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ON REGULARITY CRITERIA IN TERMS OF PRESSURE FOR THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^3$

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Abstract. In this paper we establish a Serrin-type regularity criterion on the gradient of pressure for the weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. It is proved that if the gradient of pressure belongs to $L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma \leq 3$, $1 \leq \gamma \leq \infty$, then the weak solution is actually regular. Moreover, we give a much simpler proof of the regularity criterion on the pressure, which was showed recently by Berselli and Galdi (Proc. Amer. Math. Soc. 130 (2002), no. 12, 3585–3595).

1. Introduction

We consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0,T)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, \\ \mathrm{div} u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with $\mathrm{div} u_0 = 0$ in the sense of distribution is the initial velocity field.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history (see [5, 19]). In the pioneering work [12] and [7], Leray and Hopf proved the existence of its weak solutions $u(x,t) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$ for given $u_0(x) \in L^2(\mathbb{R}^3)$. However, we do not yet know whether or not the solution develops singularities in finite time even if the initial datum is C^∞-smooth. In [15], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by Caffarelli, Kohn and Nirenberg in [3]. Further results can be found in [20] and the references therein.

On the other hand, the regularity of a given weak solution u can be shown under additional conditions. In 1962, Serrin [16] proved that if u is a Leray-Hopf weak solution belonging to $L^{\alpha,\gamma} \equiv L^\alpha(0,T;L^\gamma(\mathbb{R}^3))$ with $2/\alpha + 3/\gamma < 1, 2 < \alpha < \infty, 3 < \gamma < \infty$, then the solution $u(x,t)$ belongs to $C^\infty(\mathbb{R}^3 \times (0,T])$, while the limit case $2/\alpha + 3/\gamma = 1$ was covered much later by H. Sohr [17] (recently,

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Beirão da Veiga [1] added Serrin's condition on only two components of the velocity field). From then on, there are many criterion results added on u. In [21] and [6], von Wahl and Giga showed that if u is a weak solution in $C([0,T);L^3(\mathbb{R}^3))$, then $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T])$. Struwe [18] proved the same regularity of u in $L^\infty(0,T;L^3(\mathbb{R}^3)$ provided $\sup_{0<t\leq T} \|u(x,t)\|_{L^3}$ is sufficiently small, and Kozono and Sohr [10] obtained the regularity for the weak solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T])$ provided $u(x,t)$ is left continuous with respect to the L^3-norm for every $t \in (0,T)$. Recently Kozono and Taniuchi [11] showed that if a Leray-Hopf weak solution $u(x,t) \in L^2(0,T;BMO)$, then $u(x,t)$ is actually a strong solution of (1.1) on $(0,T]$. Recent progress concerning another limit case $u \in L^\infty(0,T;L^3)$ can be found in [8]

It is well known that if (u,p) solves the Navier-Stokes equations, then so does (u_λ, p_λ) for all $\lambda > 0$, where $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, $p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$. The class of Serrin's type is important from a viewpoint of scaling invariance, which implies that $\|u_\lambda\|_{L^{\alpha,\gamma}} = \|u\|_{L^{\alpha,\gamma}}$ holds for all $\lambda > 0$ if and only if $2/\alpha + 3/\gamma = 1$, and we say that the norm $\|u\|_{L^{\alpha,\gamma}}$ has the scaling dimension zero [3].

It is easy to check that if $2/\alpha + 3/\gamma = 3$, $\|\nabla p\|_{L^{\alpha,\gamma}}$ has scaling dimension zero. As far as we know, there are only few regularity criteria in terms of ∇p; see [2, 14]. The best result [2] for the whole space is that

$$\nabla p \in L^\alpha(0,T;L^\gamma(\mathbb{R}^3)) \ \text{ with } \frac{2}{\alpha} + \frac{3}{\gamma} = 3, \quad \text{for } \gamma \in [9/7, 3].$$

In [2], regularity criteria were established not only for the whole space, but also for a domain with boundary (bounded, exterior or the half-space). The purpose of this paper is to establish a final regularity criterion in terms of the gradient of pressure. Our main theorem reads

Theorem 1.1. *Let $u_0(x) \in L^2(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, for $q \geq 4$, and let $\operatorname{div} u_0 = 0$ in the sense of distribution. Suppose that $u(x,t)$ is a Leray-Hopf weak solution of* (1.1). *If*

$$\nabla p \in L^\alpha\left(0,T;L^\gamma(\mathbb{R}^3)\right) \quad \textit{with } \frac{2}{\alpha} + \frac{3}{\gamma} \leq 3,\ \frac{2}{3} < \alpha < \infty,\ 1 < \gamma < \infty,$$

or $\nabla p \in L^{2/3,\infty}$, or else $\|\nabla p\|_{L^{\infty,1}}$ is sufficiently small, then $u(x,t)$ is a regular solution on $[0,T]$.

Remark 1.1. For Navier-Stokes equations in a domain $\Omega \subsetneq \mathbb{R}^3$, it is very difficult; cf. [2, 22].

In section 3, we will give a much simpler proof for the following known result. Moreover, our method can treat γ uniformly instead of a different trick for different γ as done in [2, 4].

Theorem 1.2 ([2]). *Under the same assumption as Theorem* 1.1*, if*

$$p \in L^\alpha\left(0,T;L^\gamma(\mathbb{R}^3)\right) \quad \textit{with } 2/\alpha + 3/\gamma \leq 2,\ 1 < \alpha < \infty,\ 3/2 < \gamma < \infty,$$

then $u(x,t)$ is a regular solution on $[0,T]$.

Remark 1.2. The limit cases $p \in L^{1,\infty}$ or $\|p\|_{L^{\infty,3/2}}$ being sufficiently small were treated in [4].

2. Proof of Theorem 1.1

First, we should establish an a priori estimate.

Taking $\nabla\mathrm{div}$ on both sides of (1.1) for smooth (u,p), one can obtain

$$-\Delta(\nabla p)=\sum_{i,j=1}^{3}\partial_i\partial_j\left(\nabla(u_iu_j)\right).$$

Therefore the Calderon-Zygmund inequality

$$\|\nabla p\|_{L^q}\le C_1\,\||u||\nabla u|\|_{L^q} \tag{2.1}$$

holds for any $1<q<\infty$. This relation (2.1) between ∇p and derivatives of the velocity plays a very important role in the following proof. As far as we know, no one has used (2.1) before.

Multiply both sides of equation (1.1) by $4u|u|^2$, and integrate over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{L^4}^4+4\||\nabla u||u|\|_{L^2}^2+2\|\nabla|u|^2\|_{L^2}^2\\
\le\;& 4\int_{\mathbb{R}^3}|\nabla p||u|^3dx\le 4\|\nabla p\|_{L^2}^{1/2}\|\nabla p\|_{L^\gamma}^{1/2}\|u\|_{L^{12\gamma/(3\gamma-2)}}^3\\
\le\;& \epsilon\|\nabla p\|_{L^2}^2+C(\epsilon)\|\nabla p\|_{L^\gamma}^{2/3}\|u\|_{L^{12\gamma/(3\gamma-2)}}^4\\
\le\;& \epsilon C\||\nabla u||u|\|_{L^2}^2+C(\epsilon)\|\nabla p\|_{L^\gamma}^{2/3}\|u\|_{L^4}^{4(1-1/\gamma)}\|u\|_{L^{12}}^{4/\gamma}\\
\le\;& \epsilon C\||\nabla u||u|\|_{L^2}^2+C(\epsilon,\delta)\|\nabla p\|_{L^\gamma}^{2\gamma/3(\gamma-1)}\|u\|_{L^4}^4+\delta\|u\|_{L^{12}}^4,
\end{aligned}\tag{2.2}$$

where we used (2.1) for $q=2$. Since

$$\|u\|_{L^{12}}^4=\left\||u|^2\right\|_{L^6}^2\le C\||\nabla u||u|\|_{L^2}^2,$$

after choosing suitable ϵ and δ, it follows from (2.2) that

$$\frac{d}{dt}\|u\|_{L^4}^4\le C\|\nabla p\|_{L^\gamma}^{2\gamma/3(\gamma-1)}\|u\|_{L^4}^4. \tag{2.3}$$

Then applying Gronwall inequality on (2.3), we have

$$\sup_{0\le t\le T}\|u(.,t)\|_{L^4}^4\le\|u_0\|_{L^4}^4\exp\left\{\int_0^T\|\nabla p(.,\tau)\|_{L^\gamma}^{2\gamma/3(\gamma-1)}d\tau\right\}.$$

If $1<\alpha,\gamma<\infty$, note that $2\gamma/3(\gamma-1)\le\alpha$. Due to the integrability of ∇p, it follows that

$$\sup_{0\le t\le T}\|u(.,t)\|_{L^4}^4\le C(T)\|u_0\|_{L^4}^4. \tag{2.4}$$

For $(\alpha,\gamma)=(2/3,\infty)$, by taking the limit case in (2.2), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{L^4}^4+4\||\nabla u||u|\|_{L^2}^2+2\|\nabla|u|^2\|_{L^2}^2\\
\le\;& \epsilon C\||\nabla u||u|\|_{L^2}^2+C(\epsilon,\delta)\|\nabla p\|_{L^\infty}^{2/3}\|u\|_{L^4}^4+\delta\|u\|_{L^{12}}^4.
\end{aligned}\tag{2.5}$$

Then by Gronwall inequality, (2.4) follows from (3.9).

Similarly, for $(\alpha,\gamma)=(\infty,1)$, we have

$$\begin{aligned}
&\frac{d}{dt}\|u\|_{L^4}^4+4\||\nabla u||u|\|_{L^2}^2+2\|\nabla |u|^2\|_{L^2}^2 \\
&\le \epsilon C\||\nabla u||u|\|_{L^2}^2+C(\epsilon)\|\nabla p\|_{L^1}^{2/3}\||\nabla u||u|\|_{L^2}^2. \qquad (2.6)
\end{aligned}$$

So if $\sup_{0\le t\le T}\|\nabla p(.,t)\|_{L^1}$ is sufficiently small, say $\epsilon C\le 2$ and

$$C(\epsilon)\sup_{0\le t\le T}\|\nabla p(.,t)\|_{L^1}^{2/3}\le 2,$$

then

$$\sup_{0\le t\le T}\|u(.,t)\|_{L^4}^4\le \|u_0\|_{L^4}^4. \qquad (2.7)$$

In order to prove Theorem 1.1 we recall a result of Giga [6] (see also [9]).

Theorem 2.1 ([6])**.** *Suppose $u_0\in L^s(\mathbb{R}^3)$, $s\ge 3$. Then there exists T_0 and a unique classical solution $u\in BC([0,T_0);L^s(\mathbb{R}^3))$. Moreover, let $(0,T_*)$ be the maximal interval such that u solves* (1.1) *in $C((0,T_*);L^s(\mathbb{R}^3))$, $s>3$. Then*

$$\|u(.,\tau)\|_{L^s}\ge \frac{C}{(T_*-\tau)^{(s-3)/2s}} \qquad (2.8)$$

with constant C independent of T_* and s.

Proof of Theorem 1.1. Since $u_0\in L^2(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$ for some $q\ge 4$, then $u_0\in L^4(\mathbb{R}^3)$. Due to Theorem 2.1, there is a maximal interval $[0,T_*)$ such that there exists a unique solution $\tilde u(x,t)\in BC([0,T_*);L^4(\mathbb{R}^3))$. Since u is a Leray-Hopf weak solution which satisfies the energy inequality, we have by the uniqueness criterion of Serrin-Masuda [16], [13]

$$u\equiv\tilde u \quad \text{on} \quad [0,T_*).$$

By the a priori estimate, (2.4) or (2.7), and combined with the standard continuation argument, we can continue our local smooth solution corresponding to $u_0\in L^4(\mathbb{R}^3)$ to obtain $u\in BC([0,T];L^4(\mathbb{R}^3))\cap C^\infty(\mathbb{R}^3\times(0,T])$. This completes the proof of Theorem 1.1. □

Remark 2.1. By the same trick as that used in section 3, one can establish an a priori estimate for $\|\nabla p\|_{L^s}$ with $3\le s<4$.

3. A new proof for Theorem 1.2

The first step is to give an interpolation inequality.

Lemma 3.1. *Suppose a measurable function $f\in L^{\infty,s}\cap L^{s,3s}$ on $(\mathbb{R}^3\times[0,T))$. Then $f\in L^{p,q}$ with $s\le p, s\le q\le 3s$ and $\frac{s}{p}+\frac{3s}{2q}\ge\frac{3}{2}$, and*

$$\|f\|_{L^{p,q}}\le C(p,q,T)\|f\|_{L^{\infty,s}}^{\frac{3s-q}{2q}}\|f\|_{L^{s,3s}}^{(3q-3s)/2q}, \qquad (3.1)$$

where $C(s,p,q,T)$ depends on s,p,q,T, and $C(p,q,T)=1$ if $\frac{s}{p}+\frac{3s}{2q}=\frac{3}{2}$.

Proof.

$$\begin{aligned}
\|f\|_{L^{p,q}} &= \Big(\int_0^T\|f(.,\tau)\|_{L^q}^p d\tau\Big)^{1/p}\le\Big(\int_0^T\|f(.,\tau)\|_{L^s}^{\theta p}\|f(.,\tau)\|_{L^{3s}}^{(1-\theta)p}d\tau\Big)^{1/p}\\
&\le C(s,p,q,T)\|f\|_{L^{\infty,s}}^{\theta}\|f\|_{L^{s,3s}}^{(1-\theta)},
\end{aligned}$$

where we use the interpolation theorem

$$\frac{1}{q} = \frac{\theta}{s} + \frac{1-\theta}{3s}, \quad s \le q \le 3s, \tag{3.2}$$

and Hölder's inequality, provided $(1-\theta)p \le s$.

From (3.2), $1-\theta = \frac{3q-3s}{2q}$, we obtain $\frac{s}{p} + \frac{3s}{2q} \ge \frac{3}{2}$. If $\frac{s}{p} + \frac{3s}{2q} = \frac{3}{2}$, which implies $1-\theta = \frac{s}{p}$, then obviously $C(s,p,q,T) = 1$. □

The idea of the proof of Theorem 1.2 is similar to that of Theorem 1.1. Now the only thing we need is the following a priori estimate.

Theorem 3.2. *Let $s \ge 3, 1 < \alpha < \infty$ and $\frac{3}{2} < \gamma < \infty$ be given. Suppose $u_0 \in L^s(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$. Assume (u,p) is a smooth solution of (1.1) in $\mathbb{R}^3 \times (0,T)$ with $u \in L^{\infty,2}$ and $\nabla u \in L^{2,2}$. If $p \in L^{\alpha,\gamma}$ with $\frac{2}{\alpha} + \frac{3}{\gamma} = 2$, then $u \in L^{\infty,s} \cap L^{s,3s}$*

$$\sup_{0 \le t \le T} \|u(.,t)\|_{L^s}^s \le 2^{[C\|p\|_{L^{\alpha,\gamma}}]+1} \|u_0\|_{L^s}^s, \tag{3.3}$$

where $C = C(s,\alpha,\gamma)$.

Proof. In order to prove (3.3) we multiply both sides of equation (1.1) by $su|u|^{s-2}$, and integrate over $\mathbb{R}^3 \times (0,t)$, $0 < t \le T$. After suitable integration by parts, we obtain

$$\begin{aligned} \|u(.,t)\|_{L^s}^s + s\int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 |u|^{s-2} dx d\tau + \frac{4(s-2)}{s} \|\nabla |u|^{s/2}\|_{L^{2,2}}^2 \\ \le \quad 2(s-2)\int_0^t \int_{\mathbb{R}^3} |p||u|^{s/2-1} |\nabla |u|^{s/2}| dx d\tau + \|u_0\|_{L^s}^s, \end{aligned} \tag{3.4}$$

where we used

$$\begin{aligned} -s\int_0^t \int_{\mathbb{R}^3} \nabla p \cdot u|u|^{s-2} dx d\tau &= s(s-2)\int_0^t \sum_{i,j=1}^3 \int_{\mathbb{R}^3} p\frac{\partial u_j}{\partial x_i} u_i u_j |u|^{s-4} dx d\tau \\ &\le 2(s-2)\int_0^t \int_{\mathbb{R}^3} |p||u|^{s/2-1} |\nabla |u|^{s/2}| dx d\tau. \end{aligned}$$

If we use the fact that

$$|\nabla |u|^{s/2}| \le \frac{s}{2} |u|^{s/2-1} |\nabla u|,$$

then (3.4) will be reduced as follows:

$$\begin{aligned} \|u(.,t)\|_{L^s}^s + 2\|\nabla |u|^{s/2}\|_{L^{2,2}}^2 &\le 2(s-2)\int_0^t \int_{\mathbb{R}^3} |p||u|^{s/2-1} |\nabla |u|^{s/2}| dx d\tau + \|u_0\|_{L^s}^s \\ &\equiv A + \|u_0\|_{L^s}^s. \end{aligned} \tag{3.5}$$

Before going to estimate A, we recall the well-known equality given by

$$-\Delta p = \sum_{i,j=1}^3 \partial_i \partial_j (u_i u_j). \tag{3.6}$$

The Calderon-Zygmund inequality implies

$$\|p\|_{L^\gamma} \le C_1 \|u\|_{L^{2\gamma}}^2, \quad 1 < \gamma < \infty. \tag{3.7}$$

Now,

$$
\begin{aligned}
A &\leq C_2 \int_0^t \|p\|_{L^a} \|u\|_{L^b}^{s/2-1} \|\nabla |u|^{s/2}\|_{L^2} d\tau \\
&\quad \Big(\text{Hölder's inequality} \quad \frac{1}{a} + \frac{s/2-1}{b} = \frac{1}{2}\Big) \\
&\leq \frac{1}{2} C_2 \int_0^t \|p\|_{L^a}^2 \|u\|_{L^b}^{s-2} d\tau + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau \quad \Big(\text{Young's inequality}\Big) \\
&\leq \frac{1}{2} C_2 \int_0^t \|p\|_{L^\gamma}^{2(1-\theta)} \|p\|_{L^{b/2}}^{2\theta} \|u\|_{L^b}^{s-2} d\tau + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau \\
&\quad \Big(\text{interpolation inequality} \quad \frac{1}{a} = \frac{1-\theta}{\gamma} + \frac{\theta}{b/2}\Big) \\
&\leq C_3 \int_0^t \|p\|_{L^\gamma}^{2(1-\theta)} \|u\|_{L^b}^{4\theta+s-2} d\tau + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau \quad \Big(\text{by} \quad (3.7)\Big) \\
&\leq C_3 \|p\|_{L^{\alpha,\gamma}}^{2(1-\theta)} \|u\|_{L^{q,b}}^{4\theta+s-2} + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau \\
&\quad \Big(\text{Hölder's inequality} \quad \frac{2(1-\theta)}{\alpha} + \frac{4\theta+s-2}{q} = 1\Big).
\end{aligned}
$$

We can choose the number $\theta = \frac{1}{2}$; then

$$
a = \frac{2\gamma s}{2\gamma + s - 2}, \quad b = \frac{\gamma s}{\gamma - 1}, \quad q = \frac{\alpha s}{\alpha - 1}. \tag{3.8}
$$

From (3.8), by direct computation, q and b satisfy

$$
\frac{s}{q} + \frac{3s}{2b} = \frac{5}{2}\left(\frac{1}{\alpha} + \frac{3}{2\gamma}\right) \geq \frac{3}{2}, \quad s < q, \quad s < b < 3s,
$$

so we can use inequality (3.1). Therefore

$$
\begin{aligned}
A &\leq C_3 \|p\|_{L^{\alpha,\gamma}} \|u\|_{L^{q,b}}^s + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau \\
&\leq C_4 \|p\|_{L^{\alpha,\gamma}} \|u\|_{L^{\infty,s}}^{\frac{2\gamma-3}{2\gamma}s} \|u\|_{L^{s,3s}}^{\frac{3}{2\gamma}s} + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau \\
&\leq C_5 \|p\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{2\gamma-3}} \|u\|_{L^{\infty,s}}^s + C_6 \|u\|_{L^{s,3s}}^s + \int_0^t \|\nabla |u|^{s/2}\|_{L^2}^2 d\tau,
\end{aligned}
$$

where C_5 is constant depending only on α, γ and s, while C_6 is an absolute constant to be determined later. Substituting the above inequalities into (3.5) and using the Sobolev inequality for suitable C_6,

$$
C_6 \|u\|_{L^{3s}}^s = C_6 \||u|^{s/2}\|_{L^6}^2 \leq \|\nabla |u|^{s/2}\|_{L^2}^2,
$$

one has

$$
\|u(.,t)\|_{L^s}^s \leq C_5 \|p\|_{L^{\alpha,\gamma}}^{\frac{2\gamma}{2\gamma-3}} \|u\|_{L^{\infty,s}}^s + \|u_0\|_{L^s}^s. \tag{3.9}
$$

Theorem 3.1 follows from (3.9) and the integrability of p. □

Remark 3.1. From the proof of Theorem 1.2, it is obvious that Theorem 1.2 holds for arbitrary dimension N, $N \geq 3$.

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