ON REGULARISED QUANTUM DIMENSIONS OF THE SINGLET VERTEX OPERATOR ALGEBRA AND FALSE THETA FUNCTIONS

THOMAS CREUTZIG, ANTUN MILAS AND SIMON WOOD

Abstract. By working in the generality of the singlet vertex operator algebras we connect several important concepts in the theory of vertex operator algebras, quantum modular forms, and modular tensor categories. More precisely, starting from explicit formulae for characters of modules over the singlet vertex operator algebra, which can be expressed in terms of false theta functions and their derivatives, we first deform these characters by using a complex parameter ϵ. We then apply modular transformation properties of regularised partial theta functions to study asymptotic behaviour of regularised characters of irreducible modules and compute their regularised quantum dimensions. We also give a purely geometric description of the regularisation parameter as a uniformisation parameter of the fusion variety coming from atypical blocks. It turns out that the quantum dimensions behave very differently depending on the sign of the real part of ϵ. The map from the space of characters equipped with the Verlinde product to the space of regularised quantum dimensions turns out to be a genuine ring isomorphism for positive real part of ϵ while for negative real part of ϵ its surjective image gives the fusion ring of a rational vertex operator algebra. The category of modules of this rational vertex operator algebra should be viewed as obtained through the process of a semi-simplification procedure widely used in the theory of quantum groups. Interestingly, the modular tensor category structure constants of this vertex operator algebra, can be also detected from vector valued quantum modular forms formed by distinguished atypical characters.

1. Introduction

Vertex operator algebras and modular objects are intimately connected through the study of their graded dimensions of modules, also known as characters. If the vertex operator algebra is rational, and satisfies a certain cofiniteness condition, this connection is well known, while in the non-rational case the situation is far from being understood. Let us briefly recall the rational story.

1.1. **Quantum dimensions and the Verlinde formula of rational vertex operator algebras.** In the late 80's Verlinde conjectured an intriguing relation between the modular properties of characters of a certain two-dimensional conformal field theory and its fusion ring [V]. Shortly after, Moore and Seiberg explained how this relation arises from the axioms of rational conformal field theory [MS]. The algebraic axiomatisation of (chiral) two-dimensional conformal field theory is given by the representation theory of vertex operator algebras. The important question was thus to understand precisely the connection between modular forms and rational vertex operator algebras on a mathematical level of rigour. In the early 90's, Zhu [Z] was proved that the space of torus one point functions, and especially

T.C. was supported by an NSERC Research Grant (RES0020460).

A.M. was supported by a Simons Foundation grant.

S.W. was supported by an ARC Discovery Early Career Researcher Award (DE140101825).

characters of modules, of a rational vertex operator algebra satisfying the C_2-cofiniteness condition, is finite dimensional and carries an action of the modular group $SL(2;\mathbb{Z})$. Finally, in 2005, Huang [Hu] proved that the representation category of a rational C_2-cofinite vertex operator algebra (with some additional mild properties) has the structure of a modular tensor category.

Equivalence classes of simple objects in a modular tensor category carry a projective action of the modular group and it turns out that this categorical projective $SL(2;\mathbb{Z})$ action descends to the modular one. An important property of a modular tensor category is that the Verlinde formula holds (see e.g the book [T]). Let V be a rational vertex operator algebra with simple modules $M_0 = V, M_1, \ldots, M_n$, then the characters are the graded (by conformal dimension) trace functions

$$\mathrm{ch}\big[M_i\big](\tau) := \mathrm{tr}_{M_i}\left(q^{L_0-\frac{c}{24}}\right),$$

with $q = e^{2\pi i\tau}$ and c the central charge of V. The modular S-transformation defines a $(n+1)\times(n+1)$ matrix S_{ij} via

$$\mathrm{ch}\big[M_i\big]\left(-\frac{1}{\tau}\right) = \sum_{j=0}^{n} S_{ij}\mathrm{ch}\big[M_j\big](\tau).$$

The Verlinde formula explicitly computes the fusion coefficients in terms of the S matrix coefficients. On the space of equivalence classes of modules $[M_i]$, the product

$$[M_i]\times[M_j] = \sum_{k=0}^{n} {N_{ij}}^{k}[M_k]$$

is given by

$${N_{ij}}^{k} = \sum_{\ell=0}^{n} \frac{S_{i\ell}S_{j\ell}\left(S^{-1}\right)_{\ell k}}{S_{0\ell}}.$$

This formula implies that for each $r = 0, \ldots, n$ the map

$$[M_i] \mapsto \frac{S_{ir}}{S_{0r}}$$

is a ring homomorphism. The numbers $\frac{S_{ir}}{S_{0r}}$ are called generalised quantum dimensions of the simple object M_i, while for $r = 0$ they are just called quantum dimensions. Let us assume that the vacuum vector is the element of least conformal dimension of all simple modules of V, then the corresponding term dominates in the $q \to 0^+$ limit and hence the categorical dimension $\frac{S_{i0}}{S_{00}}$ satisfies

$$\frac{S_{i0}}{S_{00}} = \lim_{\tau\to 0^+} \frac{\mathrm{ch}\big[M_i\big]\,(\tau)}{\mathrm{ch}\big[M_0\big]\,(\tau)}, \tag{1}$$

where τ approaches 0 from the upper half plane. This relation between quantum dimensions and asymptotic behaviour of characters was first noted in [DV]. For a thorough discussion of quantum dimensions in rational vertex operator algebras see [DJX] (see also [M] and [BM1]), where a striking relation to indices of subfactors is also pointed out.

1.2. **Modularity and non-C_2-cofinite vertex operator algebras.** Non-rational vertex operator algebras can further be subdivided into two classes: C_2-cofinite and non-C_2-cofinite vertex operator algebras. C_2-cofiniteness is a condition that implies that the vertex operator algebra has only finitely many simple modules, up to isomorphism. Even though there is no general Verlinde-type formula for C_2-cofinite vertex operator algebras, results of Miyamoto [Miy] guarantee that at least the space of characters, supplemented by pseudo characters, is modular invariant. The (p_+, p_-)-triplet W-algebra is the best understood family of this type [AM2, AM3, FGST1, FGST2, Kau, GK, NT, TW]. In these cases, irreducible characters and their modular properties have been computed. They are sums of vector valued modular forms of weight zero, one and two. So in some sense we get a picture that resembles the rational case. For the $(1, p)$-triplet algebra there is even a proposal for a generalised Verlinde formula indirectly obtained from the action of $SL(2;\mathbb{Z})$ on characters [FHST].

Verlinde-type algebras in the case of non-rational, non C_2-cofinite vertex operator algebras have been studied by the authors and D. Ridout [CR1, CR2, CR3, CM1, RW]. These vertex operator algebras are expected to have uncountably many isomorphism classes of modules. The best known example is the rank one Heisenberg vertex operator algebra. The modular S-transformation of Heisenberg characters defines an integral kernel, the S-kernel, in terms of which the Verlinde formula can be expressed. This is explained in great detail in Section 1.2 of [CR3].

One general feature of these irrational theories is that they admit typical modules (labelled by a continuous parameter) and atypical modules (parametrised by a discrete set). When it comes to modular transformation properties, the S-transformation on the character should produce both typical and atypical characters. So we expect

$$\mathrm{ch}\big[M\big](-1/\tau) = \underbrace{\int_\Omega S_{M,\nu}\mathrm{ch}\big[M_\nu\big](\tau)d\nu}_{\text{continuous part}} + \underbrace{\sum_i \alpha_{M,i}\mathrm{ch}\big[M_i\big](\tau)}_{\text{discrete part}}, \tag{2}$$

where $\mathrm{ch}\big[M_i\big]$ are certain atypical characters and $S_{M,\nu}$ is the S-kernel defined on some domain Ω parametrising typicals. The reader should exercise caution here as the above relation might only exist in a distributional setting (see for instance [CR1], [CR2], [RW]); however, see also below. In the case of the Heisenberg vertex operator algebras, only the continuous Gaussian integral appears as there are no atypical modules. This is no surprise as the representation category of the Heisenberg algebra is semi-simple under suitable conditions on modules. Other more complicated examples of non C_2-cofinite vertex operator algebras include reducible but indecomposable modules. In many examples interesting modular-like objects appear in characters. For example, mock modular forms appear in the atypical modules of many super vertex operator algebras due to important work of Kac and Wakimoto, see e.g. [KW1]. In [AC], the mock modular properties were related to the fusion ring of a family of such theories. Meromorphic Jacobi forms of both positive and negative index also appear as the analytic continuations of highest-weight module characters of affine vertex operator algebras at admissible, rational level [KW2]. Here the relation of modular transformations to the Verlinde algebra is subtle as one has to work with the literal trace functions, that is, series expansions where the coefficients are dimensions of weight spaces and not their analytic continuations, see [CR1, CR2].

The case we are interested in here concerns certain vertex operator algebras whose atypical characters are essentially determined by certain false theta functions and their derivatives. A classical *partial* theta series is the theta like sum given by

$$P_{a,b}(q) = \sum_{m=0}^{\infty} q^{a(m+\frac{b}{2a})^2}, \quad a, b \in \mathbb{N}.$$

A *false* theta function is a difference of two partial theta functions with fixed a. These objects appear in connection to representation theory [BM1, CM1], mock modular forms, and as Fourier coefficients of negative index meromorphic Jacobi forms [BCR]. These appearances are related as the $\mathcal{M}(p)$ singlet algebras are the coset vertex operator algebras of a family of W-algebras whose atypical characters can be analytically continued to meromorphic Jacobi forms and their Fourier coefficients are essentially singlet algebra characters [CRW]. The definition of false theta functions can also be modified to higher ranks [BM2].

1.3. **The ϵ-plane and quantum dimensions of the singlet vertex operator algebra.** In this paper we are primarily concerned with the $\mathcal{M}(p_+, p_-)$ singlet algebra, which is defined as the intersection of the kernels of two screening operators. Additionally we will also present results for the $\mathcal{M}(p)$ singlet algebra studied previously in [CM1].

As previously shown in [CM1], a modular group action on $\mathcal{M}(p)$ atypical characters only exists, in the sense of (2) being a genuine equality of functions rather than distributions, if we introduce a regularisation parameter ϵ and view $\mathrm{ch}\big[M^\epsilon\big](\tau)$ as a function of both ϵ and τ. There are several reasons why regularisation is useful. For example, it allows us to distinguish each irreducible module by its regularised character. Moreover, if we require $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$, then atypicals transform as

$$\mathrm{ch}\big[M^\epsilon\big](-1/\tau) = \underbrace{\int_{\mathbb{R}} S^\epsilon_{M,\nu} \mathrm{ch}\big[\mathcal{F}^\epsilon_\nu\big](\tau) d\nu}_{\text{continuous part}} + \underbrace{G^\epsilon_M(\tau)}_{\substack{\text{discrete} \\ \text{correction part}}}, \tag{3}$$

where $\mathcal{F}_\nu$ are Fock modules and $S^\epsilon_{M,\nu}$ is now the regularised S-kernel. Our first main result gives such a formula for the characters of atypical $\mathcal{M}(p_+, p_-)$ and $\mathcal{M}(p)$ modules including explicit formulae for the correction term expressed in theta functions (see Proposition 7 together with equation (9) and Theorem 8). A similar formula, but without correction term or regularisation parameter, was found in [RW] by considering characters as algebraic distributions. The regularisation scheme considered in this article has the advantage of seeing the correction term which gives rise to the rich structure discussed below as well as to parametrisation of the fusion variety discussed below. In contrast to the $\mathcal{M}(p)$ vertex operator algebra this term does *not* vanish when ϵ goes to zero.

Keeping in mind formula (1), which is valid for rational models, we *define* the regularised quantum dimension of M to be

$$\mathrm{qdim}\left[M^\epsilon\right] = \lim_{\tau \to 0^+} \frac{\mathrm{ch}\big[M^\epsilon\big](\tau)}{\mathrm{ch}\big[V^\epsilon\big](\tau)},$$

where V is the vertex operator algebra; in our case $\mathcal{M}(p_+, p_-)$ or $\mathcal{M}(p)$. As was shown in [BM1], (non-regularised) quantum dimensions should enjoy nice properties even beyond rational vertex operator algebras.

The aim of this article is to compute a conjectural formula for the fusion ring of the singlet algebra by means of regularised quantum dimensions. While computing these formulae, we also obtain many analytic results which are purely number theoretic in nature, for example, asymptotic properties of quotients of characters and modular-like transformation properties of "weight" $\frac{3}{2}$ false theta functions. What is interesting is that the quantum dimensions behave very differently in the two disjoint domains:

$$\mathrm{Re}(\epsilon) < 0 \qquad \text{and} \qquad \mathrm{Re}(\epsilon) > 0,$$

separated by the "wall" $\mathrm{Re}(\epsilon) = 0$.[1]

Our findings can be summarised as follows. For $\mathrm{Re}(\epsilon) > 0$, the discrete part in the S-transformation formula vanishes and only the continuous part contributes to the asymptotic properties of regularised characters. Regularised quantum dimensions of irreducibles in this regime are defined everywhere, and are given explicitly by (17). We also show that the regularised Verlinde-type formula for irreducible modules matches the main result of [RW] (see Theorem 19). In particular, $\mathrm{ch}\big[M^\epsilon\big](\tau) \to \mathrm{qdim}\,[M^\epsilon]$ is injective (see Theorem 25). This ring is conjecturally the Grothendieck (or fusion) ring of a suitable quotient of the module category of the singlet algebra.

The region $\mathrm{Re}(\epsilon) < 0$ is more subtle. Here the quantum dimensions are strip-wise constant functions of ϵ. This sharp difference comes from the fact that the correction part now not only contributes to the modular transformation properties of regularised characters, but actually dominates the continuous part in the asymptotics.

Regularised quantum dimensions of atypicals in this regime are given in formula (18). Moreover, the fusion ring structure leaks through the wall $\mathrm{Re}(\epsilon) = 0$, meaning that at the end of each open strip there is a point on the wall where the limit from the left and right agree. The surjective image of the (conjectural) fusion ring is then isomorphic to the fusion ring of minimal models (see Theorem 25 for the precise statement).

1.4. **Semi-simplification and quantum modular forms.** The results obtained in Proposition 14 and Theorem 25 are somewhat surprising because the correction term appears to capture modular data of a rational VOA, that is, of the Virasoro minimal models and the $A_1^{(1)}$ WZW models, depending on ϵ for the $\mathcal{M}(p_+, p_-)$ algebra; and of $A_1^{(1)}$ WZW models for the $\mathcal{M}(p)$ algebra.

We provide two explanations of this phenomenon. One is purely categorical, but requires that we postulate the existence of an appropriate categorical structure on the tensor category of modules. While the other is a computational approach that reproduces the modular data of the minimal models and that of the $A_1^{(1)}$ WZW models directly from irreducible atypical characters.

Let us first consider the $\mathcal{M}(p)$ case in the categorical approach. Here we expect the category of modules to be rigid, which we assume for the moment. The representation category of $\mathcal{M}(p)$ contains standard modules $\mathcal{F}_\lambda$. As one can see in Section 6.1, their quantum dimensions in the half-plane where the correction term dominates is zero. Thus, we can regard the standard modules $\mathcal{F}_\lambda$ as negligible objects in the tensor category and they form an ideal.Then possible now to replace the category of all $\mathcal{M}(p)$-modules with its atypical blocks, that is, those with finite composition series of atypicals. We then form a new

[1] Other conventions exists in the literature use $i\epsilon$ as a regularisation parameter instead of ϵ. This leads to the disjoint domains $\mathrm{Im}(\epsilon) > 0$ and $\mathrm{Im}(\epsilon) < 0$, respectively.

category where the Hom spaces are moded out by negligible morphisms. This procedure of moding out negligible morphisms is well-known in the theory of quantum groups (see [AP] for details). In particular, in the quotient category the standard modules $\mathcal{F}_\lambda$ are isomorphic to the zero object. Then this quotient category (also known as the "semi-simplification") is expected to be a modular tensor category.

For the $\mathcal{M}(p_+, p_-)$ vertex operator algebra, things are more subtle. We do not expect the category of modules to be rigid and therefore there is no notion of a categorical trace. Instead, we first mod out the full category by the tensor ideal of minimal models (both [RW] and our paper provide sufficient evidence for the existence of such an ideal). Then the resulting quotient category - after restricted to a full subcategory - is conjecturally braided and rigid. Assuming that, we proceed as above and after semi-simplification, we expect to get the (p_+, p_-)-minimal models. In summary, we conjecture that:

(a) The quotient category of $\mathcal{M}(p_+, p_-)$ singlet modules with respect to the tensor ideal of Virasoro minimal models is braided and rigid. Its "semi-simplified" category is equivalent to the modular tensor category of (p_+, p_-)-Virasoro minimal models.
(b) The category of $\mathcal{M}(p)$ singlet modules is braided and rigid. Its "semi-simplified" category is equivalent to the modular tensor category of $A_1^{(1)}$ at level $p-2$.

Although elegant, this categorical approach is largely conjectural. So we turn to the more concrete computational approach. The main issue when dealing with singlet characters is that they do not look anything like finite-dimensional vector valued modular forms. However, the problem of infinite dimensionality is easily cured by taking the quotient of the space of characters by the subspace of characters of modules with vanishing quantum dimension. This leaves us with a finite-dimensional space of distinguished atypical characters. In order to extract an S-matrix from this space we make use of quantum modular forms (*qmf*) introduced in [Za]. Quantum modular forms are of interest in connection to mock modular forms and invariants of 3-manifolds, so it is not a priori clear whether they play an important role in the representation theory of irrational vertex operator algebras (see [BCR, BM1] for some indications that the two subjects are indeed related). The main property of qmf, at least for purposes of this paper, is that they live both in the upper and the lower half-plane; and that radial limits from both sides agree on a distinguished subset of the rationals - the *quantum set*. It is already known, that one can form a quantum modular form on the upper half-plane by choosing a false theta function, while on the lower half it is an Eichler integral. Then we prove that there is a vector valued quantum modular form of weight $\frac{1}{2}$ that transforms in exactly the same way as characters of the $A_1^{(1)}$ WZW models at level $p-2$ (see Theorem 30). Similar constructions for the $\mathcal{M}(p_+, p_-)$ singlets give rise to a vector valued quantum modular form of weight $\frac{3}{2}$ whose S-matrix is the one of the (p_+, p_-)-minimal models (see Theorem 31).

1.5. **Fusion varieties.** The fusion rings of the singlet algebras in the $\mathrm{Re}(\epsilon) > 0$ regime are infinite-dimensional commutative algebras with uncountably many generators due to the continua of standard modules. If we restrict ourselves to the subcategory discussed in Section 1.4 we find that its fusion ring can be described as a quotient of a certain polynomial ring (see Proposition 22). As such it can be viewed as the ring of functions on an algebraic variety - the *fusion variety*. For all singlet vertex operator algebras we explicitly describe this singular curve, which is always of genus zero (see Theorem 23). From the perspective of fusion varieties, the ϵ-parameter is essentially the uniformisation parameter. This is another reason why ϵ-regularisation is an important ingredient in studying irrational vertex operator algebras.

2. The Heisenberg and singlet vertex operator algebras

In this section, we recall well known structures needed to describe the singlet vertex operator algebra. We use the notation of [FGST1, FGST2] (see also [AM3], [TW] for related results). Let $p_+, p_- \geq 1$, $p_+ \neq p_-$ and $\gcd(p_+, p_-) = 1$. In later sections we will occasionally consider the case when $p_+, p_- \geq 2$ and the case when either $p_+ = 1$ or $p_- = 1$ separately. Additionally let

$$\alpha_+ = \sqrt{\frac{2p_-}{p_+}}, \qquad \alpha_- = -\sqrt{\frac{2p_+}{p_-}}, \qquad \alpha_0 = \alpha_+ + \alpha_-, \qquad \alpha = p_+\alpha_+ = -p_-\alpha_- = \sqrt{2p_+p_-},$$

and

$$\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_- = \alpha_{r+p_+,s+p_-}, \qquad \alpha_{r,s;n} = \alpha_{r,s+np_-} = \alpha_{r-np_+,s} = \alpha_{r,s} + \frac{n}{2}\alpha\,,$$

for $r, s, n \in \mathbb{Z}$.

Let $\hat{\mathfrak{h}}$ be the rank 1 extended Heisenberg Lie algebra with generators $a_n, n \in \mathbb{Z}$, satisfying the commutation relations

$$[a_m, a_n] = m\delta_{m,-n}C$$

where C is central. We denote by $\hat{\mathfrak{h}}_+$ and $\hat{\mathfrak{h}}_-$ the commutative Lie algebras generated by Heisenberg generators labelled by positive and negative integers respectively. Let $\mathcal{F}_\lambda, \lambda \in \mathbb{C}$ be the standard Fock module over $U(\hat{\mathfrak{h}})$ generated by a highest weight vector v_λ such that

$$a_n \cdot v_\lambda = \delta_{n,0}\lambda v_\lambda,\ n \geq 0$$
$$U(\hat{\mathfrak{h}}_-)v_\lambda = \mathcal{F}_\lambda$$

where C acts as 1.

The Fock module $\mathcal{F}_0$ carries the structure of a vertex operator algebra, which is generated by the field

$$Y(a_{-1}\mathbf{1}, z) = a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},$$

and where the vacuum vector is $\mathbf{1} = v_0$. This vertex operator algebra becomes a vertex operator algebra by choosing the conformal vector to be

$$\omega = \frac{1}{2}(a_{-1}^2 + \alpha_0 a_{-2})\mathbf{1}.$$

The Fourier modes $L(n)$ of ω then span the Virasoro algebra with central charge

$$c = 1 - 3\alpha_0^2 = 1 - 6\frac{(p_+ - p_-)^2}{p_+p_-}\,.$$

This choice of conformal structure determines the conformal dimension of the highest weight vectors $v_\lambda \in \mathcal{F}_\lambda$ to be

$$\Delta_\lambda = \tfrac{1}{2}\lambda(\lambda - \alpha_0) = \tfrac{1}{2}(\lambda - \tfrac{\alpha_0}{2})^2 - \tfrac{\alpha_0}{8}.$$

The minimal model vertex operator algebras at $c = 1 - 6\frac{(p_+-p_-)^2}{p_+p_-}, p_+, p_- \geq 2$ can be realised as subquotients of Fock modules [IK]. The representation theory of these vertex operator algebras is completely reducible and the distinct isomorphism classes of irreducible representations are labelled by the set

$$\mathcal{T}_{p_+,p_-} = \{(r,s) \mid 1 \leq r < p_+, 1 \leq s < p_-, sp_+ > rp_-\}\,.$$

We denote these irreducible representations by $\mathcal{L}_{r,s}$ and their conformal dimension is $\Delta_{r,s} = \frac{(p_+r-p_-s)^2-(p_+-p_-)^2}{4p_+p_-}$. In this parametrisation the vacuum module is given by $\mathcal{L}_{1,1}$. Note that $\Delta_{r,s} = \Delta_{p_+-r,p_--s}$ which is why the irreducible minimal model representations are labelled by $\mathcal{T}_{p_+,p_-}$ rather than the set $\{(r,s)|1 \leq r < p_+, 1 \leq s < p_-\}$.

2.1. **The singlet vertex operator algebra.** The Feigin-Fuchs modules $\mathcal{F}_{r,s;n} := \mathcal{F}_{\alpha_{r,s;n}}$ organise into so called Felder complexes [Fel] by means of the two commuting screening operators $\mathcal{Q}_+$ and $\mathcal{Q}_-$. For $1 \leq r < p_+$, $1 \leq s \leq p_-$, $n \in \mathbb{Z}$

$$\cdots \xrightarrow{\mathcal{Q}_+^{[r]}} \mathcal{F}_{p_+-r,s;n-1} \xrightarrow{\mathcal{Q}_+^{[p_+-r]}} \mathcal{F}_{r,s;n} \xrightarrow{\mathcal{Q}_+^{[r]}} \mathcal{F}_{p_+-r,s;n+1} \xrightarrow{\mathcal{Q}_+^{[p_+-r]}} \cdots \tag{4}$$

while for $1 \leq r \leq p_+$, $1 \leq s < p_-$, $n \in \mathbb{Z}$

$$\cdots \xrightarrow{\mathcal{Q}_-^{[s]}} \mathcal{F}_{r,p_--s;n+1} \xrightarrow{\mathcal{Q}_-^{[p_--s]}} \mathcal{F}_{r,s;n} \xrightarrow{\mathcal{Q}_-^{[s]}} \mathcal{F}_{r,p_--s;n-1} \xrightarrow{\mathcal{Q}_-^{[p_--s]}} \cdots\,. \tag{5}$$

We omit explicit formulas for the "powers" $\mathcal{Q}_\pm^{[i]}$ as they involve contour integrals (see [TW] and references therein for more details). These complexes are exact everywhere but $\mathcal{F}_{r,s;0}$ if $r < p_+, s < p_-$. If either $r = p_+$ or $s = p_-$, then they are exact everywhere.

These two commuting screening operators conveniently allow one to define three vertex operator subalgebras of $\mathcal{F}_0 = \mathcal{F}_{1,1;0}$.

Definition 1. For $p_+, p_- \geq 2$, let

$$\begin{aligned}
\mathcal{M}(p_+,p_-)^+ &= \ker \mathcal{Q}_+ : \mathcal{F}_{1,1;0} \to \mathcal{F}_{p_+-1,1;1}, \quad p_+ \geq 2,\\
\mathcal{M}(p_+,p_-)^- &= \ker \mathcal{Q}_- : \mathcal{F}_{1,1;0} \to \mathcal{F}_{1,p_--1;-1}, \quad p_- \geq 2,\\
\mathcal{M}(p_+,p_-) &= \mathcal{M}(p_+,p_-)^+ \cap \mathcal{M}(p_+,p_-)^-, \quad p_+,p_- \geq 2,
\end{aligned}$$

while for $p_- = 1$ or $p_+ = 1$, let

$$\begin{aligned}
\mathcal{M}(p_+) &= \ker \mathcal{Q}_+ : \mathcal{F}_{1,1;0} \to \mathcal{F}_{p_+-1,1;1}, p_+ \geq 2,\\
\mathcal{M}(p_-) &= \ker \mathcal{Q}_- : \mathcal{F}_{1,1;0} \to \mathcal{F}_{1,p_--1;1}, p_- \geq 2,
\end{aligned}$$

respectively.

The vertex operator algebra $\mathcal{M}(p_+,p_+)$ is called the (p_+,p_-)-singlet algebra while $\mathcal{M}(p_+)$ and $\mathcal{M}(p_-)$ are the $(p_+,1)$- and $(1,p_-)$-singlet algebras, respectively. It was proven in [AM3] for $p_- = 2$ and for general p_- in [TW] that $\mathcal{M}(p_+,p_-)$ is a $\mathcal{W}$-algebra of the singlet type, in the sense that is strongly generated by the Virasoro field and another primary field. The vertex operator algebras $\mathcal{M}(p_+,p_-)^+$ and $\mathcal{M}(p_+,p_-)^-$ have not been studied in the literature perhaps because they are not $\mathcal{W}$-algebras according to standard definitions unless either p_+ or p_- is 1. If $p_+ = 1$ then $\mathcal{M}(p_-) = \mathcal{M}(1,p_-)^-$ and if $p_- = 1$ then $\mathcal{M}(p_+) = \mathcal{M}(p_+,1)^+$.

The minimal model vertex operator algebra is given by the (equivalent) cohomologies of either $\mathcal{Q}_+$ or $\mathcal{Q}_-$ at $\mathcal{F}_{1,1;0}$, while the remaining minimal model representations are given by the cohomologies at $\mathcal{F}_{r,s;0}$.

2.2. **Representation theory.** Since $\mathcal{M}(p_+,p_-)^\pm$ and $\mathcal{M}(p_+,p_-)$ are vertex operator subalgebras of $\mathcal{F}_0$, the Fock modules $\mathcal{F}_\lambda$ are also representation over these subalgebras. When considered as representations over $\mathcal{M}(p_+,p_-)$ or $\mathcal{M}(p_\pm)$, we call the the Fock modules *standard modules*. Additionally we define the *typical modules* to be the standard modules that are irreducible over the vertex operator algebra being considered. All modules that are not typical modules are called *atypical modules*.

Definition 2. For $n \in \mathbb{Z}$ let $\mathcal{K}^\pm_{r,s;n}$, $\mathcal{I}^\pm_{r,s;n}$, $\mathcal{K}_{r,s;n}$ and $\mathcal{I}_{r,s;n}$ be the following subspaces of the $\mathcal{F}_{r,s;n}$:

$$\begin{aligned}
\mathcal{K}^+_{r,s;n} &= \ker \mathcal{Q}_+^{[r]} : \mathcal{F}_{r,s;n} \to \mathcal{F}_{p_+-r,s;n+1}, && p_+ \geq 2, 1 \leq r < p_+, 1 \leq s \leq p_-,\\
\mathcal{K}^-_{r,s;n} &= \ker \mathcal{Q}_-^{[s]} : \mathcal{F}_{r,s;n} \to \mathcal{F}_{r,p_--s;n-1}, && p_- \geq 2, 1 \leq r \leq p_+, 1 \leq s < p_-,\\
\mathcal{K}_{r,s;n} &= \mathcal{K}^+_{r,s;n} \cap \mathcal{K}^-_{r,s;n}, && p_+,p_- \geq 2, 1 \leq r < p_+, 1 \leq s < p_-.
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}^+_{r,s;n} &= \operatorname{im} \mathcal{Q}_+^{[p_+-r]} : \mathcal{F}_{p_+-r,s;n-1} \to \mathcal{F}_{r,s;n}, && p_+ \geq 2, 1 \leq r < p_+, 1 \leq s \leq p_-,\\
\mathcal{I}^-_{r,s;n} &= \operatorname{im} \mathcal{Q}_-^{[p_--s]} : \mathcal{F}_{r,p_--s;n+1} \to \mathcal{F}_{r,s;n}, && p_- \geq 2, 1 \leq r \leq p_+, 1 \leq s < p_-,\\
\mathcal{I}_{r,s;n} &= \mathcal{I}^+_{r,s;n} \cap \mathcal{I}^-_{r,s;n}, && p_+,p_- \geq 2, 1 \leq r < p_+, 1 \leq s < p_-.
\end{aligned}$$

Note that the singlet vacuum is $\mathcal{K}_{1,1;0} = \mathcal{M}(p_+,p_-)$ and that $\mathcal{K}^\pm_{1,1;0} = \mathcal{M}(p_+,p_-)^\pm$. The $\mathcal{K}_{r,s;n}$ and the $\mathcal{I}^\pm_{r,s;n}$ are modules over $\mathcal{M}(p_+,p_-)^\pm$ and since $\mathcal{M}(p_+,p_-) = \mathcal{M}(p_+,p_-)^+ \cap \mathcal{M}(p_+,p_-)^-$, both $\mathcal{K}_{r,s;n}$ and $\mathcal{I}_{r,s;n}$ are $\mathcal{M}(p_+,p_-)$-modules. For notational simplicity it turns out to be convenient to extend the range of the r,s label in $\mathcal{I}_{r,s;n}$ to include the border cases when $r = p_+$ or $s = p_-$, so we define

$$\mathcal{I}_{p_+,s;n} = \mathcal{I}^-_{p_+,s;n}, \qquad \mathcal{I}_{r,p_-;n} = \mathcal{I}^+_{r,p_-;n}, \qquad \mathcal{I}_{p_+,p_-;n} = \mathcal{F}_{p_+,p_-;n},$$

where $1 \leq r < p_+$, $1 \leq s < p_-$ and $n \in \mathbb{Z}$.

In [TW] (cf. also [AM3]), Zhu's associative algebra of $\mathcal{M}(p_+,p_-)$ was computed. By using its structure one can prove that every irreducible module appears as subquotient of a Fock representation. This implies the following theorem.

Theorem 3. *Let $p_+,p_- \geq 2$. An irreducible ($\mathbb{Z}_{\geq 0}$-gradable) $\mathcal{M}(p_+,p_-)$-module is isomorphic to one of the following:*

(a) Typical*: Standard modules $\mathcal{F}_\lambda$, where $\lambda \notin \frac{1}{\sqrt{2p_+p_-}}\mathbb{Z}$ or $\mathcal{F}_{p_+,p_-;n}, n \in \mathbb{Z}$.*

(b) Atypical submodules of standard modules*: $\mathcal{I}_{r,s;n}$, where $1 \leq r \leq p_+$ and $1 \leq s \leq p_-$ and $n \in \mathbb{Z}$.*

(c) Atypical subquotients of standard modules*: irreducible Virasoro modules $\mathcal{L}_{r,s}$, where $(r,s) \in \mathcal{T}_{p_+,p_-}$.*

Recall that the Felder complexes are exact away from $n = 0$, thus

$$\mathcal{K}^\mu_{r,s;n} \cong \mathcal{I}^\mu_{r,s;n},\ \mu = \pm, \emptyset\,.$$

However, at $n = 0$ when $1 \le r < p_+, 1 \le s < p_-$, the modules $\mathcal{K}_{r,s;0}, \mathcal{I}_{r,s;0}$ satisfy the following exact sequences

$$0 \longrightarrow \mathcal{I}_{r,s;0} \longrightarrow \mathcal{K}_{r,s;0} \longrightarrow \mathcal{L}_{r,s} \longrightarrow 0\,. \tag{6}$$

If we consider the $\mathcal{Q}_+$ Felder complex, but restrict to $\mathcal{I}^-_{r,s;n} \subset \mathcal{F}_{r,s;n}$, then because the screening operators $\mathcal{Q}_+$ and $\mathcal{Q}_-$ commute, we obtain the complex

$$\cdots \xrightarrow{\mathcal{Q}_+^{[r]}} \mathcal{I}^-_{p_+-r,s;n-1} \xrightarrow{\mathcal{Q}_+^{[p_+-r]}} \mathcal{I}^-_{r,s;n} \xrightarrow{\mathcal{Q}_+^{[r]}} \mathcal{I}^-_{p_+-r,s;n+1} \xrightarrow{\mathcal{Q}_+^{[p_+-r]}} \cdots$$

which happens to be exact for all n.[2] Similarly for the $\mathcal{Q}_-$ Felder complex we have the exact sequence

$$\cdots \xrightarrow{\mathcal{Q}_-^{[s]}} \mathcal{I}^+_{r,p_--s;n+1} \xrightarrow{\mathcal{Q}_-^{[p_--s]}} \mathcal{I}^+_{r,s;n} \xrightarrow{\mathcal{Q}_-^{[s]}} \mathcal{I}^+_{r,p_--s;n-1} \xrightarrow{\mathcal{Q}_-^{[p_--s]}} \cdots\,,$$

Remark 4. The structure of Felder's complex is somewhat simpler if either p_+ or p_- is 1. If $p_+ = 1$ then it is given by

$$\cdots \xrightarrow{\mathcal{Q}_-^{[s]}} \mathcal{F}_{r,s} \xrightarrow{\mathcal{Q}_-^{[p_--s]}} \mathcal{F}_{r+1,p_--s} \xrightarrow{\mathcal{Q}_-^{[s]}} \mathcal{F}_{r+2,s} \xrightarrow{\mathcal{Q}_-^{[p_--s]}} \cdots\,,$$

and is exact everywhere. Irreducible atypical $\mathcal{M}(p_-)$ modules are obtained in the range $1 \le s \le p_- - 1$:

$$M_{r,s} = \ker(\mathcal{Q}_-^{[s]} : \mathcal{F}_{r,s} \to \mathcal{F}_{r+1,p_-s}) = \operatorname{im}(\mathcal{Q}_-^{[p-s]} : \mathcal{F}_{r-1,s} \to \mathcal{F}_{r,s}).$$

A classification theorem for $\mathcal{M}(p_-)$-modules analogous to Theorem 3 can be found in [CM1].

3. Characters of singlet algebra modules

In this section we will use the Felder complexes to derive character formulae for atypical modules in terms of the characters of standard modules. These character formulae will then offer a simple means of regularising the characters of atypical modules in later sections.

The partial theta function $P_{a,b}(u,\tau)$ is the power series

$$P_{a,b}(u,\tau) = \sum_{k\ge 0} z^{k+\frac{b}{2a}} q^{a(k+\frac{b}{2a})^2},\ q = e^{2\pi i\tau}, z = e^{2\pi i u}\,,$$

where $a, b \in \mathbb{N}, \tau \in \mathbb{H}, u \in \mathbb{C}$, while the theta function is given by

$$\theta_{a,b}(u,\tau) = \sum_{k\in\mathbb{Z}} z^{k+\frac{b}{2a}} q^{a(k+\frac{b}{2a})^2},\ q = e^{2\pi i\tau}, z = e^{2\pi i u}\,.$$

Furthermore, we define

$$\theta_{a,b}(\tau) = \theta_{a,b}(0,\tau)\,, \qquad P_{a,b}(\tau) = P_{a,b}(0,\tau)\,,$$

and

$$\theta_{a,b}(\tau)' = z\partial_z\theta_{a,b}(u,\tau)\,|_{u=0}, \qquad P_{a,b}(\tau)' = z\partial_z P_{a,b}(u,\tau)\,|_{u=0}.$$

[2]The exactness can be seen be comparing socle sequence decompositions.

The theta functions satisfy the well-known modular transformation formula

$$\theta_{a,b}\left(\frac{u}{\tau},-\frac{1}{\tau}\right)=\sqrt{\frac{-i\tau}{2a}}e^{\pi i\frac{u^2}{2a\tau}}\sum_{c=0}^{2a-1}e^{\frac{-2\pi ibc}{2a}}\theta_{a,c}(u,\tau), \tag{7}$$

while the modular transformation properties of the partial theta functions $P_{a,b}(u,\tau)$ are much more delicate and involve the ϵ-regularisation procedure introduced in [CM1].

The character of the standard module $\mathcal{F}_\lambda$ is given by

$$\mathrm{ch}\big[\mathcal{F}_\lambda\big](u,\tau)=\mathrm{tr}_{\mathcal{F}_\lambda}\,q^{L(0)-c/24}=\frac{q^{\Delta_\lambda-\frac{c}{24}}}{(q)_\infty}=\frac{q^{(\lambda-\frac{\alpha_0}{2})^2/2}}{\eta(q)}.$$

Note that both $\mathcal{F}_\lambda$ and $\mathcal{F}_{\alpha_0-\lambda}$ have identical characters. We will distinguish these two standard modules by introducing a deformation parameter ϵ in Section 4.

The Virasoro minimal model representations are given by the cohomologies of the Felder complexes and therefore by the Euler-Poincaré principle their characters are the alternating sums of the characters of all the entries of the complexes

$$\begin{aligned}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]&=\sum_{k\in\mathbb{Z}}\mathrm{ch}\big[\mathcal{F}_{r,s;2k}\big]-\mathrm{ch}\big[\mathcal{F}_{p_+-r,s;2k+1}\big]=\sum_{k\in\mathbb{Z}}\mathrm{ch}\big[\mathcal{F}_{r,s;2k}\big]-\mathrm{ch}\big[\mathcal{F}_{r,p_--s;2k+1}\big]\\&=\frac{\theta_{p_+p_-,-rp_-+sp_+}(\tau)-\theta_{p_+p_-,rp_-+sp_+}(\tau)}{\eta(\tau)}=\frac{\theta_{p_+p_-,rp_--sp_+}(\tau)-\theta_{p_+p_-,rp_-+sp_+}(\tau)}{\eta(\tau)},\end{aligned}$$

where $(r,s)\in\mathcal{T}_{p_+,p_-}$.

By choosing resolutions for the $\mathcal{I}^+_{r,s;n}$ and co-resolutions for the $\mathcal{I}^-_{r,s;n}$ in the Felder complexes above, the Euler-Poincaré principle yields the following character formulae

$$\begin{aligned}\mathrm{ch}\big[\mathcal{I}^+_{r,s:n}\big]&=(1-\delta_{s,p_-})\delta_{n\geq1}\left(\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]-\delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]\right)\\&\quad+\sum_{k\geq0}\mathrm{ch}\big[\mathcal{F}_{p_+-r,s:n-2k-1}\big]-\mathrm{ch}\big[\mathcal{F}_{r,s:n-2k-2}\big]\\&=(1-\delta_{s,p_-})\delta_{n\geq1}\frac{1}{\eta(q)}\left(\delta_{n,\mathrm{even}}(\theta_{p_+p_-,-rp_-+sp_+}(\tau)-\theta_{p_+p_-,rp_-+sp_+}(\tau))\right.\\&\quad\left.-\delta_{n,\mathrm{odd}}(\theta_{p_+p_-,rp_-+sp_+-p_+p_-}(\tau)-\theta_{p_+p_-,-rp_-+sp_++p_+p_-}(\tau))\right)\\&\quad+\frac{P_{p_+p_-,(2-n)p_+p_--rp_--sp_+}(\tau)-P_{p_+p_-,(2-n)p_+p_-+rp_--sp_+}(\tau)}{\eta(q)},\end{aligned}$$

where $1\leq r<p_+, 1\leq s\leq p_-$,

$$\begin{aligned}\mathrm{ch}\big[\mathcal{I}^-_{r,s:n}\big]&=(1-\delta_{r,p_+})\delta_{n\geq0}\left(\delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]-\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]\right)\\&\quad+\sum_{k\geq0}\mathrm{ch}\big[\mathcal{F}_{r,s:n-2k}\big]-\mathrm{ch}\big[\mathcal{F}_{r,p_--s:n-2k-1}\big]\\&=(1-\delta_{r,p_+})\delta_{n\geq0}\frac{1}{\eta(q)}\left(\delta_{n,\mathrm{odd}}(\theta_{p_+p_-,rp_-+sp_+-p_+p_-}(0,\tau)-\theta_{p_+p_-,-rp_-+sp_++p_+p_-}(0,\tau)\right.\\&\quad\left.-\delta_{n,\mathrm{even}}(\theta_{p_+p_-,-rp_-+sp_+}(0,\tau)-\theta_{p_+p_-,rp_-+sp_+}(0,\tau)))\right)\\&\quad+\frac{P_{p_+p_-,-np_+p_-+rp_--sp_+}(0,\tau)-P_{p_+p_-,-np_+p_-+rp_-+sp_+}(0,\tau)}{\eta(q)},\end{aligned}$$

where $1 \leq r \leq p_+, 1 \leq s < p_-$ and

$$\delta_{n\geq k} = \begin{cases} 1 & n \geq k \\ 0 & n < k \end{cases}, \quad k = 0, 1.$$

These formulae are structurally similar to the characters of modules over the $\mathcal{M}(p)$ singlet algebra studied in [CM1], that is, they are expressible as differences of partial thetas divided by the Dedekind η-function.

For the $\mathcal{I}_{r,s;n}$ we again apply the Euler-Poincaré principle to obtain the characters in terms of $\mathcal{I}^{\pm}_{r,s;n}$ characters.

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{I}_{r,s;n}\big] &= \sum_{k\geq 0} \mathrm{ch}\big[\mathcal{I}^+_{r,s;n-2k}\big] - \mathrm{ch}\big[\mathcal{I}^+_{r,p_- - s;n-2k-1}\big] \\
&= \sum_{k\geq 0} \mathrm{ch}\big[\mathcal{I}^-_{p_+ - r,s;n-2k-1}\big] - \mathrm{ch}\big[\mathcal{I}^-_{r,s;n-2k-2}\big] \\
&= n \cdot \delta_{n\geq 0} \left(\delta_{n,\mathrm{even}} \mathrm{ch}\big[\mathcal{L}_{r,s}\big] - \delta_{n,\mathrm{odd}} \mathrm{ch}\big[\mathcal{L}_{r,p_- - s}\big]\right) \\
&\quad + \sum_{k\geq 0} (k+1) \left(\mathrm{ch}\big[\mathcal{F}_{p_+ - r,s:n-2k-1}\big] + \mathrm{ch}\big[\mathcal{F}_{r,p_- - s:n-2k-3}\big]\right. \\
&\qquad\qquad \left. - \mathrm{ch}\big[\mathcal{F}_{r,s:n-2k-2}\big] - \mathrm{ch}\big[\mathcal{F}_{p_+ - r,p_- - s:n-2k-2}\big]\right) \\
&= \delta_{n\geq 1} \frac{n}{\eta(q)} \left(\delta_{n,\mathrm{even}}(\theta_{p_+p_-,-rp_- + sp_+}(0,\tau) - \theta_{p_+p_-,rp_- + sp_+}(0,\tau))\right. \\
&\quad \left. - \delta_{n,\mathrm{odd}}(\theta_{p_+p_-,rp_- + sp_+ - p_+p_-}(0,\tau) - \theta_{p_+p_-,-rp_- + sp_+ + p_+p_-}(0,\tau))\right) \\
&\quad \frac{1}{\eta(\tau)} \sum_{k\geq 0} (k+1) \left(q^{p_-p_+\left(k + \frac{(2-n)p_+p_- - p_+s - p_-r}{2p_+p_-}\right)^2} + q^{p_-p_+\left(k + \frac{(2-n)p_+p_- + p_+s + p_-r}{2p_+p_-}\right)^2} \right. \\
&\qquad\qquad \left. - q^{p_-p_+\left(k + \frac{(2-n)p_+p_- - p_+s + p_-r}{2p_+p_-}\right)^2} - q^{p_-p_+\left(k + \frac{(2-n)p_+p_- + p_+s - p_-r}{2p_+p_-}\right)^2} \right),
\end{aligned}$$

where $1 \leq r < p_+, 1 \leq s < p_-$. By the identity

$$\sum_{k\geq 0} (k+1) q^{a(k+\frac{b}{2a})^2} = (1 - \tfrac{b}{2a}) P_{a,b}(0,\tau) + P_{a,b}(0,\tau)',$$

we obtain the following character formulae in terms of partial theta functions:

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{I}_{r,s;n}\big] &= \delta_{n\geq 1} \frac{n}{\eta(q)} \left(\delta_{n,\mathrm{even}}(\theta_{p_+p_-,-rp_- + sp_+}(0,\tau) - \theta_{p_+p_-,rp_- + sp_+}(0,\tau))\right. \\
&\quad \left. - \delta_{n,\mathrm{odd}}(\theta_{p_+p_-,rp_- + sp_+ - p_+p_-}(0,\tau) - \theta_{p_+p_-,-rp_- + sp_+ + p_+p_-}(0,\tau))\right) \\
&\quad + \frac{1}{\eta(q)} \left[\frac{1}{2p_+p_-} \left((np_+p_- + rp_- + sp_+) P_{p_+p_-,(2-n)p_+p_- - rp_- - sp_+}(0,\tau) \right. \right. \\
&\qquad + (np_+p_- - rp_- - sp_+) P_{p_+p_-,(2-n)p_+p_- + rp_- + sp_+}(0,\tau) \\
&\qquad - (np_+p_- - rp_- + sp_+) P_{p_+p_-,(2-n)p_+p_- + rp_- - sp_+}(0,\tau)
\end{aligned}$$

$$-(np_+p_- + rp_- - sp_+)P_{p_+p_-,(2-n)p_+p_- - rp_- + sp_+}(0,\tau)\Big)$$

$$+ P_{p_+p_-,(2-n)p_+p_- - rp_- - sp_+}(0,\tau)' + P_{p_+p_-,(2-n)p_+p_- + rp_- + sp_+}(0,\tau)'$$

$$- P_{p_+p_-,(2-n)p_+p_- + rp_- - sp_+}(0,\tau)' - P_{p_+p_-,(2-n)p_+p_- - rp_- + sp_+}(0,\tau)'\Bigg]$$

Finally, the character of the $\mathcal{K}_{r,s;0}$ follows from the exact sequence (6)

$$\mathrm{ch}\big[\mathcal{K}_{r,s;0}\big] = \mathrm{ch}\big[\mathcal{L}_{r,s}\big] + \mathrm{ch}\big[\mathcal{I}_{r,s;0}\big]\,.$$

4. Regularised characters of singlet algebra modules

In this section we regularise the characters of irreducible $\mathcal{M}(p_+,p_-)$ modules following the methods in [CM1]. Let $\epsilon \in \mathbb{C}$. We define the regularised typical characters to be

$$\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{\lambda}\big] := e^{2\pi\epsilon(\lambda-\alpha_0/2)}\mathrm{ch}\big[\mathcal{F}_{\lambda}\big] = e^{2\pi\epsilon(\lambda-\alpha_0/2)}\frac{q^{(\lambda-\alpha_0/2)^2/2}}{\eta(\tau)}.$$

The regularised characters of atypical modules will then be defined using the resolutions of previous sections where unregularised characters of standard modules are replaced by regularised characters of standard modules. To more easily give the regularised characters of atypical modules, we introduce ϵ-regularised partial theta functions

$$P^{\epsilon}_{a,b}(u,\tau) = \sum_{k\geq 0} e^{2\pi\epsilon\left(\frac{b}{2a}+k\right)} z^{\frac{b}{2a}+k} q^{a\left(\frac{b}{2a}+k\right)^2} \tag{8}$$

as well as the mixed false theta functions

$$F^{\epsilon}_{b,c}(\tau) := \frac{1}{\eta(\tau)}\left(P^{-\sqrt{2p_+p_-}\epsilon}_{p_+p_-,b-c}(0,\tau) - P^{-\sqrt{2p_+p_-}\epsilon}_{p_+p_-,b+c}(0,\tau)\right), \quad b,c\in\mathbb{Z}, c\neq 0.$$

By replacing the characters of typical modules by regularised characters in the character formulae of the previous section, we get the following formulae for regularised atypical characters:

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{I}^{+,\epsilon}_{r,s;n}\big] &= (1-\delta_{s,p_-})\delta_{n\geq 1}\left(\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big] - \delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]\right)\\
&\quad + \sum_{k\geq 0}\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{p_+-r,s;n-2k-1}\big] - \mathrm{ch}\big[\mathcal{F}^{\epsilon}_{r,s;n-2k-2}\big]\\
&= (1-\delta_{s,p_-})\delta_{n\geq 1}\left(\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big] - \delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]\right)\\
&\quad + \frac{P^{-\sqrt{2p_-p_+}\epsilon}_{p_+p_-,(2-n)p_+p_- - rp_- - sp_+}(\tau) - P^{-\sqrt{2p_-p_+}\epsilon}_{p_+p_-,(2-n)p_+p_- + rp_- - sp_+}(\tau)}{\eta(q)}\\
&= (1-\delta_{s,p_-})\delta_{n\geq 1}\left(\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big] - \delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]\right) + F^{\epsilon}_{(2-n)p_+p_- - sp_+, rp_-}(\tau),
\end{aligned} \tag{9}$$

where $1 \le r < p_+, 1 \le s \le p_-$.

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{I}^{-,\epsilon}_{r,s;n}\big] &= (1-\delta_{r,p_+})\delta_{n\geq 0}\left(\delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]-\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]\right)\\
&\quad+\sum_{k\geq 0}\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{r,s;n-2k}\big]-\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{r,p_--s;n-2k-1}\big]\\
&= (1-\delta_{r,p_+})\delta_{n\geq 0}\left(\delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]-\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]\right)\\
&\quad+\frac{P^{-\sqrt{2p_-p_+}\epsilon}_{p_+p_-,-np_+p_-+rp_--sp_+}(\tau)-P^{-\sqrt{2p_-p_+}\epsilon}_{p_+p_-,-np_+p_-+rp_-+sp_+}(\tau)}{\eta(q)}\\
&= (1-\delta_{r,p_+})\delta_{n\geq 0}\left(\delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]-\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]\right)+F^{\epsilon}_{-np_+p_-+rp_-,sp_+}(\tau),
\end{aligned} \tag{10}$$

where $1 \le r \le p_+, 1 \le s < p_-$.

$$\begin{aligned}
\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{r,s;n}\big] &= n\cdot\delta_{n\geq 0}\left(\delta_{n,\mathrm{even}}\mathrm{ch}\big[\mathcal{L}_{r,s}\big]-\delta_{n,\mathrm{odd}}\mathrm{ch}\big[\mathcal{L}_{r,p_--s}\big]\right)\\
&\quad+\frac{1}{4}\sum_{\nu\in\{\pm 1\}}\Bigg[F^{\epsilon}_{(2-n)p_+p_--\nu sp_+,\nu rp_-}(\tau)'+F^{\epsilon}_{(2-n)p_+p_--\nu rp_-,\nu sp_+}(\tau)'\\
&\quad+\left(n+2\frac{\nu s}{p_-}\right)F^{\epsilon}_{(2-n)p_+p_--\nu sp_+,\nu rp_-}(\tau)+\left(n+2\frac{\nu r}{p_+}\right)F^{\epsilon}_{(2-n)p_+p_--\nu rp_-,\nu sp_+}(\tau)\Bigg].
\end{aligned} \tag{11}$$

where $1 \le r < p_+, 1 \le s < p_-$. Note that we do not regularise the characters of the irreducible Virasoro modules $\mathcal{L}_{r,s}$.

5. Modular-type properties of regularised characters

In this section we will develop the modular transformation properties of the characters of both standard and atypical modules.

The transformation properties of standard modules can be found in [CM1, Proposition 22] and other papers.

Proposition 5. *The modular S-transformation of typical characters is*

$$\mathrm{ch}[\mathcal{F}^{\epsilon}_{\lambda+\alpha_0/2}]\left(-\frac{1}{\tau}\right)=\int_{\mathbb{R}}S^{\epsilon}_{\lambda+\alpha_0/2}(x)\mathrm{ch}[\mathcal{F}^{\epsilon}_{x+\alpha_0/2}](\tau)dx,$$

with $S^{\epsilon}_{\lambda+\alpha_0/2}(x)=e^{2\pi\epsilon(\lambda-x)}e^{-2\pi\lambda x}$.

Next we consider the modular properties of characters of atypical modules. The easiest to consider are the characters of the minimal model representations $\mathcal{L}_{r,s}$ for which it is well know that they form a finite dimensional representation of the modular group without the need to invoke any regularisation or partial theta functions.

Proposition 6.

$$\mathrm{ch}\big[\mathcal{L}_{r,s}\big]\left(-\frac{1}{\tau}\right)=\sum_{(r',s')\in\mathcal{T}_{p_+,p_-}}S^{\mathrm{Vir}}_{(r,s),(r',s')}\mathrm{ch}\big[\mathcal{L}_{r',s'}\big](\tau),$$

where

$$S^{\mathrm{Vir}}_{(r,s),(r',s')}=(-1)^{(r+s)(r'+s')}\sqrt{\frac{8}{p_+p_-}}\sin\left(\frac{\pi rr'(p_--p_+)}{p_+}\right)\sin\left(\frac{\pi ss'(p_--p_+)}{p_-}\right), \tag{12}$$

and

$$\mathcal{T}_{p_+,p_-} = \{(r,s) \mid 1 \le r \le p_+ - 1, 1 \le s \le p_- - 1, sp_+ > rp_-\},$$

Asymptotically, as $y \to 0^+$,

$$\mathrm{ch}\big[\mathcal{L}_{r,s}\big](iy) \sim S^{\mathrm{Vir}}_{(r,s),(r_0,s_0)} e^{\frac{\pi}{12y}\left(1-\frac{6}{p_+p_-}\right)},$$

where $(r_0,s_0) \in \mathcal{T}_{p_+,p_-}$ is the unique pair such that $r_0p_- - s_0p_+ = 1$, that is, (r_0,s_0) is the label of the Virasoro highest weight module with least conformal dimension.

Proof. The first formula is the well known S-transformation of the characters of the Virasoro minimal models which is easily derived from the transformation properties of theta functions. For the second formula it is sufficient to observe that the character with the (r_0,s_0) label dominates in the expansion of $\mathrm{ch}\big[\mathcal{L}_{r,s}\big](-\frac{1}{\tau})$. □

In order to compute the modular transformations of the regularised characters of the remaining atypical irreducible modules, we need to understand those of $F^{\epsilon}_{b,c}$.

Proposition 7. *Let $\epsilon \notin i\mathbb{R}$. The S-transformation of the mixed false theta functions $F^{\epsilon}_{b,c}$ is given by*

$$F^{\epsilon}_{b,c}\left(-\frac{1}{\tau}\right) = \int_{\mathbb{R}} S^{\epsilon}_{b,c}(x)\,\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{x+\frac{\alpha_0}{2}}\big](\tau)dx + \frac{1-\mathrm{sgn}\,\mathrm{Re}(\epsilon)}{2} X^{\epsilon}_{b,c}(\tau)$$

with the "correction term"

$$X^{\epsilon}_{b,c} = \frac{iq^{-\frac{\epsilon^2}{2}}}{\sqrt{2p_+p_-}} \sum_{m=0}^{2p_+p_- -1} e^{-\pi i \frac{bm}{p_+p_-}} \sin\left(\pi \frac{cm}{p_+p_-}\right) \frac{\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\eta(\tau)}$$

and the S-kernel

$$S^{\epsilon}_{b,c}(x) = -e^{-2\pi\epsilon x} e^{2\pi i \frac{(b-p_+p_-)}{\sqrt{2p_+p_-}}(x+i\epsilon)} \frac{\sin\left(2\pi c\frac{x+i\epsilon}{\sqrt{2p_+p_-}}\right)}{\sin\left(\pi\sqrt{2p_+p_-}\,(x+i\epsilon)\right)}.$$

Proof. In [CM1] it was shown that

$$P_{\epsilon}(u,\tau) = \sum_{k\ge 0} e^{2\pi\epsilon(k+\frac{1}{2})} z^{k+\frac{1}{2}} q^{\left(k+\frac{1}{2}\right)^2/2}$$

transforms as

$$P_{\epsilon}\left(\frac{u}{\tau},\frac{-1}{\tau}\right) = \frac{e^{\pi i u^2/\tau}\sqrt{-i\tau}}{2}\left(-i\int_{\mathbb{R}} \frac{q^{x^2/2}z^x}{\sin(\pi(x+i\epsilon))}dx \right.$$
$$\left. + \frac{1}{2}(1+\mathrm{sgn}\,\mathrm{Re}(\epsilon)) \sum_{n\in\mathbb{Z}} (-1)^n z^{n-i\epsilon} q^{(n-i\epsilon)^2/2}\right).$$

Since

$$P^{\epsilon}_{a,b}(u,\tau) = z^{\frac{b}{2a}-\frac{1}{2}} e^{2\pi\epsilon\left(\frac{b}{2a}-\frac{1}{2}\right)} q^{a\left(\frac{b}{2a}-\frac{1}{2}\right)^2} P_{\epsilon}(u+(b-a)\tau, 2a\tau),$$

the transformation formulae for partial theta functions can obtained from that of $P_{\epsilon}(u,\tau)$.

$$P^{\epsilon}_{a,b}\left(\frac{u}{\tau},\frac{-1}{\tau}\right) = e^{2\pi i\left(\frac{b}{2a}-\frac{1}{2}\right)\frac{u}{\tau}} e^{2\pi\epsilon\left(\frac{b}{2a}-\frac{1}{2}\right)} e^{-2\pi i\frac{a}{\tau}\left(\frac{b}{2a}-\frac{1}{2}\right)^2} P_{\epsilon}\left(\frac{\tilde{u}}{\tilde{\tau}},\frac{-1}{\tilde{\tau}}\right), \quad \tilde{u} = \frac{u-b+a}{2a}, \tilde{\tau} = \frac{\tau}{2a}.$$

Algebraic manipulations thus yield

$$P_\epsilon\left(\frac{u}{\tau},\frac{-1}{\tau}\right)=e^{2\pi\epsilon\left(\frac{b}{2a}-\frac{1}{2}\right)}e^{\pi i\frac{u^2}{2a\tau}}\sqrt{\frac{-i\tau}{8a}}\left(-i\int_{\mathbb{R}}\frac{q^{\frac{x^2}{4a}}z^{\frac{x}{2a}}e^{2\pi i\frac{a-b}{2a}x}}{\sin(\pi(x+i\epsilon))}dx\right.$$
$$\left.+\frac{1+\operatorname{sgn}\operatorname{Re}(\epsilon)}{2}\sum_{n\in\mathbb{Z}}(-1)z^{\frac{n-i\epsilon}{2a}}e^{2\pi i\frac{a-b}{2a}(n-i\epsilon)}q^{\frac{(n-i\epsilon)^2}{4a}}\right).$$

By plugging this transformation formula into definition of the mixed false theta functions $F^\epsilon_{b,c}$, rescaling the integration variable x by a factor of $\sqrt{2p_+p_-}$ and simplifying, one sees that

$$F^\epsilon_{b,c}\left(-\frac{1}{\tau}\right)=\int_{\mathbb{R}}\frac{q^{\frac{x^2}{2}}}{\eta(\tau)}e^{-2\pi i\frac{(b-p_+p_-)}{\sqrt{2p_+p_-}}(x-i\epsilon)}\frac{\sin\left(2c/\sqrt{2p_+p_-}\pi\left(x-i\epsilon\right)\right)}{\sin\left(\sqrt{2p_+p_-}\pi\left(x-i\epsilon\right)\right)}dx+$$
$$+\frac{1-\operatorname{sgn}\operatorname{Re}(\epsilon)}{\eta(\tau)\sqrt{32p_+p_-}}\sum_{k\in\mathbb{Z}}\left(e^{\pi i\frac{ck}{p_+p_-}}-e^{-\pi i\frac{ck}{p_+p_-}}\right)e^{-\pi i\frac{b}{p_+p_-}k}q^{\frac{\left(k+i\sqrt{2p_+p_-}\epsilon\right)^2}{4p_+p_-}}.$$

The second summand can be rewritten in terms of standard theta functions:

$$\sum_{k\in\mathbb{Z}}\left(e^{\pi i\frac{ck}{p_+p_-}}-e^{-\pi i\frac{ck}{p_+p_-}}\right)e^{-\pi i\frac{bk}{p_+p_-}}q^{\frac{\left(k+i\sqrt{2p_+p_-}\epsilon\right)^2}{4p_+p_-}}$$
$$=q^{-\frac{\epsilon^2}{2}}\sum_{m=0}^{2p_+p_--1}\left(e^{\pi i\frac{cm}{p_+p_-}}-e^{-\pi i\frac{cm}{p_+p_-}}\right)e^{-\pi i\frac{bm}{p_+p_-}}\sum_{k\in\mathbb{Z}}q^{p_+p_-\left(k+\frac{m}{2p_+p_-}\right)^2}q^{i\sqrt{2p_+p_-}\epsilon\left(k+\frac{m}{2p_+p_-}\right)}$$
$$=q^{-\frac{\epsilon^2}{2}}\sum_{m=0}^{2p_+p_--1}\left(e^{\pi i\frac{cm}{p_+p_-}}-e^{-\pi i\frac{cm}{p_+p_-}}\right)e^{-\pi i\frac{bm}{p_+p_-}}\theta_{p_+p_-,m}\left(i\sqrt{2p_+p_-}\epsilon\tau,\tau\right),$$

where the second line follows by substituting $k\in\mathbb{Z}$ by $2p_+p_-k+m,\ k\in\mathbb{Z},m=0,\dots,2p_+p_--1$. The proposition then follows by changing the integration variable from x to $-x$. $\square$

We can now use the Proposition 7 to determine modular S-transformations of the remaining atypical submodules of standard modules.

Theorem 8. *For $1\le r\le p_+, 1\le s\le p_-$, the modular S-transformation of the characters of the $\mathcal{I}_{r,s;n}$ modules is*

$$\operatorname{ch}\left[\mathcal{I}^\epsilon_{r,s;n}\right]\left(-\frac{1}{\tau}\right)=n\delta_{n\ge0}\sum_{(r',s')\in\mathcal{T}_{p_+,p_-}}(-1)^{n(p_+s'+p_-r')}S^{\mathrm{Vir}}_{(r,s),(r',s')}\operatorname{ch}\left[\mathcal{L}_{r',s'}\right](\tau)$$
$$+\int_{\mathbb{R}}S^\epsilon_{r,s;n}(x)\operatorname{ch}\left[\mathcal{F}^\epsilon_{x+\frac{\alpha_0}{2}}\right](\tau)dx+\frac{1-\operatorname{sgn}\left(\operatorname{Re}(\epsilon)\right)}{2}Y^\epsilon_{r,s;n}(\tau)$$

with S-kernel

$$S^\epsilon_{r,s;n}(x)=-e^{-2\pi\epsilon x}e^{-\pi in\sqrt{2p_+p_-}(x+i\epsilon)}\frac{\sin\left(2\pi rp_-\frac{x+i\epsilon}{\sqrt{2p_+p_-}}\right)\sin\left(2\pi sp_+\frac{x+i\epsilon}{\sqrt{2p_+p_-}}\right)}{\sin\left(\pi\sqrt{2p_+p_-}(x+i\epsilon)\right)^2},$$

and the correction term

$$
\begin{aligned}
Y^{\epsilon}_{r,s;n}(\tau) = {} & \frac{1}{4\pi p_+p_-}\frac{d}{d\epsilon}q^{-\frac{\epsilon^2}{2}}\sum_{m=0}^{2p_+p_--1}(-1)^{mn}\sin\left(\pi\frac{rm}{p_+}\right)\sin\left(\pi\frac{sm}{p_-}\right)\frac{\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\eta(\tau)} \\
& -\frac{nq^{-\frac{\epsilon^2}{2}}}{\sqrt{2p_+p_-}}\sum_{m=0}^{2p_+p_--1}(-1)^{mn}\sin\left(\pi\frac{rm}{p_+}\right)\sin\left(\pi\frac{sm}{p_-}\right)\frac{\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\eta(\tau)} \\
& +\frac{ri}{p_+}\frac{q^{-\frac{\epsilon^2}{2}}}{\sqrt{2p_+p_-}}\sum_{m=0}^{2p_+p_--1}(-1)^{mn}\cos\left(\pi\frac{rm}{p_+}\right)\sin\left(\pi\frac{sm}{p_-}\right)\frac{\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\eta(\tau)} \\
& +\frac{si}{p_-}\frac{q^{-\frac{\epsilon^2}{2}}}{\sqrt{2p_+p_-}}\sum_{m=0}^{2p_+p_--1}(-1)^{mn}\sin\left(\pi\frac{rm}{p_+}\right)\cos\left(\pi\frac{sm}{p_-}\right)\frac{\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\eta(\tau)}
\end{aligned}
$$

Proof. The theorem follows directly from applying Proposition 7 – which gives the modular transformations of the mixed false theta functions $F^{\epsilon}_{b,c}$ – to the appropriate character formulae for $\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{r,s;n}\big]$, $\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{r,p_-;n}\big] = \mathrm{ch}\big[\mathcal{I}^{\epsilon,+}_{r,p_-;n}\big]$, $\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{p_+,s;n}\big] = \mathrm{ch}\big[\mathcal{I}^{\epsilon,-}_{p_+,s;n}\big]$ and $\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{p_+,p_-;n}\big] = \mathrm{ch}\big[\mathcal{F}^{\epsilon}_{p_+,p_-;n}\big]$; and by using the identities

$$
\begin{aligned}
F^{\epsilon\,\prime}_{b,c} &= -\frac{1}{\sqrt{2p_+p_-}}\frac{1}{2\pi}\frac{d}{d\epsilon}F^{\epsilon}_{b,c}, \\
S^{\mathrm{Vir}}_{(r,p_--s),(r',s')} &= -(-1)^{p_+s'+p_-r'}S^{\mathrm{Vir}}_{(r,s),(r',s')}.
\end{aligned}
$$

□

Remark 9. An interesting observation is that in the limit $\epsilon\to 0$, the correction term $Y^{\epsilon}_{r,s;n}(\tau)$ of the characters of the $\mathcal{I}_{r,s;n}$ tends to

$$
\lim_{\epsilon\to 0}Y^{\epsilon}_{r,s;n}(\tau) = -\frac{n}{2}\sum_{(r',s')\in\mathcal{T}_{p_+,p_-}}(-1)^{n(p_+s'+p_-r')}S^{\mathrm{Vir}}_{(r,s),(r',s')}\mathrm{ch}\big[\mathcal{L}_{r',s'}\big](\tau).
$$

We thus observe that the correction term carries information about the minimal model characters.

6. Quantum Dimensions

As explained in the introduction and as in [CM1], we define the regularised quantum dimension of a module M to be

$$
\mathrm{qdim}\left[M^{\epsilon}\right] := \lim_{\tau\to 0+}\frac{\mathrm{ch}\big[M^{\epsilon}\big](\tau)}{\mathrm{ch}\big[V^{\epsilon}\big](\tau)},
$$

where 0 is approached from the upper half plane and V is the vertex operator algebra itself. Then the ordinary (non-regularised) quantum dimensions can be computed as the left and right limit

$$
\mathrm{qdim}\left[V\right]^{\pm} := \lim_{\epsilon\to 0\pm}\mathrm{qdim}\left[V^{\epsilon}\right],
$$

where 0 is approached from the left or the right along the real axis. A priori it is not clear that these two limits should agree. In order to be able to compute quantum dimensions we will make use of the following trick commonly used in rational conformal field theories [DV]:

$$\operatorname{qdim}[M^\epsilon] = \lim_{\tau\to 0+} \frac{\operatorname{ch}\big[M^\epsilon\big](\tau)}{\operatorname{ch}\big[V^\epsilon\big](\tau)} = \lim_{y\to+\infty} \frac{\operatorname{ch}\big[M^\epsilon\big](-1/iy)}{\operatorname{ch}\big[V^\epsilon\big](-1/iy)}.$$

That is, we can use modular transformation formulae of regularised characters and the quantum dimension is the ratio of dominating terms in the numerator and denominator. In rational theories, this method results in $\operatorname{qdim}[M] = \frac{S_{i,\min}}{S_{0,\min}}$, where min denotes the label corresponding to the highest weight module of least conformal dimension. If all non-vacuum conformal dimensions are positive, then min is the label that denotes the vertex operator algebra itself (for this and more general constructions see [BM1]). Since the modular transformations of characters considered in this paper involve integrals over continua of modules, we make use of the following convenient fact used in asymptotic analysis [O].

Lemma 10. *Let $f(x)$ satisfies properties as in* [O], *and let $F(y) = \int_0^\infty e^{-y\pi x^2} f(x)dx$. Then asymptotically, as $y \to +\infty$,*

$$F(y) = \frac{1}{2\sqrt{y}} f(0) + O\left(\frac{1}{y}\right).$$

6.1. **Quantum dimensions of $\mathcal{M}(p)$-modules.** Before we dive into a thorough discussion of the $\mathcal{M}(p_+, p_-)$ algebra, we first discuss the $\mathcal{M}(p)$ algebra. We specialise our notation to facilitate comparisons to [CM1], where the first two authors previously studied regularised quantum dimensions of $\mathcal{M}(p)$ modules for $\operatorname{Re}(\epsilon) > 0$. For the remainder of this section let $p_+ = 1, p_- = p$ and denote the irreducible atypical modules by $M_{r,s} \cong \mathcal{I}_{1,s;r-1}, r \in \mathbb{Z}, 1 \le s \le p$. Using Lemma 10 one can easily compute (see also [CM1]):

Proposition 11. *For* $\operatorname{Re}(\epsilon) > 0$, *the the quantum dimensions of typical and atypical modules are:*

$$\begin{aligned} \operatorname{qdim}[\mathcal{F}^\epsilon_\lambda] &= e^{2\pi\epsilon(\lambda-\alpha_0/2)} \frac{\sin\left(i\pi\sqrt{2p}\epsilon\right)}{\sin\left(i\pi\frac{\epsilon}{\sqrt{2p}}\right)}, \\ \operatorname{qdim}\left[M^\epsilon_{r,s}\right] &= e^{\pi\epsilon(r-1)\sqrt{2p}} \frac{\sin\left(2\pi\frac{s\epsilon i}{\sqrt{2p}}\right)}{\sin\left(2\pi\frac{\epsilon i}{\sqrt{2p}}\right)}, \end{aligned} \tag{13}$$

where $\alpha_0 = \sqrt{2p} - \sqrt{2/p}$.

Observe that

$$\operatorname{qdim}\left[M^{\epsilon+i\sqrt{2p}}_{r,s}\right] = \operatorname{qdim}\left[M^\epsilon_{r,s}\right],$$

so the quantum dimension is defined on a semi-infinite cylinder.

Next we consider the case when $\epsilon < 0, \epsilon \in \mathbb{R}$ for atypical irreducible modules $M_{r,s}$, where $1 \le s \le p-1, r \in \mathbb{Z}$. The correction term is thus given by

$$Y_{1,s;r-1}(\tau) = \frac{iq^{-\epsilon^2/2}}{\sqrt{2p}\eta(\tau)} \sum_{m=0}^{2p-1} (-1)^{rm} \sin\left(\pi\frac{sm}{p}\right) \sum_{k\in\mathbb{Z}} q^{i\sqrt{2p}\epsilon(k+\frac{m}{2p})} q^{p(k+\frac{m}{2p})^2};$$

see also [CM1]. Observe that the continuous part does not contribute after the modular transformation $\tau \to -\frac{1}{\tau}$ and taking limits. It is thus sufficient to only consider the correction term. For $\tau \to +i\infty$, $X_{r,s}(\tau)$ is dominated by the terms with $k=0, m=1$ and $k=-1, m=2p-1$.

$$\begin{aligned}
\operatorname{qdim}\left[M^{\epsilon}_{r,s}\right] &= \lim_{\tau\to 0+} \frac{\operatorname{ch}\left[M^{\epsilon}_{r,s}\right](\tau)}{\operatorname{ch}\left[M^{\epsilon}_{1,1}\right](\tau)} = \lim_{\tau\to +i\infty} \frac{\operatorname{ch}\left[M^{\epsilon}_{r,s}\right](-1/\tau)}{\operatorname{ch}\left[M^{\epsilon}_{1,1}\right](-1/\tau)} \\
&= \lim_{\tau\to+i\infty} \frac{(-1)^r \sin\left(\pi\frac{s}{p}\right)\left(q^{\frac{i\epsilon}{\sqrt{2p}}} - q^{-\frac{i\epsilon}{\sqrt{2p}}}\right) q^{\frac{1}{4p}}}{(-1)\sin\left(\pi\frac{1}{p}\right)\left(q^{\frac{i\epsilon}{\sqrt{2p}}} - q^{-\frac{i\epsilon}{\sqrt{2p}}}\right) q^{\frac{1}{4p}}} = (-1)^{r-1}\frac{\sin(\pi s/p)}{\sin(\pi/p)}.
\end{aligned} \tag{14}$$

Staying within the $\operatorname{Re}(\epsilon) < 0$ regime, consider the (open) strips

$$\mathbb{S}(k,m) = \left\{\epsilon \in \mathbb{C} \,\middle|\, k + \frac{2m-1}{4p} < \frac{\operatorname{Im}(\epsilon)}{\sqrt{2p}} < k + \frac{2m+1}{4p}\right\},$$

$k \in \mathbb{Z}$ where $m = 0, \dots, 2p-1$. If $\epsilon \in \mathbb{S}(k,m)$ then the dominating term of $Y_{1,s;r-1}$ has index k, m, however if $m=0$ or $m=p$ then $\sin\left(\pi\frac{sm}{p}\right) = 0$ and the neighbouring indices dominate.

Proposition 12. *For $\epsilon \in \mathbb{S}(k,m), k \in \mathbb{Z}, m = 0\dots, 2p-1$,*

$$\operatorname{qdim}\left[M^{\epsilon}_{r,s}\right] = \begin{cases} (-1)^{m(r-1)}\dfrac{\sin(\pi m s/p)}{\sin(\pi m/p)} & \text{if } m \neq 0, p, \\ (-1)^{(m+1)(r-1)+\frac{m}{p}(s-1)}\dfrac{\sin(\pi s/p)}{\sin(\pi/p)} & \text{if } m = 0, p. \end{cases} \tag{15}$$

An important point to make here is that for $n = 2pk + m, k \in \mathbb{Z}, m = 0, \dots 2p-1$,

$$\lim_{\epsilon\to\left(\frac{in}{\sqrt{2p}}\right)^+} \operatorname{qdim}\left[M^{\epsilon}_{r,s}\right] = (-1)^{(r-1)m}\frac{\sin(\pi n s/p)}{\sin(\pi n/p)} = \lim_{\epsilon\to\left(\frac{in}{\sqrt{2p}}\right)^-} \operatorname{qdim}\left[M^{\epsilon}_{r,s}\right], \tag{16}$$

if $m \neq 0, p$, whereas if $m = 0, p$,

$$\lim_{\epsilon\to\left(\frac{in}{\sqrt{2p}}\right)^+} \operatorname{qdim}\left[M^{\epsilon}_{r,s}\right] = s(-1)^{(r-1)m+\frac{m}{p}} \neq \lim_{\epsilon\to\left(\frac{in}{\sqrt{2p}}\right)^-} \operatorname{qdim}\left[M^{\epsilon}_{r,s}\right],$$

where the limits were taken along the line parallel to the real axis and the $\pm$ of $\left(\frac{in}{\sqrt{2p}}\right)^{\pm}$ indicates the sign of $\operatorname{Re}(\epsilon)$. Thus, the quantum dimension "leaks" across the "wall" $\operatorname{Re}(\epsilon) = 0$ from the continuous to discrete regime on a countable set.

For the quantum dimensions of typicals modules in the regime $\operatorname{Re}(\epsilon) < 0$, the denominator dominates and thus

$$\operatorname{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right] = 0.$$

Again, for $n = 2pk + m, k \in \mathbb{Z}, m = 0, \dots 2p-1$ we can compare limits.

$$\lim_{\epsilon\to\left(\frac{ni}{\sqrt{2p}}\right)^+} \operatorname{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right] = 0 = \lim_{\epsilon\to\left(\frac{ni}{\sqrt{2p}}\right)^-} \operatorname{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right],$$

if $m \neq 0, p$, whereas if $m = 0, p$

$$\lim_{\epsilon\to\left(\frac{ni}{\sqrt{2p}}\right)^+} \operatorname{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right] = e^{2\pi i(\lambda-\alpha_0/2)\frac{n}{\sqrt{2p}}} p(-1)^{\frac{m}{p}(p-1)} \neq \lim_{\epsilon\to\left(\frac{ni}{\sqrt{2p}}\right)^-} \operatorname{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right],$$

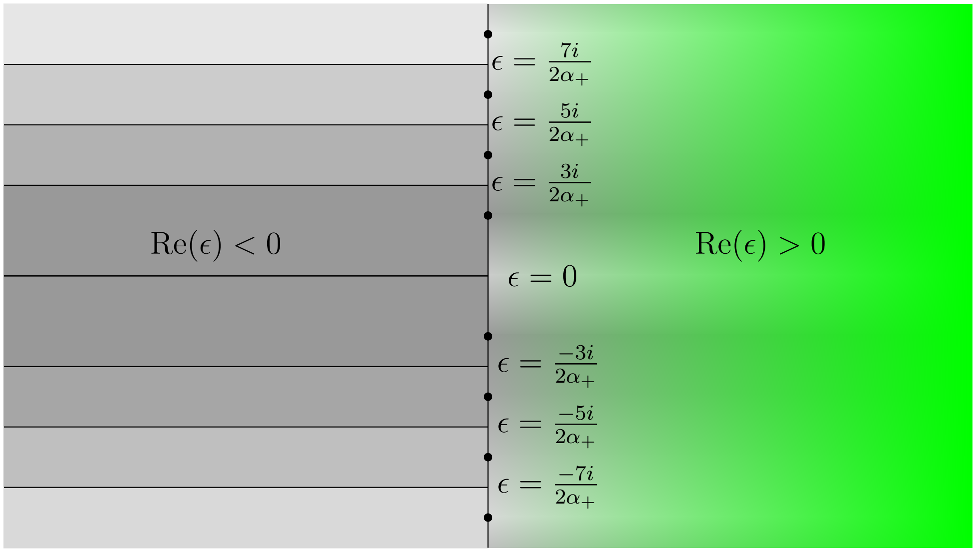

FIGURE 1. Quantum dimensions of atypical $\mathcal{M}(p)$ modules in the ϵ-plane

The right hand-side of the diagram represents the continuous region, where the quantum dimension is given by formula (13). Semi-infinite strips on the left represent regions where the quantum dimension is constant. The black dots at $\epsilon = \frac{ni}{\alpha_+}$, $n \in \mathbb{Z}$, denote "drip" points where the left and right limits of quantum dimensions coincide.

In [CGP, BCGP], an infinite dimensional quantum group at even root of unity $\bar{U}_q^H(sl_2)$, $q = e^{\pi i/p}$ was studied. The category of finite-dimensional weight modules for this quantum group is expected to be equivalent to a certain (tensor) subcategory of modules for the $\mathcal{M}(p)$-singlet. One strong piece of evidence in support of this belief is the agreement of fusion products among irreducibles [CGP]. Following the notation from [BCGP], irreducible modules are denoted by $S_i \otimes \mathbb{C}^H_{pr}$, $i = 0, ..., p-1$, $r \in \mathbb{Z}$ (atypicals) and V_α, where $\alpha \in \mathbb{C}\backslash\mathbb{Z} \cup r\mathbb{Z}$ (typicals). The $M_{r,i+1}$ should then correspond to $S_i \otimes \mathbb{C}^H_{pr}$, and V_α to F_α. Rather than pushing this connection further, we will instead compare our model to the $A_1^{(1)}$ WZW models.

Lemma 13. *Let $P^k_+ = \{0, 1..., k\}$, $k \in \mathbb{N}$ label the irreducible modules of the $A_1^{(1)}$ vertex operator algebra at level $k \in \mathbb{N}$. Then the associated S-matrix is given by*

$$S^k_{a,b} = \sqrt{\frac{2}{k+2}} \sin\left(\pi \frac{(a+1)(b+1)}{k+2}\right), \quad a, b \in P^k_+,$$

where the zeroth label corresponds to the vacuum module vertex operator algebra $L_{sl_2}(k\Lambda_0)$.

The results of this section then imply the following proposition.

Proposition 14. *There is an induced ring structure on the space of quantum dimensions of $\mathcal{M}(p)$ modules. For $\mathrm{Re}(\epsilon) < 0$, this ring is isomorphic to the fusion ring of $A_1^{(1)}$ at level $p - 2$.*

Proof. Formula (16) of Proposition 12 implies that the map $[X] \mapsto \mathrm{qdim}\,[X^\epsilon]$, where X is any $\mathcal{M}(p)$ singlet module, is a ring homomorphism for all values of ϵ. We now specialise to $\mathrm{Re}(\epsilon) < 0$ where, by Proposition 12, the space of quantum dimensions is "strip wise" constant. Next we analyse the image of the ring homomorphism. Observe that only (15) is relevant here and also the periodicity $n \equiv i \bmod 2p$, so we need only consider the values in (15) where $n = 1, ..., p-1, p+1, ..., 2p-1$. We also distinguish between r even and r odd.

By Proposition 18 for r odd and $n = 1, ..., p-1, p+1, ..., 2p-1$ we get

$$A(\text{odd}, s) = \left\{ \frac{\sin(\frac{\pi s}{p})}{\sin(\frac{\pi}{p})}, ..., \frac{\sin(\frac{\pi(p-1)s}{p})}{\sin(\frac{(p-1)\pi}{p})}, (-1)^s \frac{\sin(\frac{\pi s}{p})}{\sin(\frac{\pi(p+1)}{p})}, ..., (-1)^s \frac{\sin(\frac{\pi(p-1)s}{p})}{\sin(\frac{(2p-1)\pi}{p})}. \right\}$$

For r even, and $n = 1, ..., p-1, p+1, ..., 2p-1$, we have

$$A(\text{even}, s) = \left\{ -\frac{\sin(\frac{\pi s}{p})}{\sin(\frac{\pi}{p})}, .., (-1)^{p-1} \frac{\sin(\frac{\pi(p-1)s}{p})}{\sin(\frac{(p-1)\pi}{p})}, \right.$$

$$\left. (-1)^{p+1+s} \frac{\sin(\frac{\pi s}{p})}{\sin(\frac{\pi(p+1)}{p})}, ...(-1)^{p+s+i} ..., (-1)^{p+p-1}(-1)^s \frac{\sin(\frac{\pi(p-1)s}{p})}{\sin(\frac{(2p-1)\pi}{p})}. \right\}$$

These two sets are related by $A(\text{odd}, p-s) = -A(\text{even}, s)$. So we can ignore the quantum dimensions coming from either r even or r odd. Finally we use the fact that the $(p-1)\times(p-1)$ matrix $\frac{\sin(\frac{\pi ab}{p})}{\sin(\frac{a\pi}{p})}$ is invertible. So the image of the ring homomorphism is $p-1$-dimensional. $\square$

6.2. **Quantum dimensions of $\mathcal{M}(p_+, p_-)$-models.** For the remainder of this section we assume that $p_\pm \geq 2$. Recall that the vacuum module character is given by

$$\mathrm{ch}\big[\mathcal{K}^\epsilon_{1,1;0}\big] = \mathrm{ch}\big[\mathcal{L}_{1,1}\big] + \mathrm{ch}\big[\mathcal{I}^\epsilon_{1,1;0}\big],$$

where, as mentioned previously, we do not regularise the Virasoro part since it leads to non-existing limits.

In the limit

$$\lim_{y\to+\infty} \mathrm{ch}\big[\mathcal{K}^\epsilon_{1,1;0}\big] \left(\frac{-1}{iy}\right)$$

the minimal model terms are dominated by integration and $Y^\epsilon_{r,s;n}$ terms. Thus, in the denominator of $\mathcal{M}(p_+, p_-)$ quantum dimensions $\mathrm{ch}\big[\mathcal{K}^\epsilon_{1,1;0}\big]$ can be replaced by $\mathrm{ch}\big[\mathcal{I}^\epsilon_{1,1;0}\big]$. This immediately implies that the $\mathcal{M}(p_+, p_-)$ quantum dimensions of the irreducible minimal module modules vanish for all ϵ.

Corollary 15. *The $\mathcal{M}(p_+, p_-)$ quantum dimensions vanish for all ϵ, that is,*

$$\mathrm{qdim}\left[\mathcal{L}_{r,s}\right] = 0.$$

For the remaining quantum dimensions we first consider $\mathrm{Re}(\epsilon) > 0$, that is, when the correction term $Y^\epsilon_{r,s;n}$ does not appear in the modular transformation formula of the $\mathcal{I}_{r,s;n}$.

Proposition 16. *For* $\mathrm{Re}(\epsilon) > 0$ *the typical and non-Virasoro irreducible atypical* $\mathcal{M}(p_+, p_-)$ *quantum dimensions are given by*

$$\begin{aligned}
\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda+\alpha_0/2}\right] &= \lim_{y\to+\infty} \frac{\mathrm{ch}\left[\mathcal{F}^{\epsilon}_{\lambda+\alpha_0/2}\right]}{\mathrm{ch}\left[\mathcal{I}^{\epsilon}_{1,1;0}\right]} = e^{2\pi\lambda\epsilon} \frac{\sin(i\pi\sqrt{2p_+p_-}\epsilon)\sin(i\pi\sqrt{2p_+p_-}\epsilon)}{\sin(i\pi\sqrt{2p_+/p_-}\epsilon)\sin(i\pi\sqrt{2p_-/p_+}\epsilon)}. \\
\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right] &= \lim_{y\to\infty+} \frac{\mathrm{ch}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right](-1/iy)}{\mathrm{ch}\left[\mathcal{K}^{\epsilon}_{1,1;0}\right](-1/iy)} = \frac{S^{\epsilon}_{r,s;n}(0)}{S^{\epsilon}_{1,1;0}(0)} \\
&= e^{-\sqrt{2p_+p_-}\pi n\epsilon} \frac{\sin(ir\pi\sqrt{2p_-/p_+}\epsilon)\sin(is\pi\sqrt{2p_+/p_-}\epsilon)}{\sin(i\pi\sqrt{2p_-/p_+}\epsilon)\sin(i\pi\sqrt{2p_+/p_-}\epsilon)}.
\end{aligned} \tag{17}$$

Proof. The quantum dimensions follow directly from applying Proposition 5, Theorem 8 and Lemma 10. ☐

Clearly, in the limit $\epsilon \to 0$, we get

$$\mathrm{qdim}\left[\mathcal{F}_{\lambda+\alpha_0/2}\right] = p_+p_-,$$
$$\mathrm{qdim}\left[\mathcal{I}_{r,s;n}\right] = rs.$$

Next we consider $\mathrm{Re}(\epsilon) < 0$.

Corollary 17. *For* $\mathrm{Re}(\epsilon) < 0$ *the* $\mathcal{M}(p_+, p_-)$ *quantum dimensions of standard modules vanish, that is,*

$$\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right] = 0, \quad \forall\lambda \in \mathbb{C}.$$

Proof. Since we are considering $\mathrm{Re}(\epsilon) < 0$, the numerator of $\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right]$ is dominated by the $Y^{\epsilon}_{1,1;0}$ term in the denominator. ☐

We define the strip

$$\mathbb{S}(k, m) = \left\{\epsilon \in \mathbb{C} \,\middle|\, k + \frac{2m-1}{4p_+p_-} < \frac{\mathrm{Im}(\epsilon)}{\sqrt{2p_+p_-}} < k + \frac{2m+1}{4p_+p_-}\right\},$$

where $k \in \mathbb{Z}$ and $m \in \{0, 1, \dots, 2p_+p_- - 1\}$.

Proposition 18. *For* $\mathrm{Re}(\epsilon) < 0$ *and* $\epsilon \in \mathbb{S}(k, m)$*,* $k \in \mathbb{Z}$ *and* $m \in \{0, 1, \dots, 2p_+p_- - 1\}$*, then*

$$\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right] = \begin{cases}
(-1)^{mn} \dfrac{\sin\left(\frac{\pi m}{p_+} r\right)\sin\left(\frac{\pi m}{p_-} s\right)}{\sin\left(\frac{\pi m}{p_+}\right)\sin\left(\frac{\pi m}{p_-}\right)} & p_+, p_- \text{ do not divide } m, \\
s\,(-1)^{t(p_-n+1+s)} \dfrac{\sin\left(\frac{\pi t p_-}{p_+} r\right)}{\sin\left(\frac{\pi t p_-}{p_+}\right)} & \text{only } p_- \text{ divides } m = tp_-, \\
r\,(-1)^{t(p_+n+1+r)} \dfrac{\sin\left(\frac{\pi t p_+}{p_-} s\right)}{\sin\left(\frac{\pi t p_+}{p_-}\right)} & \text{only } p_+ \text{ divides } m = tp_+, \\
(-1)^{n+m} \dfrac{\sin\left(\frac{\pi r}{p_+}\right)\sin\left(\frac{\pi s}{p_-}\right)}{\sin\left(\frac{\pi}{p_+}\right)\sin\left(\frac{\pi}{p_-}\right)} & m = 0, p_+p_-,
\end{cases} \tag{18}$$

where $1 \le r \le p_+$*,* $1 \le s \le p_-$ *and* $n \in \mathbb{Z}$*.*

Proof. The quantum dimensions are easily computed case by case using the transformation formulae of Theorem 8.

1. case:

$$\begin{aligned}
\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right] &= \lim_{\tau\to i\infty}\frac{Y^{\epsilon}_{r,s;n}(\tau)}{Y^{\epsilon}_{1,1;0}(\tau)}\\
&= \lim_{\tau\to i\infty}\frac{\frac{d}{d\epsilon}q^{-\frac{\epsilon^2}{2}}\sum_{m=0}^{2p_+p_--1}(-1)^{mn}\sin\left(\pi\frac{rm}{p_+}\right)\sin\left(\pi\frac{sm}{p_-}\right)\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\frac{d}{d\epsilon}q^{-\frac{\epsilon^2}{2}}\sum_{m=0}^{2p_+p_--1}\sin\left(\pi\frac{m}{p_+}\right)\sin\left(\pi\frac{m}{p_-}\right)\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}\\
&= (-1)^{mn}\,\frac{\sin\left(\frac{\pi m}{p_+}r\right)\sin\left(\frac{\pi m}{p_-}s\right)}{\sin\left(\frac{\pi m}{p_+}\right)\sin\left(\frac{\pi m}{p_-}\right)},
\end{aligned}$$

2. case:

$$\begin{aligned}
\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right] &= \lim_{\tau\to i\infty}\frac{Y^{\epsilon}_{r,s;n}(\tau)}{Y^{\epsilon}_{1,1;0}(\tau)}\\
&= \lim_{\tau\to i\infty}\frac{\frac{si}{p_-}\sum_{m=0}^{2p_+p_--1}(-1)^{mn}\sin\left(\pi\frac{rm}{p_+}\right)\cos\left(\pi\frac{sm}{p_-}\right)\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}{\frac{i}{p_-}\sum_{m=0}^{2p_+p_--1}\sin\left(\pi\frac{m}{p_+}\right)\cos\left(\pi\frac{m}{p_-}\right)\theta_{p_+p_-,m}(i\sqrt{2p_+p_-}\epsilon\tau,\tau)}\\
&= s\,(-1)^{t(p_-n+1+s)}\,\frac{\sin\left(\frac{\pi t p_-}{p_+}r\right)}{\sin\left(\frac{\pi t p_-}{p_+}\right)}.
\end{aligned}$$

3. case (analogous to the 2. case):

4. case: For $m=0, p_+p_-$ the corresponding coefficients in $Y_{r,s,;n}$ vanish and so the $m\pm 1$ terms dominate. □

When comparing (17) and (18), we see that the two limits $\epsilon\to 0$ agree on a discrete set of points for $r<p_+$ and $s<p_-$:

$$\lim_{\epsilon\to\frac{im}{\sqrt{2p_+p_-}}^{+}}\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right] = \lim_{\epsilon\to\frac{im}{\sqrt{2p_+p_-}}^{-}}\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right]. \tag{19}$$

Theorem 19. *The regularised quantum dimensions satisfy*

$$\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda}\right]\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\mu}\right] = \sum_{j_+=0}^{p_+-1}\sum_{j_-=0}^{p_--1}\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda+\mu+j_+\alpha_++j_-\alpha_-}\right]$$

$$\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right]\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\mu}\right] = \sum_{j_+=0}^{r-1}\sum_{j_-=0}^{s-1}\mathrm{qdim}\left[\mathcal{F}^{\epsilon}_{\lambda+\alpha_{r-2j_+,s-2j_-;n}}\right]$$

$$\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{1,1;m}\right]\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right] = \mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;m+n}\right]$$

$$\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{2,1;0}\right]\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right]=\begin{cases}\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{2,s;n}\right] & \textit{if } r=1\\ \operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r-1,s;n}\right]+\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r+1,s;n}\right] & \textit{if } 1<r<p_+\\ \operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{1,s;n-1}\right]+2\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{p_+-1,s;n}\right]+ & \textit{if } r=p_+\\ \operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{1,s;n+1}\right] & \end{cases}$$

$$\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{1,2;0}\right]\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,s;n}\right]=\begin{cases}\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,2;n}\right] & \textit{if } s=1\\ \operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,s-1;n}\right]+\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,s+1;n}\right] & \textit{if } 1<s<p_-\\ \operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,1;n-1}\right]+2\operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,p_--1;n}\right]+ & \textit{if } s=p_-\\ \operatorname{qdim}\left[\mathcal{I}^{\epsilon}_{r,1;n+1}\right] & \end{cases}.$$

Proof. This result follows from expanding

$$\frac{\sin(ipx)}{\sin(ix)}=\sum_{j=0}^{p-1}e^{(p-i-2j)x},$$

and the computations follow very closely those of Chapter 4 of [CM1] so we omit them here. ☐

One can check directly that the above formulas induce a ring structure (we provide another proof below).

Remark 20. As mentioned in the introduction, we expect that an appropriate generalisation of the Verlinde formula holds for many irrational and even non C_2-cofinite vertex operator algebras. Considering characters as algebraic distributions instead of as functions on the upper-half plane, in Section 4 [RW], the third author and D. Ridout found a Verlinde-type ring of characters $(\mathcal{V}_{ch},+,\times)$, where $\mathcal{V}_{ch}$ is the free abelian group generated by irreducible characters. We will prove in Theorem 25 that this ring agrees with the one in Theorem 19, in the sense that

$$\operatorname{ch}\big[X\big]\to\operatorname{qdim}\left[X^{\epsilon}\right],$$

defines an isomorphism of rings. But there is even a third point of view here. By using the Verlinde product of *regularised* characters defined in [CM1], as in the case of $\mathcal{M}(p)$-modules, we can define a ring of regularised $\mathcal{M}(p_+,p_-)$-characters. Again, this ring is isomorphic to $(\mathcal{V}_{ch},+,\times)$.

The previous remark gives more than sufficient evidence for the correctness of the following conjecture.

Conjecture 21. *The relations in Theorem 19 or Section 4* [RW], *obtained at the level of characters, remain valid inside the Grothendieck ring of a suitable quotient category of $\mathcal{M}(p_+,p_-)$-singlet modules.*

7. Fusion rings and fusion varieties

In this part, we study the behaviour of the Verlinde algebra of characters as we vary the parameter ϵ throughout the complex plane. We also explain that the ϵ parameter, used in [CM1] merely as a computational tool, is actually a uniformisation parameter of an interesting algebraic variety - a further indication of the conceptual importance of the parameter.

Let us start first start with a brief discussion of rational theories. It is known that the fusion ring of a rational vertex operator algebra (obtained either as the Grothendieck ring in the category or as the Verlinde formula of characters) is of finite rank. After an extension of scalars, we easily infer that the corresponding fusion algebra is semi-simple; therefore the set of maximal ideals is precisely of the size of the set of equivalence classes of irreducible modules. This algebra can be also be viewed as the algebra of functions of a zero-dimensional variety - the *fusion variety*. There are various conjectures about the realisation of this fusion ring in terms of generators and relations, especially in the case of WZW models [Ge, BR].

As we already discussed in Section 1.4, from the categorical perspective the categories of $\mathcal{M}(p_+,p_-)$ and $\mathcal{M}(p)$-modules are too large for practical purposes. That is why we prefer the category constructed only from atypical blocks. One important advantage of this category is that it admits only countably many equivalence classes of irreducible modules. As such, after we pass to its Grothendieck ring, it has direct relevance to algebraic geometry because it defines the ring of functions of an n-dimensional complex algebraic variety with $n \geq 1$.

The first statement of the next result follows easily from [CM1], while the second statement is taken from Section 4, [RW].

Proposition 22. *The Verlinde algebra of atypical blocks has the following presentation*

(i)

$$\mathbb{V}(1,p) = \frac{\mathbb{C}[X,Z,Z^{-1}]}{\langle U_p(\frac{X}{2}) - U_{p-2}(\frac{X}{2}) - Z - Z^{-1}\rangle},$$

(ii)

$$\mathbb{V}(p_+,p_-) = \frac{\mathbb{C}[X,Y,Z,Z^{-1}]}{\langle U_{p_+}(\frac{X}{2}) - U_{p_+-2}(\frac{X}{2}) - Z - Z^{-1}, U_{p_-}(\frac{Y}{2}) - U_{p_--2}(\frac{Y}{2}) - Z - Z^{-1}\rangle},$$

if one identifies

$$X \leftrightarrow \mathrm{ch}\big[\mathcal{I}_{2,1;0}\big], \qquad Y \leftrightarrow \mathrm{ch}\big[\mathcal{I}_{1,2;0}\big], \qquad Z^{\pm 1} \leftrightarrow \mathrm{ch}\big[\mathcal{I}_{1,1;\pm 1}\big],$$

where the U_i are Chebyshev polynomials of the second kind.

Clearly, Z^{-1} can be eliminated from each presentation by introducing z^+ and z^- variables with the relation $z^+z^- - 1$. Observe that the above presentation also holds over integers. This is the main reason why we used Chebyshev polynomials of the second kind instead of the first kind.

Next, we analyse varieties $\mathcal{X}_{1,p}$ and $\mathcal{X}_{p_+,p_-}$, whose rings of functions are $\mathbb{V}(1,p)$ and $\mathbb{V}(p_+,p_-)$, respectively.

Theorem 23. *For every $p \geq 2$, the fusion varieties $\mathcal{X}_{1,p}$ is an irreducible rational curve with $p-1$ ordinary singular points, while $\mathcal{X}_{p_+,p_-}$ is a genus zero curve with $\frac{(p_--1)(p_+-1)}{2}$ singular points. Moreover, these singularities can be viewed as pinched cycles obtained from a Riemann surface of higher genus.*

Proof. For $\mathcal{X}_{1,p}$, observe that our variety can be described as the set of points (x,z) such that $z \neq 0$ and

$$2zT_p\left(\frac{x}{2}\right) - z^2 - 1 = 0, \tag{20}$$

where $T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x))$ are Chebyshev polynomials of the first kind. From the Jacobian we see that possible singular points are $z = \pm 1$ and $U'_{p-1}(x/2) = 0$. We have

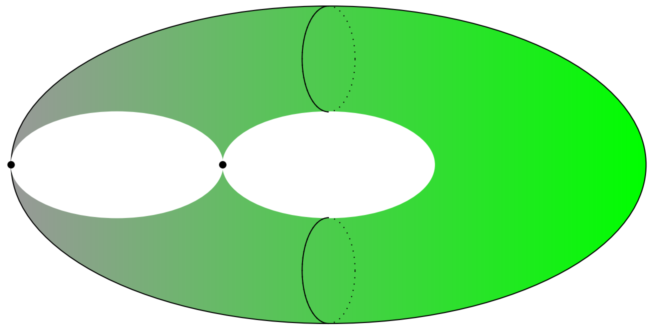

FIGURE 2. Fusion variety of the $(2,5)$-singlet algebra.

$2(p-1)$ solutions to this equation: $(x_k, z) = (2\cos(\frac{k\pi}{p}), \pm 1)$, $k = 1, ..., p-1$. By plugging these into (20) and using the fact that $T_p(x_k) = \pm 1$, we see that either $(x_k, 1)$ or $(x_k, -1)$ are a singular points, all together $p-1$. These are all ordinary double points. By using the formula $T_n(cos(\theta)) = cos(n\theta)$ we easily see that $x(t) = \frac{1}{t} + t$, $z(t) = \frac{1}{t^n}$ is a rational parametrisation of the curve and it is thus of genus zero and clearly irreducible.

For $\mathcal{X}_{p_+,p_-}$ a similar analysis applies. Observe that this curve is defined as all (x, y, z) such that $z \neq 0$ and $2zT_{p_+}(\frac{x}{2}) - z^2 - 1 = 2zT_{p_-}(\frac{x}{2}) - z^2 - 1 = 0$. For the singular locus we first solve $z = \pm 1$, $U_{p_+-1}(x/2) = U_{p_--1}(y/2) = 0$, resulting in $2(p_+-1)(p_--1)$ possible singular points. As these points are supposed to lie on the curve we have to eliminate everything but $\frac{1}{2}(p_+-1)(p_--1)$ points. The genus is again zero due to the rational parametrisation $x(t) = \frac{1}{t^{p_-}} + t^{p_-}$, $y(t) = \frac{1}{t^{p_+}} + t^{p_+}$, $z(t) = \frac{1}{t^{p_+p_-}}$. $\square$

Remark 24. Based on analysis of $\mathcal{X}_{p_+,p_-}$ for low p_+ and p_-, we believe that $\mathcal{X}_{p_+,p_-}$ is also irreducible. In fact, if we extend the definition of $\mathcal{X}_{p_+,p_-}$ to all positive integers, the curve seems to be irreducible precisely when p_+ and p_- are relatively prime.

Now we see that quantum dimensions in $\mathrm{Re}(\epsilon) > 0$ (at least those associated to generators of the fusion ring) provide a uniformisation of the fusion ring with ϵ being the uniformisation parameter. For example, by using formula (17), we easily see that $\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{1,2;0}\right]$, $\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{2,1;0}\right]$, $\mathrm{qdim}\left[\mathcal{I}^{\epsilon}_{1,1;1}\right]$ give a parametrisation mentioned the proof of Theorem 23 with $t = e^{\frac{\pi\epsilon}{\sqrt{2p_+p_-}}}$. So ϵ should be viewed as a uniformisation parameter of the fusion variety.

Theorem 25. *The map*

$$\mathrm{ch}\big[M\big] \mapsto \mathrm{qdim}\left[M^{\epsilon}\right]$$

from the Verlinde ring of characters (as computed in [RW]*) to the space of quantum dimensions as functions of ϵ is:*

(1) *For the domain* $\mathrm{Re}(\epsilon) > 0$*: a ring isomorphism*
(2) *For the domain* $\mathrm{Re}(\epsilon) < 0$ *a ring homomorphism with image isomorphic to the fusion ring of the (p_+, p_-) Virasoro minimal model.*
(3) *For the domain* $\mathrm{Re}(\epsilon) < 0$ *and $m \neq 0$ divisible by p_-: a ring homomorphism with image isomorphic to the fusion ring of $A_1^{(1)}$ at level $p_+ - 2$.*
(4) *For the domain* $\mathrm{Re}(\epsilon) < 0$ *and $m \neq 0$ divisible by p_+: a ring homomorphism with image isomorphic to the fusion ring of $A_1^{(1)}$ at level $p_- - 2$.*

Proof. (1) Comparing with [RW] we see that the map is a ring homomorphism. Also, it can be easily seen that the space of characters is spanned by $\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{\lambda}\big], \mathrm{ch}\big[\mathcal{I}^{\pm,\epsilon}_{r,s;0}\big]$ and $\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{r,s;0}\big]$, and that the space of characters in [RW] is spanned by characters with the same labels. The corresponding quantum dimensions are linearly independent since there are no relations between the functions $e^{2\pi\epsilon\lambda}$, so we have an isomorphism.

(2) We first compare quantum dimensions in $\mathrm{Re}(\epsilon) > 0$ to those in $\mathrm{Re}(\epsilon) < 0$ and observe that the values of quantum dimension at each semi-infinite strip can be computed as the limiting value at a certain point in $\mathrm{Re}(\epsilon) > 0$; see formula (19). Limits of course preserve products of quantum dimensions. Because of (1), we naturally get a ring structure on the same span of quantum dimensions. Here, the kernel contains the ideal generated by the $\mathrm{ch}\big[\mathcal{F}^{\epsilon}_{\lambda}\big], \mathrm{ch}\big[\mathcal{I}^{\pm,\epsilon}_{r,s;n}\big]$ and $\mathrm{ch}\big[\mathcal{I}^{\epsilon}_{r,s;0}\big] - \mathrm{ch}\big[\mathcal{I}^{\epsilon}_{p_+-r,p_--s;0}\big]$. The product of the quantum dimensions of the $\mathcal{I}^{\epsilon}_{r,s;0}$ for r, s in $\mathcal{T}_{p_+,p_-}$ is given by the corresponding Virasoro minimal model fusion coefficients. So that they must span a homomorphic image of the Virasoro minimal model fusion ring. Linear independence now follows, since the matrix of generalised quantum dimensions of a modular tensor category is related to the categorical S-matrix via elementary matrix operations, but the S-matrix is invertible. For $m = 0$ the left and right limits do not agree at $\epsilon = 0$ but instead we can use a different point (still on the same strip). Again we get a ring homomorphism whose image is isomorphic to the Virasoro fusion ring.

For (3) and (4), the previous argument with limits apply and we get a homomorphism from the fusion ring. Recall the definition of t via $m = tp_-$ (case (3)) and $m = tp_+$ (case (4)). Finally, to prove that the image is isomorphic to the corresponding $A_1^{(1)}$ fusion ring we note that t takes only values in $\{1, \ldots, p_+ - 1, p_+ + 1, \ldots, 2p_+ - 1\}$ (case (3)) or $\{1, \ldots, p_- - 1, p_- + 1, \ldots, 2p_- - 1\}$ (case (4)). So that the quantum dimensions in this case span at most a $p_+ - 1$ respectively $p_- - 1$ dimensional vector space. The same reasoning as in Proposition 14 applies here as well so the proof follows. □

Remark 26. There is a purely geometrical interpretation of Theorem 25. Because every $\mathbb{C}$-algebra homomorphism between finitely generated reduced $\mathbb{C}$-algebras is the pull-back of the corresponding regular map of the corresponding affine varieties, the observed surjective homomorphisms of rings from $\mathrm{Re}(\epsilon) > 0$ to (parts of) $\mathrm{Re}(\epsilon) < 0$ corresponds simply to an embedding of finite number of special points (zero dimensional variety) to $\mathcal{X}_{p,p'}$ (or $\mathcal{X}_{1,p}$).

Remark 27. The varieties $\mathcal{X}_{p_+,p_-}$ (more precisely, the plane curve defined by $T_{p_+}(x) - T_{p_-}(y) = 0$), have appeared in the physics literature in the problem of (p_+, p_-) minimal string theory [SS]. Points on the fusion variety $\mathcal{X}_{p_+,p_-}$ are in one-to-one correspondence with branes of the (p_+, p_-) minimal string theory and the uniformisation parameter is expressible in terms of the boundary cosmological constant of the string theory.

Remark 28. In [BM1], the full asymptotic expansion of $\mathrm{ch}\big[X\big](\tau)$ was studied for atypical modules X as $\tau \to 0^+$. In particular, by using a completely different method, the second author and K. Bringmann obtained results on unregularised quantum dimensions. More about the geometry of fusion varieties including higher rank vertex operator algebras [BM2] will be a subject of [CM2].

8. Quantum modularity and quantum dimensions

In this section, we associate to each singlet vertex operator algebra a vector-valued quantum modular form such that its associated S-matrix is the S-matrix that arises in the

$\mathrm{Re}(\epsilon) < 0$ regime studied earlier (see Theorem 25). In other words, as explained in the introduction, we provide another approach to "semi-simplification" from the limiting properties of characters as we approach the real axis.

We recall the definition of quantum modular forms, originally due to Zagier [Za].

Definition 29. A weight k quantum modular form is a complex-valued function f on $\mathbb{Q}$, or possibly smaller infinite set $\mathbb{Q} \setminus S$, the so-called *quantum set*, such that the function

$$h_\gamma(x) := f(x) - \epsilon_\gamma (cx+d)^{-k} f(\frac{ax+b}{cx+d}), \quad x \in \mathbb{Q} \setminus S$$

$\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2,\mathbb{Z})$, satisfies a suitable property of continuity and analyticity.

Our understanding is that this definition is intentionally ambiguous so that it can accommodate more examples. But all known examples of quantum modular forms enjoy slightly nicer properties. For *strong* quantum modular forms, there is an additional requirement that such a function extends to an analytic function defined in the both upper and the lower half-plane and that limits coincide to all orders at roots of unity. This way one obtains a fairly non-standard object: an analytic function in the upper half-plane which leaks throughout the quantum set (typically a proper subset of $\mathbb{Q}$) into the lower half-plane.

8.1. $\mathcal{M}(p)$**-models.** For brevity, let

$$F_{j,p}(\tau) := \sum_{n \in \mathbb{Z}} \mathrm{sgn}(n) q^{p\left(n+\frac{j}{2p}\right)^2},$$

where we use the convention $\mathrm{sgn}(n) = 1$, for $n \geq 0$ and -1 otherwise. We are only interested in the (quotient) space of characters moded out by the subspace of characters of standard modules. As easily be seen, the character $\mathrm{ch}\big[M_{r,s}\big](\tau)$ admits a unique decomposition $q_{r,s}(\tau)+$ $\mathrm{ch}\big[M_{1,s'}\big](\tau)$ where $q_{r,s}(\tau)$ is a finite q-series divided by the Dedekind eta function. So we are left with $\mathrm{ch}\big[M_{1,s}\big]$, where $1 \leq s \leq p-1$. One can easily verify that

$$\mathrm{ch}\big[M_{1,s}\big](\tau) = \frac{F_{p-s,p}(\tau)}{\eta(\tau)}.$$

In [BM1] it was proved that $\mathrm{ch}\big[M_{1,s}\big](\tau)$ is a quantum modular form with quantum set $\mathbb{Q}$. We recall the construction here. As in Zagier's paper [Za], define for $\tau \in \bar{\mathbb{H}} := \{\tau \in \mathbb{C}; \mathrm{Im}(\tau) < 0\}$ the non-holomorphic Eichler integral

$$F^*_{j,p}(\tau) := \sqrt{2i} \int_{\overline{\tau}}^{i\infty} \frac{f_{j,p}(z)}{(z-\tau)^{\frac{1}{2}}} dz,$$

where $f_{j,p}(z) := \sum_{n \in \mathbb{Z}} \left(n + \frac{j}{2p}\right) q^{p\left(n+\frac{j}{2p}\right)^2}$. A key step in the proof of Theorem 4.1 of [BM1] is to show that $F_{j,p}(\tau)$ agrees for $\tau = \frac{h}{k} \in \mathbb{Q}$ with $F^*_{j,p}(\tau)$ up to infinite order (in the sense that appropriate limits agree) and that $F^*_{j,p}(\tau)$ satisfies nice transformation law under a particular congruence subgroup. Here we are interested in transformation properties under the full modular group $\Gamma(1)$.

We shall ignore the η factor here as it does not affect the S-matrix. We form the vector valued function $\mathbf{F}(\tau) = (F_{1,p}(\tau), ...F_{p-1,p}(\tau))$. To find a quantum S-matrix we consider $\mathbf{F}^*(\tau) = (F^*_{1,p}(\tau), ..., F^*_{p-1,p}(\tau))$. Then we get the following transformation formula

$$\begin{aligned} f_{j,p}(-1/\tau) &= (-\tau)\sqrt{\frac{-i\tau}{2p}} \sum_{c=1}^{2p-1} e^{-\pi i jc/p} f_{c,p}(\tau) \\ &= (-\tau)\sqrt{\frac{-i\tau}{2p}}(-2i) \sum_{c=1}^{p-1} \sin\left(\frac{cj\pi}{p}\right) f_{c,p}(\tau). \end{aligned}$$

where we used the relation $f_{j,p}(\tau) = -f_{2p-j,p}(\tau)$. We now get

$$\begin{aligned} F^*_{j,p}(-1/\tau) &= \sqrt{2i}\int_{-1/\bar\tau}^{i\infty} \frac{f_{j,p}(z)}{(z+1/\tau)^{\frac{1}{2}}}dz = \sqrt{2i}\int_{\bar\tau}^{0} \frac{u^{-2}f_{j,p}(-1/u)}{(-1/u+1/\tau)^{\frac{1}{2}}}du \\ &= -\sqrt{2i}\int_0^{\bar\tau} \frac{\sqrt{u\tau}}{u^2\sqrt{-\tau+u}}(-u)\sqrt{\frac{-iu}{2p}} \sum_{c=1}^{2p-1} e^{-\pi i jc/p} f_{c,p}(u)du \\ &= -\sqrt{\frac{2i\tau}{p}} \sum_{c=0}^{p-1} \sin\left(\frac{\pi cj}{p}\right) \sqrt{2i}\int_0^{\bar\tau} \frac{f_{c,p}(u)}{\sqrt{u-\tau}}du, \end{aligned}$$

after we take $u = -1/z$, $\frac{du}{u^2} = dz$. Form a $(p-1)\times(p-1)$ matrix $\mathbf{S}(p)$, where $[\mathbf{S}(p)]_{c,j} = \sqrt{\frac{2}{p}}\sin(\frac{\pi cj}{p})$.

Theorem 30. *For $w \in \mathbb{H} \cup \overline{\mathbb{H}} \cup \mathbb{Q}$, we have that $\mathbf{F}(w)$ is a weight $\frac{1}{2}$ vector-valued quantum modular form. In particular, we have that*

$$\sqrt{\frac{1}{wi}}\mathbf{F}\left(-\frac{1}{w}\right) - [\mathbf{S}(p)]\mathbf{F}(w) = [\mathbf{S}(p)]g(w), \tag{21}$$

where $g(w) = -\sqrt{2i}\int_0^{i\infty} \frac{f(u)du}{\sqrt{u-w}}$, $f(u) = (f_{1,p}(u), ..., f_{p-1,p}(u))$ and $\mathbf{S}(p)$ is the S-matrix of $A_1^{(1)}$ at level $p-2$.

Proof. This follows from [BM1] and the discussion above. It was already proved in [BM1] that the quantum set of $\mathbf{F}$ is $\mathbb{Q}$ by computing radial limits at each point in $\mathbb{Q}$. It is also explained in [BM1] and elsewhere that $g_\alpha(w)$, $\alpha \in \Gamma(1)$, is a smooth function for $\alpha \in \mathbb{R}$. Although $g(w)$ is a priori only defined in $\overline{\mathbb{H}}$, we may take any path L connecting points 0 to $i\infty$. Then we can holomorphically continue $g(w)$ to all of $\mathbb{C} \setminus L$. Thus, we obtain a continuation of g which is smooth on $\mathbb{R}$ and analytic on $\mathbb{R} \setminus \{0\}$. $\square$

8.2. **$\mathcal{M}(p_+, p_-)$-models.** In [BM1], it was shown that all atypical characters are mixed quantum modular forms with quantum set $\mathbb{Q}$; see also [BCR] where similar quantities appear. In the spirit of the $\mathcal{M}(p)$ algebra we consider the vector spaces spanned by characters of irreducible modules for the $\mathcal{M}(p_+, p_-)$ algebra. Consider its quotient space modulo the subspace spanned by characters of standard modules $\mathcal{F}_\lambda$ and of the irreducible atypical module $\mathcal{I}^\nu_{r,s;n}$ and $\mathcal{L}_{r,s}$. By the relations used in section 5.1 of [BM1] and in previous sections, easy inspection shows that for a spanning set of the quotient space of we can choose

$$\begin{aligned}\tilde{\chi}_{r,s}(\tau) &:= \eta(\tau)\mathrm{ch}\big[\mathcal{I}_{r,s;0}\big](\tau) - \left(\sum_{k=0}^{\infty} q^{p_-p_+\left(k+\frac{2p_+p_- + p_+s + p_-r}{2p_+p_-}\right)^2} - \sum_{k=0}^{\infty} q^{p_-p_+\left(k+\frac{2p_+p_- - p_+s + p_-r}{2p_+p_-}\right)^2}\right) \\ &= \sum_{k\geq 0}(k+1)\left(q^{p_-p_+\left(k+\frac{2p_+p_- - p_+s - p_-r}{2p_+p_-}\right)^2} + q^{p_-p_+\left(k+\frac{2p_+p_- + p_+s + p_-r}{2p_+p_-}\right)^2}\right. \\ &\qquad \left. - q^{p_-p_+\left(k+\frac{2p_+p_- - p_+s + p_-r}{2p_+p_-}\right)^2} - q^{p_-p_+\left(k+\frac{2p_+p_- + p_+s - p_-r}{2p_+p_-}\right)^2}\right),\end{aligned}$$

where $(r,s) \in \mathcal{T}_{p_+,p_-}$. This space is precisely $\frac{(p_+-1)(p_--1)}{2}$-dimensional.

Similarly, we introduce for $\tau \in \overline{\mathbb{H}}$,

$$\mathbf{G}^*(\tau) = \sqrt{2i}\int_{\bar{\tau}}^{i\infty} \frac{\mathbf{f}(z)}{(z-\tau)^{3/2}}dz,$$

where $\mathbf{f}(z) = (..., \eta(z)\mathrm{ch}\big[\mathcal{L}_{r,s}(z)\big], ...)$, $(r,s) \in \mathcal{T}_{p_+,p_-}$ and the entries are suitably ordered. We also let

$$\mathbf{G}(\tau) = (\ldots, \tilde{\chi}_{r,s}(\tau), \ldots), \quad \tau \in \mathbb{H},$$

where the (r,s) are ordered as in $\mathbf{G}^*$. Now we extend $\mathbf{G}$ as a function defined on $\mathbb{H} \cup \overline{\mathbb{H}}$, where we let $\mathbf{G}(w) := \mathbf{G}^*(w)$ for $w \in \overline{\mathbb{H}}$.

Theorem 31. *For $w \in \mathbb{H} \cup \overline{\mathbb{H}} \cup \mathbb{Q}$, we have that $\mathbf{G}(w)$ is a weight $\frac{3}{2}$ quantum vector-valued modular form. In particular, we have that*

$$\left(\frac{1}{wi}\right)^{3/2}\mathbf{G}(-\frac{1}{w}) + [\mathbf{SM}(p_+,p_-)]\mathbf{G}(w) = [\mathbf{SM}(p_+,p_-)]g(w), \tag{22}$$

where $g(w) = -\sqrt{2i}\int_0^{i\infty}\frac{\mathbf{f}(u)}{(u-w)^{3/2}}du$. and $[\mathbf{SM}(p_+,p_-)]$ is the S-matrix of (p_+,p_-) minimal models.

Proof. It was already proven in [BM1] that each component of $\mathbf{G}(z)$ is a quantum modular form. In particular, to prove that radial limits of $\mathbf{G}(z)$ and $\mathbf{G}^*(z)$ agree as z approaches a rational value from each side, it is sufficient to decompose $\mathbf{G}^*(\tau)$ into a sum of 4 Eichler integrals associated to partial thetas and to use results obtained in [BCR] to compute these limits which are expressed in terms of L-function values. The remaining computations follow as in the previous section by using modular transformation properties of Eichler's integrals and those of the characters of the minimal models. Real analyticity is handled as in Theorem 30.

□

Remark 32. Of course, we could have worked with the "more natural" basis $\mathrm{ch}\big[\mathcal{I}_{r,s;0}\big](\tau)$ instead of $\tilde{\chi}_{r,s}(\tau)$. But it turns out that $\eta(\tau)\mathrm{ch}\big[\mathcal{I}_{r,s;0}\big](\tau)$ is a quantum modular form of mixed weight $\frac{3}{2}$ and $\frac{1}{2}$, so the formulation of Theorem 31 would be significantly messier.

We should also remark that Theorem 31 was independently discovered in [HK] in the context of quantum knot invariants.

References

[AM1] D. Adamovic and A. Milas, Logarithmic intertwining operators and $W(2, 2p-1)$-algebras, J. Math. Phys., **48** (2007), p. 073503.

[AM2] D. Adamovic and A. Milas, On W-algebras associated to $(2, p)$ minimal models and, International Mathematics Research Notices (2010) 20, 3896-3934.

[AM3] D. Adamovic and A.Milas, On W-algebra extensions of $(2, p)$ minimal models: $p > 3$, *Journal of Algebra* **344** (2011) 313-332.

[AC] C. Alfes and T. Creutzig, The mock modular data of a family of superalgebras, Proc. Amer. Math. Soc. 142 (2014), 2265-2280

[AP] H. H. Anderson and J. Paradowski, Fusion categories arising from semisimple Lie algebras, *Comm. Math. Physics* **169** (1995), 563-588.

[BM1] K. Bringmann and A.Milas, W-algebras, False Theta Series and Quantum Modular forms, Part I, submitted.

[BM2] K. Bringmann and A.Milas, W-algebras, False Theta Series and Quantum Modular forms, Part II, preprint.

[BCR] K. Bringmann, T. Creutzig and L. Rolen, Negative index Jacobi forms and quantum modular forms, `arXiv:1401.7189`.

[BCGP] C. Blanchet, F. Costantino, N. Geer, and B. Patureau-Mirand, Non semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants, `arXiv:1404.7289`

[BR] P. Bouwknegt and D. Ridout, Presentations of Wess-Zumino-Witten Fusion Rings, *Rev. Math. Phys.* **18** 201 (2006).

[CGP] F. Costantino, N. Geer, and B. Patureau-Mirand, Some remarks on the unrolled quantum group of $sl(2)$, `arXiv:1406.0410` .

[CM1] T. Creutzig and A.Milas, False Theta Functions and the Verlinde formula, *Advances in Mathematics* **520** (2014), 520-545.

[CM2] T. Creutzig and A.Milas, in preparation.

[CR1] T. Creutzig and D. Ridout, Modular Data and Verlinde Formulae for Fractional Level WZW Models I, *Nucl. Phys. B* **865** (2012) 83

[CR2] T. Creutzig and D. Ridout, Modular Data and Verlinde Formulae for Fractional Level WZW Models II, *Nucl. Phys. B* **875** (2013) 423.

[CR3] T. Creutzig and D. Ridout, Logarithmic Conformal Field Theory: Beyond an Introduction, J. Phys. A 46 (2013), no. 49, 494006.

[CRW] T. Creutzig, D. Ridout and S. Wood, Coset Constructions of Logarithmic (1, p) Models, Lett. Math. Phys. **104** (2014) 553.

[DJX] C. Dong, X. Jiao and F. Xu, Quantum dimensions and quantum Galois theory, *Trans Amer Math Soc* 365 (2013) 12, 6441–6469.

[DV] R. Dijkgraaf and E. Verlinde, Modular Invariance and the Fusion Algebra, *Nuclear Physics B* (Proc. Suppl.) 5B (1988) 87-97.

[Fel] G. Felder, BRST approach to minimal models, *Nucl Phys* B **317** (1989), 215236

[FHST] J. Fuchs, S. Hwang, A.M. Semikhatov, and I. Yu. Tipunin, Nonsemisimple Fusion Algebras and the Verlinde Formula, *Comm. Math. Phys.* **247** (2004), no. 3, 713–742.

[FGST1] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov and I.Yu. Tipunin, Logarithmic extensions of minimal models: Characters and modular transformations, *Nucl Phys* B **757** (2006), 303343

[FGST2] B.L. Feigin, A.M. Gainutdinov, A.M. Semikhatov and I.Yu. Tipunin, Kazhdan-Lusztig-dual quantum group for logarithimic extensions of Virasoro minimal models, *J Math Phys* **48** (2007), 032303

[GK] M.R. Gaberdiel and H.G. Kausch, A Rational Logarithmic Conformal Field Theory, *Phys.Lett.* B **386** (1996) 131-137.

[Ge] D. Gepner, Fusion rings and geometry, *Comm. Math. Phys.* **141** (1991), 381-411.

[HK] K. Hikami and A. Kirillov, Torus Knot and Minimal Model, *Phys.Lett.* B **575** (2003) 343-348

[Hu] Y.-Z. Huang, Vertex operator algebra and the Verlinde conjecture, Commun. Contemp. Math., **10** (2008), 108.

[IK] K. Iohara and Y. Koga, Representation theory of the Virasoro algebra, Springer Monographs in Mathematics, Springer-Verlag (2011).

[KW1] V. Kac and M. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell's function, *Comm. Math. Phys.* **215** (2001) 631–682.

[KW2] V. Kac and M. Wakimoto, Modular invariant representations of infinite-dimensional Lie algebras and superalgebras, *Proc. Nat. Acad. Sci.* **85** (1988) 4956–4960.

[Kau] H.G. Kausch, Curiosities at $c = -2$, preprint DAMTP 95-52, `arXiv:hep-th/9510149`.

[M] A.Milas, Characters of modules of irrational vertex algebras, *Proceedings of the Conference on Vertex Algebras and Automorphic Forms*, Heidelberg, 2011 **8** MATCH, Springer, (2014).

[Miy] M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C_2-cofiniteness*. Duke Math. J. **122** (2004), 51–91.

[MS] G. W. Moore and N. Seiberg, Classical and Quantum Conformal Field Theory, *Commun Math Phys* **123** (1989) 177.

[NT] K. Nagatomo and A. Tsuchiya, The triplet vertex operator algebra W(p) and the restricted quantum group at root of unity, *Advanced Studies in Pure Mathematics 61: Exploring New Structures and Natural Constructions in Mathematical Physics*, American Mathematical Society (2011), 1-49.

[O] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[RW] D. Ridout and S. Wood, Modular Transformations and Verlinde Formulae for Logarithmic (p_+, p_-)-Models, *Nucl Phys* B **880** (2014), 175-202.

[SS] N. Seiberg and D. Shih, Branes, rings and matrix models in minimal (super)string theory, JHEP **0402** (2004) 021.

[T] V. G.. Turaev, Quantum invariants of knots and 3-manifolds, de Gruyter Studies in Mathematics 18, Second revised edition, Walter de Gruyter & Co., Berlin (2010) xii+592.

[TW] A. Tsuchiya and S. Wood, On the extended W-algebra of type $\mathfrak{sl}_2$ at positive rational level, `arXiv:1302.6435`.

[V] E. P. Verlinde, Fusion Rules and Modular Transformations in 2D Conformal Field Theory, *Nucl Phys* B **300** (1988) 360.

[Za] D. Zagier, Quantum modular forms, In Quanta of Maths: Conference in honor of Alain Connes, Clay Mathematics Proceedings 11, AMS and Clay Mathematics Institute 2010, 659-675.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J Amer Math Soc* 9 (1996) 237–302.

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada. *email: creutzig@ualberta.ca*

Department of Mathematics and Statistics, SUNY-Albany, 1400 Washington Avenue, Albany, NY 12222, USA. *email: amilas@albany.edu*

Department of Theoretical Physics, Research School of Physics and Engineering; and Mathematical Sciences Institute; The Australian National University, Acton, ACT 2601, Australia. *email:simon.wood@anu.edu.au*