On Regulation Under Sampling

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***Abstract*—The paper deals with linear and nonlinear regulation under sampling. It is shown that digital solutions exist under assumptions which are closely related to the existence of robust solutions to the continuous problem. Approximated solutions are computed starting from the continuous ones.**

***Index Terms*—Nonlinear systems, regulation problem, sampled systems.**

I. Introduction

Regulation provides an elegant framework for setting asymptotic disturbance compensation and tracking. Starting from the fundamental linear results in [6], the nonlinear problem was first studied in [9], while the discrete-time problem was recently addressed in [2].

On the basis of the results there stated, hereinafter the authors study the regulation problem for a discrete-time system resulting from the sampling of a continuous-time one. Assuming the solvability of the continuous-time regulation problem, under which extra conditions does a solution exist to the sampled problem? It is shown that the existence of a continuous robust solution suffices to solve the problem; robustness is preserved in the linear context, while the solution of the nonlinear problem satisfies a property which is conjectured to be necessary for robustness. Moreover, an approximated solution at any prefixed order can be computed starting from the continuous solution.

The result obtained is quite intuitive and suggests that one think of the sampled problem as an "approximation" of the given continuous problem, so requiring robustness. As a matter of fact, the sampled system can be considered perturbed with respect to the continuous dynamics, since references and perturbations are approximated by piecewise constant signals. It must be pointed out that the statement of the problem in a digital context, where references and perturbations are assumed piecewise constant, appears to be coherent with respect to references which are usually generated by digital devices but may be not satisfactory with respect to perturbations.

Some basic results on regulation and the problem statement are the subjects of the next section. In Section III the linear sampled regulation problem is studied. The result stated provides an elementary introduction to the nonlinear problem which is developed, following the same lines, in Section IV.

II. Some Basic Facts and Problem Formulation

The nonlinear system usually considered for studying the regulation problem is the following:

$$\begin{aligned}\dot{x} &= f(x,u,w)\\ \dot{w} &= s(w)\\ e &= h(x,w)\end{aligned} \tag{1}$$

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$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and input variables respectively, f, s, and h are analytic functions of their arguments. The signal w takes into account the external disturbances and references modeled by an *exosystem* defined on $\mathbb{R}^s$, while e denotes the tracking error and the effect of external disturbances. It is assumed that $(x,w) = (0,0)$ is an equilibrium.

For this system, the state feedback regulator problem (SFRP) consists of finding, if possible, a controller $u = \alpha(x,w)$ such that:

- (S_C) $x = 0$ is asymptotically stable for the closed-loop unperturbed dynamics $\dot{x} = f(x, \alpha(x,0), 0)$;
- (R_C) for each initial condition $(x(0), w(0))$ in a neighborhood U of the origin, $\lim_{t\to\infty} e(t) = 0$.

The linear approximation of (1) is the system

$$\begin{aligned}\dot{x} &= Ax + Bu + Pw\\ \dot{w} &= Sw\\ e &= Cx + Qw\end{aligned} \tag{2}$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=0\\u=0\\w=0}}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=0\\u=0\\w=0}}, \qquad P = \left.\frac{\partial f}{\partial w}\right|_{\substack{x=0\\u=0\\w=0}}$$
$$S = \left.\frac{\partial s}{\partial w}\right|_{w=0}, \qquad C = \left.\frac{\partial h}{\partial x}\right|_{\substack{x=0\\w=0}}, \qquad Q = \left.\frac{\partial h}{\partial w}\right|_{\substack{x=0\\w=0}}.$$

A solution to the nonlinear continuous-time SFRP, as proposed in [9], is based on the following assumptions.

- (H_1) The pair (A,B) is stabilizable.
- (H_2) The equilibrium $w = 0$ of the exosystem is stable in the sense of Lyapunov, and S has all the eigenvalues on the imaginary axis.

Theorem 1 [9]: Under (H_1) and (H_2), the nonlinear SFRP is locally solvable if and only if there exist two maps $\pi(w)$ and $\gamma(w)$ at least C^2, defined in a neighborhood of $w = 0$ with $\pi(0) = 0$ and $\gamma(0) = 0$ which solve the regulator equations

$$\begin{aligned}\frac{\partial \pi}{\partial w} s(w) &= f(\pi(w), \gamma(w), w)\\ 0 &= h(\pi(w), w).\end{aligned} \tag{3}$$

The control law takes the form $u = \alpha(x,w) = \gamma(w) + K(x - \pi(w))$, with K any matrix such that $\sigma(A+BK) \in \mathbb{C}^-$.

In the linear context, where the controller takes the form $u = K_1 x + K_2 w$, the existence of a solution is set in the more general framework of antistable signals w. More precisely, (H_2) is replaced by the following.

(H_2^L)*:* The eigenvalues of S are in the closed right-side of the complex plane.

Theorem 2 [6]: Under (H_1), (H_2^L), the linear SFRP is solvable if and only if there exist two matrices (Π, Γ) solutions of

$$\begin{aligned}\Pi S &= A\Pi + B\Gamma + P\\ 0 &= C\Pi + Q.\end{aligned} \tag{4}$$

The control law takes the form $u = \Gamma w + K(x - \Pi w)$, with K any matrix such that $\sigma(A+BK) \in \mathbb{C}^-$.

As far as discrete-time control systems are concerned, a nonlinear result was stated in [2]. Given

$$\begin{aligned}x_D(k+1) &= f_D(x_D(k), u_D(k), w_D(k))\\ w_D(k+1) &= s_D(w_D(k))\\ e_D(k) &= h(x_D(k), w_D(k))\end{aligned} \tag{5}$$

the nonlinear discrete-time SFRP consists of finding a controller of the form $u_D(k) = \alpha_D(x_D, w_D)$, $\alpha_D(\cdot,\cdot)$ a smooth map with $\alpha_D(0,0) = 0$, such that:

(S_D) $x = 0$ is locally exponentially stable for the closed-loop unperturbed dynamics

$$x(k+1) = f_D(x_D, \alpha_D(x_D, 0), 0);$$

(R_D) for each initial condition $(x_D(0), w_D(0))$ in a neighborhood U of the origin, $\lim_{k\to\infty} e_D(k) = 0$.

The linear approximation of (5) takes the form

$$\begin{aligned} x_D(k+1) &= A_D x_D(k) + B_D u_D(k) + P_D w_D(k) \\ w_D(k+1) &= S_D w_D(k) \\ e_D(k) &= C_D x_D(k) + Q_D w_D(k) \end{aligned} \tag{6}$$

where

$$A_D = \left.\frac{\partial f_D}{\partial x_D}\right|_{\substack{x_D=0\\u_D=0\\w_D=0}}, \qquad B_D = \left.\frac{\partial f_D}{\partial u_D}\right|_{\substack{x_D=0\\u_D=0\\w_D=0}}$$

$$P_D = \left.\frac{\partial f_D}{\partial w_D}\right|_{\substack{x_D=0\\u_D=0\\w_D=0}}, \qquad S_D = \left.\frac{\partial s_D}{\partial w_D}\right|_{w_D=0}$$

$$C_D = \left.\frac{\partial h_D}{\partial x_D}\right|_{\substack{x_D=0\\w_D=0}}, \qquad Q_D = \left.\frac{\partial h_D}{\partial w_D}\right|_{\substack{x_D=0\\w_D=0}}.$$

Under assumptions which are the discrete-time versions of (H_1) and (H_2), denoted by (H_1^D) and (H_2^D), the following result holds [2].

Theorem 3 [2]: Under (H_1^D) and (H_2^D), the nonlinear discrete-time SFRP is locally solvable if there exist two maps $\pi_D(w_D)$ and $\gamma_D(w_D)$, at least C^2, defined in a neighborhood of $w_D = 0$ with $\pi_D(0) = 0$ and $\gamma_D(0) = 0$, which solve the regulator equations

$$\pi_D(s_D(w_D)) = f_D(\pi_D(w_D), \alpha_D(\pi_D(w_D), w_D), w_D) \tag{7a}$$

$$0 = h(\pi_D(w_D), w_D). \tag{7b}$$

The control law takes the form $u_D(k) = \alpha_D(x_D, w_D) = \gamma_D(w_D) + K_D(x_D - \pi_D(w_D))$, with K_D any matrix such that $A_D + B_D K_D$ has eigenvalues inside the unitary circle.

With reference to the class of antistable references w_D, i.e., the discrete-time version of (H_2^L), say (H_2^{DL}), the following well-known linear result is obtained.

Theorem 4: Under $(H_1^D), (H_2^{DL})$ the linear discrete-time SFRP is solvable if and only if there exist two matrices (Π_D, Γ_D) solutions of

$$\begin{aligned} \Pi_D S_D &= A_D \Pi_D + B_D \Gamma_D + P_D \\ 0 &= C_D \Pi_D + Q_D. \end{aligned} \tag{8}$$

The control law takes the form $u_D(k) = \Gamma_D w_D + K_D(x_D - \Pi_D w_D)$, where K_D is any matrix such that $A_D + B_D K_D$ has eigenvalues inside the unitary circle.

The present paper is devoted to the study of the regulator problem for sampled systems. More precisely, on the basis of the recalled results, it will be shown that under some additional conditions, Theorems 3 or 4 apply to the discrete-time systems resulting from the sampling of continuous-time ones satisfying Theorems 1 or 2. The underlying hypothesis under which we study the existence of digital solutions is that w may be assumed constant on the sampling time intervals.

With this in mind, the problem statement is the following: *the (robust) sampled SFRP is solvable if there exists a positive number* δ_0 *such that for almost all sampling intervals* $\delta \in (0, \delta_0]$, *the (robust) discrete-time SFRP is solvable for the discrete-time sampled system.*

The robustness of the solution considered here reflects the effectiveness of the controller with respect to mismatches of the parameters of the given plant as in [6] and [3] in linear and nonlinear context, respectively.

III. The Regulation Problem for Sampled Linear Systems

Consider the sampled system associated to (2), forced by piecewise constants u and w on time intervals of amplitude δ; it takes the form (6) with

$$A_D = e^{\delta A}, \qquad B_D = \int_0^\delta e^{sA} B ds = \sum_{i=1}^\infty \frac{\delta^i}{i!} A^{i-1} B, \qquad C_D = C$$

$$S_D = e^{\delta S}, \qquad Q_D = Q, \qquad P_D = \sum_{i=0}^\infty \frac{\delta^{i+1}}{(i+1)!} P_i \tag{9}$$

where P_i can be iteratively computed according to the relationships

$$P_0 = P, \qquad P_i = AP_{i-1} + PS^i, \qquad i = 1, 2, \cdots. \tag{10}$$

Starting from a linear continuous-time solution, we discuss the existence of a solution to the associated sampled linear problem. To this aim, we need the following assumptions which, as well known, ensure the existence of a robust solution to the linear problem in the sense of Francis [6].

(H_3^L) For all $\lambda \in \sigma(S)$,

$$\operatorname{rank}\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} = n + p.$$

(H_3^{LD}) For all $\lambda \in \sigma(S_D)$,

$$\operatorname{rank}\begin{pmatrix} A_D - \lambda I & B_D \\ C_D & 0 \end{pmatrix} = n + p.$$

Next theorem states that the existence of a robust solution implies the solvability of the robust sampled linear problem.

Theorem 5: Under $(H_1), (H_2^L)$, and (H_3^L) the robust sampled linear SFRP is solvable.

Proof: First of all we note that (H_1) implies (H_1^D) for almost all δ [12]; moreover, (H_2^L) implies (H_2^{LD}). The proof will be achieved by proving that under (H_3^L) there exists a solution to (8) which admits an expansion in powers of δ around the continuous-time solution (Π_0, Γ_0). With this in mind, let us consider the following equations of the form (4):

$$\begin{aligned} \Pi_i S &= A\Pi_i + B\Gamma_i + \tilde{P}_i \qquad i = 0, 1, 2, \cdots \\ 0 &= C\Pi_i + \tilde{Q}_i \end{aligned} \tag{11}$$

with $\tilde{P}_i$ and $\tilde{Q}_i$ given by

$$\begin{aligned} &\tilde{P}_0 = P_0 = P, \qquad \tilde{Q}_0 = Q, \qquad \tilde{Q}_i = 0 \\ &\tilde{P}_i = \frac{1}{i+1}\Bigg[P_i + \sum_{h=2}^{i+1}\binom{i+1}{h} \\ &\qquad \times \left(A^h \Pi_{i+1-h} + A^{h-1} B \Gamma_{i+1-h} - \Pi_{i+1-h} S^h\right)\Bigg] \\ &i \geq 1. \end{aligned} \tag{12}$$

Because of (H_3^L), these equations can be solved in the unknowns Π_i's and Γ_i's. Let us now consider the following power expansions around (Π_0, Γ_0):

$$\Pi_D = \sum_{i=0}^\infty \frac{\delta^i}{i!} \Pi_i, \qquad \Gamma_D = \sum_{i=0}^\infty \frac{\delta^i}{i!} \Gamma_i \tag{13}$$

which are convergent for δ small enough, and the control law

$$u_D(k) = \Gamma_D w_D + K_{D_1}(x_D - \Pi_D w_D) \tag{14}$$

where K_{D_1} is such that $(A_D + B_D K_{D_1})$ has eigenvalues inside the unitary circle. Since $u_D(k) = \Gamma_D w_D$ when $x_D = \Pi_D w_D$, it is a matter of computations to verify that (13) solves (8). In fact, substituting (13) into (8), one has

$$\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\Pi_i S_D = A_D\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\Pi_i + B_D\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\Gamma_i + P_D$$
$$0 = C_D\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\Pi_i + Q_D.$$

Expanding $A_D,\ B_D,\ P_D,\ S_D,\ C_D$, and Q_D in powers of δ and regrouping the coefficients in the same power of δ, one recovers for any $i \geq 0$, the previous equations (11). Since (H_3^L) implies (H_3^{LD}), the robustness of the solution follows. □

It is interesting to point out that by approximating the power expansions (13) by means of finite ones, approximated solutions are obtained as shown hereinafter, where signals w of bounded amplitude are considered. Let

$$\Pi_D^r = \sum_{i=0}^{r}\frac{\delta^i}{i!}\Pi_i, \qquad \Gamma_D^r = \sum_{i=0}^{r}\frac{\delta^i}{i!}\Gamma_i \tag{15}$$

with

$$\Pi_D = \Pi_D^r + \mathcal{O}(\delta^{r+1}), \qquad \Gamma_D = \Gamma_D^r + \mathcal{O}(\delta^{r+1}) \tag{16}$$

where $\mathcal{O}(\delta^{r+1})$ contains the higher order remaining terms.

Corollary 1: Under the conditions of Theorem 5, for any integer r, the control law

$$u_D^r(k) = \Gamma_D^r w_D + K_{D_1}\big(x_D - \Pi_D^r w_D\big) \tag{17}$$

guarantees that $\lim_{k\to\infty}\|e_D(k)\| = \mathcal{O}(\delta^{r+1})$, for any w_D of bounded amplitude.

Proof: The controller (17) induces a map $\tilde{\Pi}$ which satisfies

$$\tilde{\Pi}S_D = (A_D + B_D K_{D_1})\tilde{\Pi} + B_D\big(\Gamma_D^r - K_{D_1}\Pi_D^r\big) + P_D$$

while substituting (13) into the first equation of (8), and considering (16) one obtains

$$\Pi_D^r S_D - (A_D + B_D K_{D_1})\Pi_D^r - B_D\big(\Gamma_D^r - K_{D_1}\Pi_D^r\big) - P_D = \mathcal{O}(\delta^{r+1})$$

so that, by the Center Manifold Theorem [1] one gets $\tilde{\Pi} = \Pi_D^r + \mathcal{O}(\delta^{r+1})$. As far as the error is concerned, note first that $\lim_{k\to\infty}\|x_D(k) - \Pi_D^r w_D(k)\| \leq \lim_{k\to\infty}(\|x_D(k) - \tilde{\Pi}w_D(k)\| + \|\tilde{\Pi} - \Pi_D^r\|\|w_D(k)\|)$. Since $\lim_{k\to\infty}\|x_D(k) - \tilde{\Pi}w_D(k)\| = 0$, we have that $\lim_{k\to\infty}\|x_D(k) - \Pi_D^r w_D(k)\| \leq \ell_1\varrho\delta^{r+1}$ for a positive constant ℓ_1. Now, since $0 = C_D\Pi_D + Q_D = C_D\Pi_D^r + Q_D + \mathcal{O}(\delta^{r+1})$, we have

$$\begin{aligned}\|e_D(k)\| &= \|C_D x_D + Q_D w_D\| \\ &\leq \|C_D\|\|x_D - \Pi_D^r w_D(k)\| + \|C_D\Pi_D^r + Q_D\|\|w_D(k)\| \\ &\leq \ell_2\|x_D - \Pi_D^r w_D(k)\| + \ell_1\varrho\delta^{r+1}\end{aligned}$$

where $\ell_2 = \|C_D\|$. Taking the limits, one gets $\lim_{k\to\infty}\|e_D(k)\| \leq \varrho\ell\delta^{r+1}$, with $\ell = \ell_1(1+\ell_2)$ an appropriate constant. □

IV. The Regulation Problem for Sampled Nonlinear Systems

In this section we consider the sampling of the nonlinear system (1), forced by piecewise constant u and w on time intervals of amplitude δ. It takes the form (5) with

$$\begin{aligned} f_D(\delta, x_D(k), u_D(k), w_D(k)) &= x\Big|_{\substack{x_D(k)\\u_D(k)\\w_D(k)}} + \delta(L_f + L_s)(x)\Big|_{\substack{x_D(k)\\u_D(k)\\w_D(k)}} + \frac{\delta^2}{2!}(L_f+L_s)^2(x)\Big|_{\substack{x_D(k)\\u_D(k)\\w_D(k)}} + \cdots \\ &= \sum_{i=0}^{\infty}\frac{\delta^i}{i!}(L_f+L_s)^i(x)\Big|_{\substack{x_D(k)\\u_D(k)\\w_D(k)}} = e^{\delta(L_f+L_s)}(x)\Big|_{\substack{x_D(k)\\u_D(k)\\w_D(k)}}\end{aligned}$$

$$\begin{aligned} s_D(\delta, w_D(k)) &= w\Big|_{w_D(k)} + \delta L_s(w)\Big|_{w_D(k)} + \frac{\delta^2}{2!} + L_s^2(w)\Big|_{w_D(k)} \cdots \\ &= \sum_{i=0}^{\infty}\frac{\delta^i}{i!}L_s^i(w)\Big|_{w_D(k)} = e^{\delta L_s}(w)\Big|_{w_D(k)}\end{aligned}$$

where the explicit dependence on δ has been put in evidence, and where $(L_f+L_s)^i$ is the ith application of $L_f + L_s := \frac{\partial(\cdot)}{\partial x}f + \frac{\partial(\cdot)}{\partial w}s$.

As in the linear case, one seeks a solution $(\pi_D(w_D), \gamma_D(w_D))$ fulfilling (7), under the hypothesis of solvability of the SFRP for the continuous system (1). To this end, let us consider the following extra condition [3], [4].

(H_3) For every pair of analytic functions $\tilde{f}(x,u,w)$ and $\tilde{h}(x,w)$ computed in the Appendix, there exist in a neighborhood of $(x,w) = 0$ two mappings $\pi(w), \gamma(w)$ such that the following equations are satisfied:

$$\begin{aligned}\frac{\partial\pi}{\partial w}s(w) &= A\pi(w) + B\gamma(w) + \tilde{f}(\pi(w),\gamma(w),w) \\ 0 &= C\pi(w) + \tilde{h}(\pi(w),w).\end{aligned} \tag{18}$$

Hypothesis (H_3) represents the nonlinear counterpart of (H_3^L), in the sense that it is a necessary condition for the existence of a robust controller [3], [4].

As a matter of fact, rewriting (1) as

$$\begin{aligned}\dot{x} &= f(x,u,w) = Ax + Bu + f_2(x,u,w) \\ \dot{w} &= s(w) \\ e &= h(x,w) = Cx + h_2(x,w)\end{aligned}$$

the underlying idea is to substitute a solution expressed as a power expansion of δ around the continuous-time solution (π_0, γ_0) into (7), to get equations of the form (18).

Theorem 6: Under $(H_1), (H_2)$, and (H_3), the nonlinear sampled SFRP is locally solvable.

Proof: As in the linear case, (H_1) implies (H_1^D) for almost all δ, and (H_2) implies (H_2^D). Referring the reader to the appendix for the expression of $\tilde{f}_i(\cdot,\cdot,\cdot)$ and $\tilde{h}_i(\cdot,\cdot)$, by introducing the equations

$$\frac{\partial\pi_i(w)}{\partial w}s(w) = A\pi_i(w) + B\gamma_i(w) + \tilde{f}_i(\pi_i(w),\gamma_i(w),w) \tag{19a}$$
$$0 = C\pi_i(w) + \tilde{h}_i(\pi_i(w),w) \tag{19b}$$

solvable with respect to π_i and γ_i because of (H_3), we show that a solution to (7) can be expressed as a power expansion in δ around the continuous-time solution (π_0,γ_0), making use of the solutions (π_i,γ_i) of (19). To do so, let us consider the following series:

$$\pi_D(\delta, w_D) = \sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i(w_D), \qquad \gamma_D(\delta, w_D) = \sum_{i=0}^{\infty}\frac{\delta^i}{i!}\gamma_i(w_D) \tag{20}$$

convergent for δ small enough, with $\pi_i(\cdot)$ and $\gamma_i(\cdot)$ solutions of (19). Let us define the control law

$$\begin{aligned}u_D(k) &= \gamma_D(\delta, w_D) + K_{D_1}(x_D - \pi_D(\delta, w_D)) \\ &= \alpha_D(\delta, x_D, w_D)\end{aligned} \tag{21}$$

with K_{D_1} such that $(A_D + B_D K_{D_1})$ has eigenvalues inside the unitary circle. Noting that $\alpha_D(\delta, \pi_D(\delta, w_D), w_D) = \gamma_D(\delta, w_D)$, substitution of (20) into (7) results in

$$\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i\left(e^{\delta L_s}w|_{w_D}\right)=f_D\left(\delta,\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i(w_D),\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\gamma_i(w_D),w_D\right) \tag{22a}$$

$$0=h\left(\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i(w_D),w_D\right). \tag{22b}$$

By equating the terms of the same power in δ, (19) is iteratively derived (see the appendix for computational details), thus proving that the solution to (7) is given by (20). □

Remark 1: For $i = 0$ from (19a), (19b) one recovers the continuous-time solution

$$\begin{aligned}&\left.\frac{\partial\pi_0(w)}{\partial w}\right|_{w_D}s(w_D)\\&=f(\pi_0(w_D),\gamma_0(w_D),w_D)\\&=A\pi_0(w_D)+B\gamma_0(w_D)+\tilde{f}_0(\pi_0(w),\gamma_0(w),w)|_{w_D}\end{aligned} \tag{23a}$$

$$0=h(\pi_0(w_D),w_D)=C\pi_0(w_D)+\tilde{h}_0(\pi_0(w),w)|_{w_D} \tag{23b}$$

with $\tilde{f}_0(\pi_0(w),\gamma_0(w),w) = f_2(\pi_0(w),\gamma_0(w),w)$ and $\tilde{h}_0(\pi_0(w),w) = h_2(\pi_0(w),w)$. For $i = 1$ one gets

$$\begin{aligned}\left.\frac{\partial\pi_1(w)}{\partial w}\right|_{w_D}s(w_D)&=A\pi_1(w_D)+B\gamma_1(w_D)\\&\quad+\tilde{f}_1(\pi_1(w),\gamma_1(w),w)|_{w_D}\end{aligned} \tag{24a}$$

$$\begin{aligned}0&=\left.\frac{\partial h}{\partial x}\right|_{\substack{\pi_0(w_D)\\ w_D}}\pi_1(w_D)\\&=C\pi_1(w_D)+\tilde{h}_1(\pi_1(w),w)|_{w_D}\end{aligned} \tag{24b}$$

where $\tilde{f}_1$ and $\tilde{h}_1$ take the form

$$\begin{aligned}\tilde{f}_1(\pi_1(w),\gamma_1(w),w)=&\left.\frac{\partial f_2(x,u,w)}{\partial x}\right|_{\substack{\pi_0(w)\\ \gamma_0(w)\\ w}}\pi_1(w)\\&+\left.\frac{\partial f_2(x,u,w)}{\partial u}\right|_{\substack{\pi_0(w)\\ \gamma_0(w)\\ w}}\gamma_1(w)\\&+\left.\frac{1}{2}L_f f(x,u,w)\right|_{\substack{\pi_0(w)\\ \gamma_0(w)\\ w}}\\ \tilde{h}_1(\pi_1(w),w)=&\left.\frac{\partial h_2(x,w)}{\partial x}\right|_{\substack{\pi_0(w)\\ w}}\pi_1(w).\end{aligned}$$

It can be easily verified that $\tilde{f}_i$ and $\tilde{h}_i$ particularize to (12) if f and h are linear.

Let us now consider as in the linear context the problem of computing approximated solutions. It is worthy to point out the relevance of this problem since closed forms for the sampled dynamics do not exist in general in the nonlinear context [5]. Let

$$\pi_D^r(\delta,w_D)=\sum_{i=0}^{r}\frac{\delta^i}{i!}\pi_i(w_D),\qquad \gamma_D^r(\delta,w_D)=\sum_{i=0}^{r}\frac{\delta^i}{i!}\gamma_i(w_D) \tag{25}$$

with

$$\begin{aligned}\pi_D(\delta,w_D)&=\pi_D^r(\delta,w_D)+\varphi_1(\delta,w_D)\\ \gamma_D(\delta,w_D)&=\gamma_D^r(\delta,w_D)+\varphi_2(\delta,w_D)\end{aligned}$$

where $\varphi_i(\delta,w_D)$ for $i = 1,2$ contain the remaining higher order terms.

Corollary 2: Under the conditions of Theorem 6, for any integer r the control law

$$\begin{aligned}u_D^r(k)&=\gamma_D^r(\delta,w_D)+K_{D_1}\left(x_D-\pi_D^r(\delta,w_D)\right)\\&=\alpha_D^r(\delta,x_D,w_D)\end{aligned} \tag{26}$$

guarantees that $\lim_{k\to\infty}\|e_D(k)\| = \mathcal{O}(\delta^{r+1})$.

Proof: Under the hypotheses of Theorem 6, a solution (20) exists, and hence functions in (25) and the control (26) are computable. The controller (26) induces a C^2 map $\tilde{\pi}(\delta,w_D)$ such that $\lim_{k\to\infty}\|x_D(k)-\tilde{\pi}(\delta,w_D)\| = 0$ and [1]

$$\tilde{\pi}(\delta,s_D(\delta,w_D))=f_D\left(\delta,\tilde{\pi}(\delta,w_D),\alpha_D^r(\delta,\tilde{\pi}(\delta,w_D),w_D),w_D\right).$$

If we now consider the exact solution $\pi_D(\delta,w_D),\gamma_D(\delta,w_D)$, given by (20), and if the control (21) is applied, for the right-hand term of (7a) we have

$$\begin{aligned}\pi_D(\delta,s_D(w_D))&=\pi_D^r(\delta,s_D(w_D))+\varphi_1(\delta,s_D(w_D))\\&=\pi_D^r(\delta,s_D(w_D))+\mathcal{O}\left(\delta^{r+1},w_D\right)\end{aligned}$$

while the left-hand term of (7a) can be rewritten as

$$\begin{aligned}&f_D(\delta,\pi_D(\delta,w_D),\alpha_D(\delta,\pi_D(\delta,w_D),w_D),w_D)\\&\quad=f_D\left(\delta,\pi_D^r(\delta,w_D),\alpha_D^r\left(\delta,\pi_D^r(\delta,w_D),w_D\right),w_D\right)+\psi(\delta,w_D)\end{aligned}$$

for an appropriate function $\psi(\delta,w_D) = \mathcal{O}(\delta^{r+1},w_D)$, since the control (21) can be expressed as $u_D(k) = \gamma_D^r(\delta,w_D)+\varphi_2(\delta,w_D)+K_{D_1}(x_D - \pi_D^r(\delta,w_D) - \varphi_1(\delta,w_D)) = u_D^r(k) + \mathcal{O}(\delta^{r+1},w_D)$. Hence, putting δ explicitly in evidence, (7a) can be written as

$$\begin{aligned}\pi_D^r(\delta,s_D(w_D))-f_D\left(\delta,\pi_D^r(\delta,w_D),\alpha_D^r\left(\delta,\pi_D^r(\delta,w_D),w_D\right),w_D\right)\\=\mathcal{O}\left(\delta^{r+1},w_D\right).\end{aligned} \tag{27}$$

Then, from the Center Manifold Theorem [1] $\tilde{\pi}(\delta,w_D) = \pi_D^r(\delta,w_D)+\xi(\delta,w_D)$, with $\xi(\delta,w_D) = \mathcal{O}(\delta^{r+1},w_D)$. Now, as in the linear case, since $\lim_{k\to\infty}\|x_D(k)-\tilde{\pi}(\delta,w_D)\| = 0$ and $w_D(k) < \varrho$, one has

$$\begin{aligned}\lim_{k\to\infty}\|x_D-\pi_D^r(\delta,w_D)\|&\le\lim_{k\to\infty}\|x_D-\tilde{\pi}(\delta,w_D)\|\\&\quad+\lim_{k\to\infty}\|\tilde{\pi}(\delta,w_D)-\pi_D^r(\delta,w_D)\|\\&=\lim_{k\to\infty}\|\xi(\delta,w_D)\|\\&\le\max_{\|w_D\|\le\varrho}\|\xi(\delta,w_D)\|\le k_1\varrho\delta^{r+1}\end{aligned}$$

for an appropriate constant k_1. Therefore

$$\begin{aligned}\lim_{k\to\infty}\|e_D(k)\|&=\lim_{k\to\infty}\|h(x_D,w_D)\|=\|h(\tilde{\pi}(\delta,w_D),w_D)\|\\&\le\left\|h(\tilde{\pi}(\delta,w_D),w_D)-h\left(\pi_D^r(\delta,w_D),w_D\right)\right\|\\&\quad+\left\|h\left(\pi_D^r(\delta,w_D),w_D\right)\right\|\\&\le k_2\varrho\delta^{r+1}+k_3\varrho\delta^{r+1}=\varrho\ell\delta^{r+1}\end{aligned}$$

$\ell = k_2 + k_3$, where $h(\tilde{\pi}(\delta,w_D),w_D) = h(\pi_D^r(\delta,w_D),w_D) + \tilde{h}(\xi(\delta,w_D),w_D)\xi(\delta,w_D)$ for an appropriate function $\tilde{h}$, and

$$\begin{aligned}&\|\tilde{h}(\xi(\delta,w_D),w_D)\xi(\delta,w_D)\|\\&\quad\le\max_{\|w_D\|\le\varrho}\|\tilde{h}(\xi(\delta,w_D),w_D)\xi(\delta,w_D)\|\le k_2\varrho\delta^{r+1}\\&\|\pi_D^r(\delta,w_D)\|\\&\quad\le\max_{\|w_D\|\le\varrho}\left\|\pi_D^r(\delta,w_D)\right\|=p(\delta,\varrho)\\&\left\|h\left(\pi_D^r,w_D\right)\right\|\\&\quad\le\max_{\substack{\|w_D\|\le\varrho\\ \|\pi_D^r\|\le p(\delta,\varrho)}}\left\|h\left(\pi_D^r,w_D\right)\right\|\le k_3\varrho\delta^{r+1}\end{aligned}$$

with k_2, k_3 appropriate constants. □

V. Conclusion

It has been shown that the existence of robust solutions to the regulator problem in the continuous-time context implies the solvability of the problem under sampling. A robust solution is obtained in the linear case. The nonlinear problem admits a solution which satisfies (H_3^D) below; such a condition is conjectured to be necessary for robustness in discrete time.

$$F^{(h,r)} = \begin{cases} \pi_r(w_D) & \text{if } h=0, r>0, \\ \frac{\partial}{\partial x}\left[(L_f+L_s)^{h-1}f\Big|_{\substack{\pi_0(w_D)\\ \gamma_0(w_D)\\ w_D}}\right]\pi_r(w_D) + \frac{\partial}{\partial u}\left[(L_f+L_s)^{h-1}f\Big|_{\substack{\pi_0(w_D)\\ \gamma_0(w_D)\\ w_D}}\right]\gamma_r(w_D), & \text{if } h\geq 1, r>0, \\ (L_f+L_s)^h x\Big|_{\substack{\pi_0(w_D)\\ \gamma_0(w_D)\\ w_D}}, & \text{if } h \geq 0, r=0 \end{cases}$$

(H_3^D) For the analytic functions $\tilde{f}_D(x_D, u_D, w_D)$ and $\tilde{h}_D(x_D, w_D)$, there exist in a neighborhood of $(x_D, w_D) = 0$ two mappings $\pi_D(w_D), \gamma_D(w_D)$ such that the following equations are satisfied:

$$\begin{aligned} \pi_D(s_D(w_D)) &= A_D\pi_D(w_D) + B_D\gamma_D(w_D) \\ &\quad + \tilde{f}_D(\pi_D(w_D), \gamma_D(w_D), w_D) \\ 0 &= C_D\pi_D(w_D) + \tilde{h}_D(\pi_D(w_D), w_D). \end{aligned}$$

It directly follows from the used arguments how to compute iteratively approximated solutions. It must be noted that only approximated solutions at the first order can be computed if robustness of the continuous problem fails.

This work represents a first contribution for the effective computation of a nonlinear digital regulator. Work is in progress for a better understanding of discrete-time nonlinear robustness.

Appendix

The expressions of $\tilde{f}_i(\cdot,\cdot,\cdot)$ and $\tilde{h}_i(\cdot,\cdot)$ in (19) are obtained by developing powers of δ equations (22). As far as the left-hand term of (22a) is concerned, make use of the exchange theorem of the Lie series and compute

$$\begin{aligned} \sum_{j=0}^{\infty}\frac{\delta^j}{j!}\pi_j\left(e^{\delta L_s}w\Big|_{w_D}\right) &= \sum_{j=0}^{\infty}\frac{\delta^j}{j!}e^{\delta L_s}(\pi_j(w))\Big|_{w_D} \\ &= \sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sum_{h=0}^{j}\binom{j}{h}L_s^h(\pi_{j-h}(w))\Big|_{w_D}. \end{aligned} \tag{28}$$

For the right-hand term in (22a) one obtains

$$\begin{aligned} &f_D\left(\delta, \sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i(w_D), \sum_{i=0}^{\infty}\frac{\delta^i}{i!}\gamma_i(w_D), w_D\right) \\ &\quad = \sum_{j=0}^{\infty}\frac{\delta^j}{j!}\sum_{h=0}^{j}\binom{j}{h}F^{(h,j-h)} \end{aligned} \tag{29}$$

where we have the equation shown at the top of the page, with $F^{(i+1,0)} = (L_f + L_s)F^{(i,0)}, h \geq 0$ so that $f_D(\delta, x, u, w) = \sum_{i=0}^{\infty}\frac{\delta^i}{i!}F^{(i,0)}$. By equating the terms in (28), (29) with the same power in δ, one obtains the following relationship:

$$\begin{aligned} \frac{\partial \pi_{j-1}(w)}{\partial w}s(w)\Big|_{w_D} &= A\pi_{j-1}(w_D) + B\gamma_{j-1}(w_D) \\ &\quad + \tilde{f}_{j-1}(\pi_{j-1}(w), \gamma_{j-1}(w), w)\Big|_{w_D} \end{aligned} \tag{30}$$

with

$$\begin{aligned} &\tilde{f}_{j-1}(\pi_{j-1}(w), \gamma_{j-1}(w), w) \\ &\quad = \frac{\partial f_2(x,u,w)}{\partial x}\Big|_{\substack{\pi_0(w)\\ \gamma_0(w)\\ w}}\pi_{j-1}(w) \\ &\qquad + \frac{\partial f_2(x,u,w)}{\partial u}\Big|_{\substack{\pi_0(w)\\ \gamma_0(w)\\ w}}\gamma_{j-1}(w) \\ &\qquad + \frac{1}{j}\sum_{h=2}^{j}\binom{j}{h}\left(F^{(h,j-h)}\Big|_{\substack{\pi_0(w)\\ \gamma_0(w)\\ w}} - L_s^h\pi_{j-h}(w)\right), \qquad \text{if } j \geq 2 \end{aligned}$$

and

$$\tilde{f}_{j-1}(\pi_{j-1}(w), \gamma_{j-1}(w), w) = f_2(\pi_0(w), \gamma_0(w), w), \quad \text{if } j = 1.$$

By setting $i = j - 1$ into (30), (19a) is derived.

As far as (19b) is concerned, from (22b) one has

$$\begin{aligned} &h\left(\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i(w), w\right)\Big|_{w_D} \\ &\quad = \sum_{i=0}^{\infty}\frac{\delta^i}{i!}C\pi_i(w_D) + h_2\left(\sum_{i=0}^{\infty}\frac{\delta^i}{i!}\pi_i(w), w\right)\Big|_{w_D} \end{aligned}$$

i.e., the coefficient of the ith power in δ is $0 = C\pi_i(w_D) + \tilde{h}_i(\pi_i(w), w)|_{w_D}$, with

$$\begin{aligned} \tilde{h}_i(\pi_i(w), w) &= \bar{h}_i(\pi_0(w), \cdots, \pi_{i-1}(w), \pi_i(w), w) \\ &= \frac{\partial^i}{\partial \delta^i}h_2\left(\sum_{j=0}^{\infty}\frac{\delta^j}{j!}\pi_j(w), w\right)\Big|_{\delta=0} \end{aligned}$$

where $\pi_0(w), \cdots, \pi_{i-1}(w)$ are known since computed in the previous $i - 1$ steps.

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