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On regularity estimates for mappings between embedded manifolds

by

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Abstract: We describe a class of metric spaces such that for set-valued mappings into such spaces it is possible to give a precise expression of regularity moduli in terms of slopes of DeGiorgi-Marino-Tosques. We also show that smooth manifolds in Banach spaces endowed with the induced metric belong to this class.

Keywords: metric regularity, slope, locally coherent space.

1. Introduction

The metric theory of metric regularity (that is to say, metric regularity of set-valued mappings between metric spaces) has by now been sufficiently well developed (see Ioffe, 2000, for the latest survey). The only infinitesimal mechanism available in such a general setting is provided by the slopes of DeGiorgi-Marino-Tosques (1980). It allows to get one side estimate for regularity moduli (that is, a lower estimate for the modulus of surjection or, equivalently, an upper estimate for the modulus of metric regularity). Moreover, under a certain assumption on the range space, the estimates give the exact value of the moduli. This assumption was first introduced in Ioffe (2000) in the form of a certain geodesic property, namely that any two points of the space must have an "almost" middle point whose distances to the given points are arbitrarily close. Later, Azé and Corvellec (2004) gave an elegant infinitesimal characterization of the property in terms of slopes of the distance function and called spaces having this property coherent. Every Banach space is coherent. Nonetheless, the coherency assumption seems to be sufficiently restrictive in the general nonlinear context. For instance, as we shall see, a closed subset of a locally uniformly convex Banach space being supplied with the induced metric is a coherent space if and only if it is convex.

The main purpose of this note is to show that "localization" of the coherency property allows to get an exact expression (in terms of slopes of the distance function) for the regularity moduli of set-valued mappings for a substantially broader class of range spaces, which in particular includes all smooth submanifolds of Banach spaces with the induced metric.

2. Coherent and locally coherent spaces

We start with some basic definitions. Let (X, d) be a metric space, and let f be an extended-real-valued function, which is finite at x. The *slope* of f at x is

$$|\nabla|f(x) = \limsup_{\substack{u\to x\\ u\neq x}} \frac{(f(x) - f(u))^+}{d(x,u)\|}.$$

In other words, slope at x is the greatest speed of decrease of the function from x. The slope of a differentiable function on a normed space is the norm of its derivative; the slope of a function satisfying the Lipschitz condition near the point cannot be greater than the Lipschitz constant of the function. X is called a *coherent space* if for any $z \neq x$

$$|\nabla d(\cdot, z)|(x) = 1$$

(slope of $x \to d(x,z)$ at x). It was shown in Azé and Corvellec (2004) that coherent spaces are characterized by the following *geodesic property*: for any two points $x_1, x_2 \in X$ and any $\varepsilon > 0$ there is a z such that $d(z, x_i) \leq (1/2)d(x_1, x_2) + \varepsilon$, $i = 1, 2$.

PROPOSITION 1 *Let X be a locally uniformly convex Banach space. Then a closed subset of X is a coherent space in the induced metric if and only if it is convex.*

Proof. Clearly, a convex set endowed with the induced metric is a coherent space, so we have to check the opposite implication.

Let u and x be a pair of points such that

$$\|u\| \leq \frac{\|x\|}{2} + \varepsilon; \quad \|x - u\| \leq \frac{\|x\|}{2} + \varepsilon. \tag{1}$$

We claim that

$$\|u\| \geq \frac{\|x\|}{2} - \varepsilon, \ \|u - x\| \geq \frac{\|x\|}{2} - \varepsilon, \ \|u + \frac{x}{2}\| \geq \|x\| - \varepsilon. \tag{2}$$

Indeed, the first two inequalities are immediate from (1) as $\|u\| + \|u - x\| \geq \|x\|$. Set now $u_\lambda = (1 - \lambda)u + (\lambda/2)x$. Then, for any $\lambda \in [0, 1]$ we have

$$\|u_\lambda\| \leq (1 - \lambda)\|u\| + \lambda\frac{\|x\|}{2} \leq \frac{\|x\|}{2} + (1 - \lambda)\varepsilon$$

and

$$\|u_\lambda - x\| \leq (1 - \lambda)\|u - x\| + \lambda\frac{\|x\|}{2} \leq \frac{\|x\|}{2} + (1 - \lambda)\varepsilon.$$

Thus, any u_λ with $\lambda \in [0, 1]$ also satisfies (1) and therefore the first two inequalities in (2) (with ε replaced by $(1 - \lambda)\varepsilon$). By replacing u by $u_{1/2}$ in the first inequality of (2), we get the last inequality of (2).

Returning to the proof of the proposition, let $Q \subset X$ be closed and a coherent space in the induced metric. Take $x_1,\ x_2 \in Q$ and let $z_k \in Q,\ k = 1, 2, \dots$ be such that $\|x_i - z_k\| \le (1/2)\|x_1 - x_2\| + 1/k$. Setting $x = x_2 - x_1$ and $u_k = z_k - x_1$, we see that x and u satisfy (1) and hence (2). In other words

$$\left| \|u_k\| - \frac{\|x\|}{2} \right| \le \frac{1}{k}; \quad \|u_k + \frac{x}{2}\| \ge \|x\| + \frac{1}{k}.$$

As X is locally uniformly convex, it follows that $u_k \to (1/2)x$ and therefore $z_k \to (1/2)(x_1 + x_2)$. As Q is a closed set, it follows that $(1/2)(x_1 + x_2) \in Q$. This means that Q contains the middle point of the segment joining any two points of Q. Using again the fact that Q is a closed set, we conclude that Q contains the entire segment joining any two its points, as claimed. ■

We shall call a metric space X *locally coherent* if for any $z \in X$

$$\lim_{\substack{x,v \to z \\ x \ne v}} |\nabla d(v, \cdot)|(x) = 1.$$

Our purpose is to show that the collection of subsets of a Banach space which, being supplied with the induced metric, become locally coherent metric spaces, is much richer than the collection of sets which are just coherent spaces. Specifically, we shall show that any smooth manifold belongs to this class.

Recall (see, e.g., Aubin and Ekeland, 1984) that a subset M of a Banach space X is a *C^k-submanifold* if for any x there are a closed subspace L_x of X, open neighborhoods U_x of x and V_x of zero and a C^k-diffeomorphism φ_x of V_x onto U_x such that

$$M \bigcap U_x = \varphi_x(L_x \bigcap V_x).$$

This definition is somewhat different from the standard definition involving local charts. It has been chosen to facilitate using the induced metric structures on submanifolds along with differential structures. To emphasize the difference we shall call the triple (L_x, V_x, φ_x) a *local parameterization* of M at x. (It can be shown, however, that this definition implies the existence of local charts if all spaces L_x split X. In the context of variational analysis the latter property may not always be natural.) We shall use some very special parameterization to prove the result we need.

PROPOSITION 2 *Let $M \subset X$ be a C^k-submanifold. Then for any $x \in M$ there is a local parameterization of M at x with a C^k-diffeomorphism ψ such that $\psi'(0) = I$, the identity map.*

Proof. Let (L, V, φ) be some parameterization of M at x, that is $M \bigcap U = \varphi(L \bigcap V)$, where $U = \varphi(V)$. Set $\psi = \varphi \circ [\varphi'(0)]^{-1}$, then $(T_x M, U, \psi)$ is a desired representation. (Here $T_x M$ is the tangent space to M at x.) ■

Proposition 3 *A smooth manifold in a Banach space is locally coherent with respect to the induced metric.*

Proof. The inequality $|\nabla d(z,\cdot)|(y) \le 1$ holds unconditionally. We therefore have to establish the limit version of the opposite inequality under the assumptions.

So, let X be a Banach space, let $M \subset X$ be a C^1-submanifold, let $\overline{x} \in M$, and let (U, L, φ) be a local parameterization of M at $\overline{x}$, with L being a closed subspace of X and $\varphi'(0) = I$. As φ is continuously (hence strictly) differentiable at zero, for any $\delta > 0$ there is an $\varepsilon > 0$ such that

$$B(\overline{x},\varepsilon) \subset U \quad \& \quad \|\varphi(x) - \varphi(u) - (x-u)\| \le \delta\|x-u\|, \ \forall\, x,u \in B(\bar{x},\varepsilon). \tag{3}$$

Take a $y \in M$ with $\|y - \overline{x}\| < \varepsilon$, and let $h \in L \cap U$ be such that $\varphi(h) = y$. Then $(1-t)h \in B(\bar{x},\varepsilon)$ for small positive t, so we have

$$\|\varphi(h) - \varphi((1-t)h) - th\| \le \delta t\|h\| \tag{4}$$

and

$$\|y - \overline{x} - h\| = \|\varphi(h) - \varphi(0) - h\| \le \delta\|h\|. \tag{5}$$

The latter implies, in particular, that

$$\Big|\frac{\|y-\overline{x}\|}{\|h\|} - 1\Big| \le \delta. \tag{6}$$

It easily follows from (4) and (5) that

$$\|\varphi((1-t)h) - \overline{x} - (1-t)(y - \overline{x})\| \le 2t\delta\|h\|.$$

Let d stand for the induced metric in M. We therefore have

$$\begin{aligned}
|\nabla d(\overline{x},\cdot)|(y) &\ge \lim_{t\to 0} \frac{d(\overline{x},y) - d(\overline{x},\varphi((1-t)h)}{d(y,\varphi((1-t)h)} \\
&= \lim_{t\to 0} \frac{\|y-\overline{x}\| - \|\varphi((1-t)h - \overline{x}\|}{\|\varphi(h) - \varphi((1-t)h)\|} \\
&\ge \frac{t\|y-\overline{x}\| - 2t\delta\|h\|}{t\|y-\overline{x}\| + \delta t\|h\|} \\
&= \frac{1 - 2\delta\dfrac{\|h\|}{\|y-\overline{x}\|}}{1 + \dfrac{\delta\|h\|}{\|y-\overline{x}\|}} \ge 1 - 3\delta
\end{aligned}$$

(the last inequality following from (6)). ■

3. Regularity

Recall the definitions of the two main regularity properties (we refer to Ioffe, 2000, for details). The notation $F : X \rightrightarrows Y$ is, as usual, used to denote a set-valued mapping from X into Y. So, let an F be given, let $V \subset X \times Y$, and let $(\overline{x}, \overline{y}) \in \text{Graph } F \bigcap V$. F is said to be

(a) *open at a linear rate near* $(\overline{x}, \overline{y})$ if there are $\gamma > 0$, $\varepsilon > 0$ such that

$$B(v, \gamma t) \subset F(B(x, t))$$

($B(x, r)$ being the closed ball of radius r around x) whenever $v \in F(x)$, $d(x, \overline{x}) < \varepsilon$, $d(v, \overline{y}) < \varepsilon$, $0 \le t < \varepsilon$. The upper bound of all such γ is called the *modulus of surjection of* F *near* $(\overline{x}, \overline{y})$. We shall denote it by $\text{sur}F(\overline{x}|\overline{y})$;

(b) *metrically regular near* $(\overline{x}, \overline{y})$ if there are $K > 0$, $\varepsilon > 0$ such that

$$d(x, F^{-1}(y)) \le K d(y, F(x))$$

whenever $d(x, \overline{x}) < \varepsilon$, $d(y, \overline{y}) < \varepsilon$. The lower bound of all such K is called the *modulus of metric regularity of* F *near* $(\overline{x}, \overline{y})$. We shall denote it by $\text{reg}F(\overline{x}|\overline{y})$.

For single-valued F we shall use the notation $\text{sur}F(\overline{x})$ and $\text{reg}F(\overline{x})$. If F is not open at a linear rate near $(\overline{x}, \overline{y})$, we set $\text{sur}F(\overline{x}|\overline{y}) = 0$; if F is not metrically regular near $(\overline{x}, \overline{y})$, we set $\text{reg}F(\overline{x}|\overline{y}) = \infty$. A well known fact is that the two properties are equivalent and, moreover, we always have

$$\text{sur}F(\overline{x}|\overline{y}) \cdot \text{reg}F(\overline{x}|\overline{y}) = 1$$

(if we agree that $0 \times \infty = 1$). This allows us to simply say that the mapping is *regular near* $(\overline{x}, \overline{y})$ if the two properties are satisfied. There is a third property equivalent to these two (psedo-Lipschitz or Aubin property) but we shall not need it in this note.

What we do need is that regularity of any set-valued mapping F can be equivalently characterized in terms of regularity of a certain single-valued mapping canonically associated with F. This mapping is the restriction to Graph F of the projection $(x, y) \to y$; call it $\mathcal{P}_F$. Namely, the following is true:

PROPOSITION 4 *Let X and Y be complete metric spaces. Endow $X \times Y$ with the α-metric ($\alpha > 0$):*

$$d_\alpha((x, y), (u, v)) = \max\{d(x, u), \alpha d(y, v))\}.$$

Let $(\overline{x}, \overline{y}) \in \text{Graph } F$. *Then*

$$\text{sur}\mathcal{P}_F((\overline{x}, \overline{y})) = \min\{\text{sur}F(\overline{x}|\overline{y}), \frac{1}{\alpha}\}.$$

In particular, if F is regular near $(\overline{x}, \overline{y})$, then $\mathcal{P}_F$ is also regular near $(\overline{x}, \overline{y})$ and $\text{sur}\mathcal{P}_F(\overline{x}, \overline{y}) = \text{sur}F(\overline{x}, \overline{y})$ *if* $\text{sur}F(\overline{x}, \overline{y}) < \infty$ *and α is sufficiently small.*

The following fact serves as a basis for obtaining various regularity criteria. We supply it with a proof as it is stated here in a slightly different form (compare to Ioffe, 2000):

PROPOSITION 5 (Basic lemma) *Let f be an l.s.c. extended-real valued function on X, and let $U \subset X$ be an open set. Assume that there is a positive γ such that for any $x \in U$ with $f(x) > 0$ there is a $u \in X$, $u \neq x$ such that*

$$f(u) \leq f(x) - \gamma d(x, u). \tag{7}$$

If there is an x with $f(x) < d(x, X\backslash U)$, then $[f \leq 0] \neq \emptyset$ and

$$d(x, [f \leq 0]) \leq \gamma^{-1} f^{+}(x).$$

REMARK. If $U = X$, then the complement of U is empty and according to our general convention, the distance to the empty set is ∞. So, in this case the conclusion of the theorem is valid for all x.

Proof If $f(x) \leq 0$, we have nothing to prove. So, assume that $f(x) > 0$. Set $\varepsilon = f(x)$ and apply Ekeland's variational principle with $\lambda = \varepsilon/\gamma$. It follows that there is a w such that

$$d(w, x) \leq \gamma^{-1}\varepsilon; \quad f(u)^{+} + \gamma d(u, w) > f(w), \quad \forall\, u \neq w. \tag{8}$$

We claim that $f(w) \leq 0$. Indeed $d(w, x) \leq \gamma^{-1} f(x) \leq d(x, X\backslash U)$ which implies that $w \in U$. So, if $f(w)$ were positive then there would be a $u \neq w$ such that $f(u) \leq f(w) - \gamma d(u, w)$, in contradiction with (8). Thus, $w \in [f \leq 0] \neq \emptyset$ and

$$d(x, [f \leq 0]) \leq d(x, w) \leq \gamma^{-1}\varepsilon = \gamma^{-1} f(x). \qquad \blacksquare$$

This lemma is essentially an existence theorem, but if we already have an $\overline{x}$ with $f(\overline{x}) = 0$, and U is a neighborhood of $\overline{x}$, then it gives a sufficient condition for metric regularity of the epigraphic mapping $x \to \text{epi}\, f(x)$ near $(\overline{x}, f(\overline{x}))$. An infinitesimal version of the Basic Lemma is given below.

PROPOSITION 6 *Let f be a lower semicontinuous function on X, and let U be an open subset of X. Suppose that there is a positive γ such that $|\nabla f|(x) \geq \gamma$ whenever $f(x) > 0$, $x \in U \cap \text{dom}\, f$. If there is an $x \in U$ satisfying*

$$f(x) < \gamma d(x, X\backslash U) \tag{9}$$

then $[f \leq 0] \neq \emptyset$ and for any x satisfying (9)

$$d(x, [f \leq 0]) \leq \gamma^{-1}(f(x))^{+}.$$

Proof. If $|\nabla f|(x) > \gamma$, then there is a $u \neq x$ arbitrarily close to x such that $f(x) - f(u) > \gamma d(x, u)$. Apply Basic Lemma. $\blacksquare$

4. Main theorems

We start with a special case (still sufficiently general, as we shall see).

THEOREM 1 *[special case]. Assume that X and Y are complete metric spaces. Set $\varphi_y(x) = d(y, F(x))$ and assume that this function is lower semicontinuous for each y. Suppose further that there are $\varepsilon > 0$, $\gamma > 0$ and a neighborhood V of $(\overline{x}, \overline{y})$ such that*

$$|\nabla \varphi_y|(x) > \gamma > 0 \tag{10}$$

for any $(x, y) \in \text{Graph } F$ of the neighborhood satisfying $0 < d(y, F(x)) < \varepsilon$. Then

$$\text{sur } F(\overline{x}|\overline{y}) \geq \gamma.$$

Conversely, assume that

(i) *Y is locally coherent space;*
(ii) *for any (x, y) of a neighborhood of $(\overline{x}, \overline{y})$ the distance d(y,F(x)) is attained (that is, there is a $v \in F(x)$ with $d(y, v) = d(y, F(x))$);*
(iii) $\text{sur } F(\overline{x}) > \gamma$.

Then, for any $\delta > 0$ there is a neighborhood of $(\overline{x}, \overline{y})$ in $X \times Y$ and an $r > 0$ such that $|\nabla \varphi_y|(x) \geq (1 - \delta)\gamma$ for all (x, y) of the neighborhood satisfying $0 < d(y, F(x)) < r$. Thus, in this case

$$\text{sur } F(\overline{x}) = \liminf_{\substack{x \underset{\text{dom} F}{\to} \overline{x} \\ y \to \overline{y},\ y \neq F(x)}} |\nabla \varphi_y|(x). \tag{11}$$

REMARK. The strict inequality sign before γ in both parts of the theorem is solely due to the possibility that the quantity to the left of the sign may be ∞. If it is finite, the proof goes through if it is replaced by a non-strict inequality or even by an equality.

Proof. We shall show that F is metrically regular near $(\overline{x}, \overline{y})$ to prove the first statement, while in the proof of the second statement we shall start with openness at a linear rate.

We first observe that

$$[\varphi_y \leq 0] = F^{-1}(y).$$

Choose $r > 0$ such that (10) holds if $d(x, \overline{x}) < 2r$, $d(y, \overline{y}) \leq \gamma r/2$ and $y \notin F(x)$. Set $U = \overset{\circ}{B}(\overline{x}, 2r)$ (the open ball of radius $2r$ around $\overline{x}$). Then, for any $y \notin F(\overline{x})$ and satisfying $d(y, \overline{y}) \leq \gamma r/2$, we have $\varphi_y(\overline{x}) < \gamma d(\overline{x}, X \backslash U)$, so by Proposition 6 the set $F^{-1}(y)$ is nonempty and $d(\overline{x}, F^{-1}(y)) \leq \gamma^{-1} d(y, \overline{y}) < r/2$. For the same reason the inequality

$$d(x, F^{-1}(y)) = d(x, [\varphi_y \leq 0]) \leq \gamma^{-1} \varphi_y(x) = \gamma^{-1} d(y, F(x))$$

also holds for any x such that $d(x,\overline{x}) < r/2$ and $\varphi_y(x) < r$.

Now, take arbitrary x such that $d(x,\overline{x}) < r/2$ and $y \notin F(x)$. If $\varphi_y(x) > \gamma d(x, X\backslash U) > \gamma r$, then

$$\begin{aligned} d(x, F^{-1}(y)) &\le d(x,\overline{x}) + d(\overline{x}, F^{-1}(y)) \\ &\le d(x,\overline{x}) + \gamma^{-1} d(y, F(\overline{x})) < r \le \gamma^{-1} d(y, F(x)), \end{aligned}$$

and we again get the desired inequality. This completes the proof of the first part of the theorem.

Let us prove the second. Since Y is locally coherent, for any $\delta > 0$ there is an $\varepsilon > 0$ such that $|\nabla d(y,\cdot)|(v) \ge 1-\delta$ whenever $y \ne v$ and both y and v belong to the ε-ball around $\overline{y}$. Let us fix such an ε for every $\delta > 0$.

Since $\operatorname{sur} F(\overline{x}) > \gamma$, the inclusion

$$B(v, \gamma t) \subset F(B(x,t)) \tag{12}$$

holds for $(x,v) \in \operatorname{Graph} F$ sufficiently close to $(\overline{x},\overline{y})$ and sufficiently small t. We may assume that ε is so small that the inclusion holds, in particular, for $x,\ v,\ t$ satisfying

$$v \in F(x),\ d(x,\overline{x}) < \varepsilon,\ d(v,\overline{y}) < \varepsilon,\ 0 \le t < \varepsilon.$$

Now fix a y with $d(y,\overline{y}) < \varepsilon/2$, and let x be such that $d(x,\overline{x}) < \varepsilon$, $0 < d(y,F(x)) < \varepsilon/2$. By (ii) there is a $v \in F(x)$ such that $d(y,v) = d(y,F(x)) < \varepsilon/2$. It follows that $d(v,\overline{y}) < \varepsilon/2$, so (12) holds for the given X and v.

As both y and v belong to the open ε-ball around $\overline{y}$, there is a sequence (v_n) converging to v and such that

$$\frac{d(y,v) - d(y,v_n)}{d(v_n,v)} \to |\nabla d(y,\cdot)|(v) \ge 1-\delta. \tag{13}$$

By (12) for sufficiently large n there are x_n such that $v_n \in F(x_n)$ and $d(x_n,x) \le \gamma^{-1} d(v_n,v)$ which implies together with (4) that

$$\begin{aligned} \limsup_{n\to\infty} \frac{d(y,F(x)) - d(y,F(x_n))}{d(x,x_n)} &\ge \frac{d(y,v) - d(y,v_n)}{d(x,x_n)} \\ &\ge \gamma \frac{d(y,v) - d(y,v_n)}{d(v,v_n)} \ge \gamma(1-\delta), \end{aligned}$$

which means that $|\nabla \varphi_y|(x) \ge (1-\delta)\gamma$. ■

The theorem applies, in particular, to continuous single-valued mappings and upper-semicontinuous compact-valued mappings. In the first case the requirement that $d(y,F(x))$ be bounded from above can be dropped, since it is automatically satisfied due to continuity.

The general case of a set-valued mapping $F:\ X \rightrightarrows Y$ with closed graph is reduced, through Proposition 4, to the study of regularity of the standard projection $\mathcal{P}_F:\ (x,y)\to y$ from Graph F into Y. Being closed, Graph F is itself a complete metric space and we can apply Theorem 1 to this mapping.

Let $F:\ X\rightrightarrows Y$. Consider the family of functions

$$f_y(x,v) = d(y,v) + i_{\text{Graph } F}(x,v),$$

with y being the parameter of the family and (x,v) being the arguments. Clearly, dom f_y = Graph F for every y. For any $\alpha>0$ and any f on $X\times Y$, we denote by $|\nabla_\alpha f|$ the slope of f with respect to the α-metric d_α.

Theorem 2 (regularity criterion – general case) *Let X and Y be complete metric spaces, let the graph of $F:\ X\rightrightarrows Y$ be closed, and let $(\overline{x},\overline{y})\in$ Graph F. Suppose there are $\varepsilon>0$, and $\gamma>0$ such that for some $\alpha>0$*

$$|\nabla_\alpha f_y|(x,v) > \gamma \tag{14}$$

if

$$v\in F(x),\quad d(x,\overline{x})<\varepsilon,\quad d(y,\overline{y})<\varepsilon,\quad d(v,\overline{y})<\varepsilon,\quad v\neq y. \tag{15}$$

Then

$$\operatorname{sur} F(\overline{x},\overline{y}) \geq \gamma.$$

Conversely, let Y be a locally coherent space. Assume that $\operatorname{sur} F(\overline{x}|\overline{y})>\gamma>0$. *Take an $\alpha<\gamma^{-1}$. Then for any $\delta>0$ there is an $\varepsilon>0$ such that $|\nabla_\alpha f_y|(x,v)\geq(1-\delta)\gamma$ whenever (x,y,v) satisfy (14). Thus, in this case*

$$\operatorname{sur} F(\overline{x},\overline{y}) = \liminf_{\substack{(x,v)\underset{\text{Graph} F}{\to}(\overline{x},\overline{y})\\ y\to\overline{y},\ y\neq v}} |\nabla_\alpha f_y|(x,v). \tag{16}$$

Proof. Again we consider Graph F with the α-metric. Applying Theorem 1 to $\mathcal{P}_F$ (and keeping in mind that $\mathcal{P}_F$ is single-valued continuous), we conclude that $\operatorname{sur}\mathcal{P}_F(\overline{x},\overline{y})\geq\gamma$. On the other hand, as follows from Proposition 4, $\operatorname{sur} F(\overline{x}|\overline{y})\geq \operatorname{sur}\mathcal{P}_F((\overline{x},\overline{y}))$. This completes the proof of the first statement.

To prove the second part, we first note that $\operatorname{sur}_\alpha\mathcal{P}_F(\overline{x},\overline{y})>\gamma$ as $\alpha^{-1}>\gamma$. Therefore, by Theorem 1, $|\nabla_\alpha f_y|(x,v)\geq(1-\delta)\gamma$ for all $(x,v)\in$ Graph F sufficiently close to $(\overline{x},\overline{y})$ as claimed. ∎

Remark The relation $|\nabla_\alpha f_y|(x,v)>\gamma$ automatically implies that $1>\alpha\gamma$ as follows from the simple inequality

$$\frac{d(y,v)-d(y,w)}{d_\alpha((x,v),(u,w))} \leq \frac{d(y,v)-d(y,w)}{\alpha d(v,w)} \leq \frac{1}{\alpha}.$$

The proof of the second part of Theorem 1 suggests that for a (globally) coherent space we can probably replace (11) and (16) by the nicer expressions:

$$\operatorname{sur} F(\overline{x}) = \liminf_{x \underset{\operatorname{dom} F}{\to} \overline{x}} \inf_{y \neq F(x)} |\nabla \varphi_y|(x) \tag{17}$$

for a single-valued mapping and

$$\operatorname{sur} F(\overline{x}, \overline{y}) = \liminf_{(x,v) \underset{\operatorname{Graph F}}{\to} (\overline{x}, \overline{y})} \inf_{y \neq v} |\nabla_\alpha f_y|(x, v) \tag{18}$$

if F is set-valued.

This is indeed so, but under a slightly stronger assumption (which is still sufficiently general as it is satisfied in Banach spaces and connected locally compact Riemannian manifolds).

THEOREM 3 *Suppose that Y have the following geodesic*

$$\text{(GP)} \qquad \forall\, y_1,\ \forall\, y_2 \in Y\ \exists y:\ [d(y_i, y) = (1/2)d(y_1, y_2)].$$

Then (17) and (18) hold true (under the assumptions of Theorems 1 and 2, respectively), whenever $\alpha\gamma \leq 1$.

Proof. It is clear that (GP) implies that the space is coherent. Hence, in view of Theorem 1 we only need to show that for any z, any v and any w there is a $y \neq v$ arbitrarily close to v and such that

$$d(z, v) - d(z, w) \geq d(y, v) - d(y, w).$$

But as follows from (GP), for any natural k, we may find a y such that $d(y, v) = 2^{-k} d(z, v)$, $d(z, y) = (1 - 2^{-k}) d(z, v)$ for which the inequality is trivially valid. ■

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