscr{B}+\mu\mathscr{A}$ will be verified.

Theorem 2.1. *Let $D_l(\lambda, \mu)$ be a regular polynomial matrix and $\lambda\mathscr{B}+\mu\mathscr{A}$ the associated matrix pencil; then*

$$\text{(i)}\qquad |D_l(\lambda, \mu)| \equiv |\lambda\mathscr{B}+\mu\mathscr{A}|,$$

$$\text{(ii)}\qquad \varrho[D_l(\lambda_k)] = \varrho(\lambda_k\mathscr{B}+\mathscr{A})-n(l-1)$$

$$k = 0, 1, \dots, s; \quad (\lambda_0\mathscr{B}+\mathscr{A}) \doteq \mathscr{B},$$

(iii) *$D_l(\lambda, \mu)$ and $\lambda\mathscr{B}+\mu\mathscr{A}$ have the same elementary divisors.*

PROOF. It can be verified by means of direct calculation that the identity

$$\mathscr{S}(\lambda\mathscr{B}+\mu\mathscr{A})\mathscr{T}=\left[\begin{array}{c|c} D_l(\lambda,\mu) & \\ \hline & E_{n(l-1)} \end{array}\right] \tag{2.2}$$

holds, where S and T are the partitioned matrices

$$\mathscr{S}=\left[\begin{array}{c|c|c|c|c|c} E & \mu^{-1}(\lambda A_0+\mu A_1) & A_2 & \dots & A_{l-2} & A_{l-1} \\ \hline & E & \mu^{-1}\lambda E & & & \\ \hline & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & E & \mu^{-1}\lambda E \\ \hline & & & & & E \end{array}\right] \tag{2.3}$$

and

$$\mathscr{T}=\left[\begin{array}{c|c|c|c} E & & & \lambda^{l-1}E \\ \hline & E & & \lambda^{l-2}\mu E \\ \hline & & \ddots & \vdots \\ & & E & \lambda\mu^{l-2}E \\ \hline & & & \mu^{l-1}E \end{array}\right]\left[\begin{array}{c|c|c|c} 0 & -\mu^{-1}E & & \\ \hline & 0 & -\mu^{-1}E & \\ \hline & & \ddots & \ddots \\ & & & -\mu^{-1}E \\ \hline E & & & 0 \end{array}\right]. \tag{2.4}$$

The determinant of S and T is according to (2.3) and (2.4)

$$|S|=1,\quad |T|=1,$$

hence (2.1) is a *strictly equivalent relation.* Assertion (i)—(iii) follows immediately from (2.2) taking into account (2.3) and (2.4).

Remark. We know already from (i) and (ii) of theorem 2.1 that

(i′) $D_l(\lambda,\mu)$ and $\lambda\mathscr{B}+\mu\mathscr{A}$ have the same eigenvalues taken into account with their multiplicities,

(ii′) $D_l(\lambda,\mu)$ has exactly as many linearly independent eigenvectors — belonging to the eigenvalue λ_k of multiplicity α_k — as $\lambda\mathscr{B}+\mu\mathscr{A}$.

Therefore we obtain from (i) and (ii) the following theorem, its formulation agreeing with that of theorem 4.2 of LANCASTER [4].

Theorem 2.2. *A regular polynomial matrix $D_l(\lambda,\mu)$ is simple if and only if the associated pencil $\lambda\mathscr{B}+\mu\mathscr{A}$ is a simple pencil.*

The following theorem can be verified by simple calculation.

Theorem 2.3. *Let $D_l(\lambda, \mu)$ be a regular polynomial matrix. If u_k is a right eigenvector of $D_l(\lambda, \mu)$ belonging to the eigenvalue λ_k, then the partitioned vector*

$$[\lambda_k^{l-1} u_k^*, \lambda^{l-2} u_k^*, \ldots, \lambda_k u_k^*, u_k^*]^* \quad k = 1, \ldots, s$$

$$[u_0^*, 0, \ldots, 0, 0]^* \quad k = 0$$

is a right eigenvector of the associated matrix pencil $\lambda\mathscr{B} + \mu\mathscr{A}$ belonging to the same eigenvalue λ_k. A similar statement holds for left eigenvectors as well.

3. Simple matrix pencils

All statements and relations established ih [1] concerning regular polynomial matrices are valid of course also for regular matrix pencils; it is even true that some of the given expressions and relations receive a simpler form if formulated concerning regular matrix pencils, which are a very simple special case of regular polynomial matrices in general. In spite of this fact we will not go into details in reformulating the results of [1] concerning matrix pencils. Here we restrict ourselves to the formulation of additional relations valid for matrix pencils only. Our starting-point is theorem 1 of [1] which can be formulate concerning matrix pencils as follows:

Theorem 3.1. *Let $\lambda B + \mu A$ be a regular matrix pencil. If the polynomial $\delta(\lambda, \mu)$ has only simple roots, then for the matrices*

$$P_k = \frac{1}{\delta'(\lambda_k)} F(\lambda_k) \quad k = 0, 1, \ldots, s \tag{3.1}$$

the following assertions hold:

(i) $$\varrho(P_k) = \varrho(BP_k) = \varrho(P_k B) = \alpha_k \quad k = 1, \ldots, s$$

$$\varrho(P_0) = \varrho(AP_0) = \varrho(P_0 A) = \alpha_0 \quad k = 0$$

(ii) $$(BP_k)^2 = BP_k; \quad (P_k B)^2 = P_k B \quad k = 1, \ldots, s$$

$$(AP_0)^2 = AP_0; \quad (P_0 A)^2 = P_0 A \quad k = 0$$

(iii) $$P_i A P_k = P_i B P_k = 0 \quad \text{if} \quad i \neq k.$$

Proof. Assertions (i) and (ii) agree with those of theorem 1 of [1] considering that for matrix pencils

$$\frac{\partial(\lambda B + \mu A)}{\partial \lambda} = B; \quad \frac{\partial(\lambda B + \mu A)}{\partial \mu} = A.$$

According to (3.1) it is sufficient to prove (iii) for the matrices $F(\lambda_k)$ and $F(\lambda_i)$ $i \neq k$. We consider the relations

$$(\lambda B + \mu A) F(\lambda, \mu) = \delta(\lambda, \mu) E$$

$$F(\lambda, \mu)(\lambda B + \mu A) = \delta(\lambda, \mu) E.$$

Substituting $\lambda=\lambda_k$ and $\lambda=\lambda_i$, resp. ($k\neq i$), we obtain

$$(3.2)\qquad \begin{aligned}\lambda_k BF(\lambda_k) &= -AF(\lambda_k)\\ \lambda_i F(\lambda_i)B &= -F(\lambda_i)A.\end{aligned}$$

Premultiplying the first relation of (3.2) by $F(\lambda_i)$ and postmultiplying the second by $F(\lambda_k)$, we obtain

$$(3.3)\qquad (\lambda_k-\lambda_i)F(\lambda_i)BF(\lambda_k) = 0$$

and since $\lambda_k\neq\lambda_i$, it follows according to (3.2) and (3.3) that

$$F(\lambda_i)BF(\lambda_k) = F(\lambda_i)AF(\lambda_k) = 0 \quad i\neq k;\quad i,k\neq 0.$$

If in (3.2) we premultiply the first relation and postmultiply the second by $F(\lambda_0)$, we obtain

$$F(\lambda_0)AF(\lambda_k) = F(\lambda_0)BF(\lambda_k) = F(\lambda_k)BF(\lambda_0) = F(\lambda_k)\ AF(\lambda_0) = 0$$

since $BF(\lambda_0)=F(\lambda_0)B=0$. Assertion (iii) is proved now for all cases.

Let the matrices (3.1) be decomposed in basis factors, that is

$$(3.4)\qquad P_k = U_k V_k^*;\quad \varrho(U_k)=\varrho(V_k)=\varrho(P_k)\quad k=0,1,\dots,s,$$

where U_k and V_k are matrices of type $n\times\alpha_k$. It is known from [1] that for $\lambda B+\mu A$ the columns of U_k and the rows of V_k^* yield α_k linearly independent right and left eigenvectors resp., belonging to the eigenvalue λ_k of multiplicity α_k. Hence we obtain from theorem 3.1.

Corollary 1. If $\delta(\lambda,\mu)$ has only simple roots, then the regular matrix pencil $\lambda B+\mu A$ is simple and can be transformed into the diagonal form

$$V^*(\lambda B+\mu A)U = \langle \mu E_{\alpha_0}, (\lambda-\lambda_1\mu)E_{\alpha_1},\dots,(\lambda-\lambda_s\mu)E_{\alpha s}\rangle,$$

or by putting $\mu=1$,

$$V^*(\lambda B+A)U = \langle E_{\alpha_0}, (\lambda-\lambda_1)E_{\alpha_1},\dots,(\lambda-\lambda_s)E_{\alpha s}\rangle.$$

The transformation matrices V^* and U are obtained from the matrices in (3.4) as

$$V^* = [V_0V_1\dots V_s]^*;\quad U=[U_0U_1\dots U_s].$$

It is of interest to note here that according to theorem 3 of [1] the following two statements are equivalent for a regular matrix pencil $\lambda B+\mu A$:

α) $\lambda B+\mu A$ is simple,

β) $\delta(\lambda,\mu)$ has only simple roots.

It is stated in theorem 1 of [1] that under the assumption of $\delta(\lambda,\mu)$ having only simple roots, $D_l'(\lambda_k)P_k$ and $P_kD_l'(\lambda_k)$ are idempotent matrices of rank α_k ($k=0,1,\dots,s$). In the case of matrix pencils these idempotents are AP_0, BP_k ($k=1,\dots,s$) and P_0A, P_kB ($k=1,\dots,s$) resp., and according to (iii) of the theorem proved above *both sets of matrices are orthogonal.* Hence taking into account the well-known theorem concerning orthogonal idempotents (cf. e.g. [5] pp.), we obtain

Corollary 2. If for the regular matrix pencil $\lambda B+\mu A$ the polynomial $\delta(\lambda, \mu)$ has only simple roots, then

$$AP_0+B\sum_{k=0}^{s}P_k = E \tag{3.5}$$

$$P_0A+\sum_{k=1}^{s}P_kB = E.$$

Before finishing this section we consider the restriction $|B|\neq 0$. Then (3.5) reduces to

$$B\sum_{k=1}^{s}P_k = E; \quad \sum_{k=1}^{s}P_kB = E$$

and this yields according to (3.2) the relations

$$B^{-1} = \sum_{k=1}^{s}P_k; \quad -A = B\sum_{k=1}^{s}\lambda_kP_kB.$$

Also the matrices (3.1) can be expressed in another form. There can be established

Theorem 3.2. *Let $\lambda B+A$ ($|B|\neq 0$) be a simple matrix pencil, i.e. let the polynomial $\delta(\lambda)$ have only simple roots. If*

$$L_k(\lambda) = \frac{\delta(\lambda)}{\delta'(\lambda_k)(\lambda-\lambda_k)} \quad k = 1, \ldots, s$$

is the Lagrangean interpolation polynomial belonging to $\delta(\lambda)$, then the matrices (3.1) *can be expressed as*

$$P_k = L_k(-B^{-1}A)B^{-1}; \quad P_k = B^{-1}L_k(-AB^{-1}) \quad k = 1, \ldots, s. \tag{3.6}$$

PROOF. It is sufficient to prove one form of (3.6). Let us introduce the notation

$$G_k^{(\nu)} = \frac{1}{(\nu+1)!}F^{(\nu)}(\lambda_k) \quad \nu = 0, 1, \ldots; \quad k = 1, \ldots, s.$$

It is easy to verify the relations

$$BG_k^{(\nu)} = \frac{\delta^{(\nu+1)}(\lambda_k)}{(\nu+1)!}E-(\lambda_kB+A)G_k^{(\nu+1)} \quad \nu = 0, 1, \ldots, s.$$

Taking into account that $G_k^{(s-1)} = \dfrac{\delta^{(s)}(\lambda_k)}{s!}B^{-1}\neq 0$, we obtain successively

$$G_k^{(s-2)} = \frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!}B^{-1}-B^{-1}(\lambda_kB+A)\frac{\delta^{(s)}(\lambda_k)}{s!}B^{-1}$$

$$= \left\{\frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!}E+(-B^{-1}A-\lambda_kE)\frac{\delta^{(s)}(\lambda_k)}{s!}E\right\}B^{-1}$$

$$G_k^{(s-3)} = \left\{\frac{\delta^{(s-2)}(\lambda_k)}{(s-2)!}E+(-B^{-1}A-\lambda_kE)\left[\frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!}E+(-B^{-1}A-\lambda_kE)\frac{\delta^{(s)}(\lambda_k)}{s!}E\right]\right\}B^{-1}$$

$$\vdots$$

and finally we arrive at

(3.7)
$$F(\lambda_k) = G_k^{(0)} = \left\{\frac{\delta'(\lambda_k)}{1!} E + (-B^{-1}A - \lambda_k E)\left[\ldots + (-B^{-1}A - \lambda_k E)\left[\frac{\delta^{(s-1)}(\lambda_k)}{(s-1)!}\right] E + \right.\right.$$
$$\left.\left. + (-B^{-1}A - \lambda_k E)\frac{\delta^{(s)}(\lambda_k)}{s!} E\right]\right]\ldots\left.\right]\right\} B^{-1}.$$

The expression in the parenthesis $\{\ldots\}$ on the right side of (3.7) is the Horner-arranging of the polynomial

$$\frac{\delta(\lambda)}{(\lambda-\lambda_k)} = \frac{\delta'(\lambda_k)}{1!} + \frac{\delta''(\lambda_k)}{2!}(\lambda-\lambda_k) + \ldots + \frac{\delta^{(s)}(\lambda_k)}{s!}(\lambda-\lambda_k)^{s-1} \tag{3.8}$$

for $\lambda = -B^{-1}A$. Hence by (3.7) and (3.8) we have

$$P_k = \frac{1}{\delta'(\lambda_k)} F(\lambda_k) = \frac{\delta(-B^{-1}A)}{\delta'(\lambda_k)(-B^{-1}A - \lambda_k E)} B^{-1} = L_k(-B^{-1}A)B^{-1} \quad k = 1, \ldots, s,$$

which proves the theorem.

Remark. In the case of $B=E$ the topic discussed in [1] and in this section reduces to the usual eigenvalue/eigenvector problem of the matrix $-A$. It is of interest to point to the fact, that in this most special case our approach followed in [1] and in this section agrees with that of the known approach of the eigenvalue/eigenvector problem of matrices of simple structure (see e.g, WEDDERBURN [6] and EGERVÁRY [7]).

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