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ON RELATION BETWEEN PSEUDO-HERMITIAN SYMMETRIC PAIRS AND PARA-HERMITIAN SYMMETRIC PAIRS

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Abstract. In this paper, we investigate relation between pseudo-Hermitian symmetric pairs and para-Hermitian symmetric ones.

1. Introduction and our result. For a Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{sl}(2, \boldsymbol{R}), \mathfrak{so}(2))$ with complex structure J, there exists an elliptic element $S \in \mathfrak{g}$ which satisfies two conditions

(i) $\mathfrak{r}$ is the centralizer $\mathfrak{c}_{\mathfrak{g}}(S)$ of S in $\mathfrak{g}$,

(ii) J is induced by $\mathrm{ad}_{\mathfrak{g}}\, S$.

For example, $S = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \in \mathfrak{g}$ is such an element. Define two automorphisms θ and η of $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{R})$ by

$$\begin{cases} \theta(A) := -{}^tA \quad \text{for } A \in \mathfrak{g}\,; \\ \eta(A) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot A \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \quad \text{for } A \in \mathfrak{g}\,. \end{cases}$$

Then, θ is a Cartan involution of $\mathfrak{g}$ such that $\theta(S) = S$, and η is an involutive automorphism of $\mathfrak{g}$ such that $\eta(S) = -S$ and $\eta \circ \theta = \theta \circ \eta$. Now, let us explain that $\mathfrak{g}$, S, θ and η bring about a para-Hermitian symmetric pair $(\mathfrak{su}(1, 1), \mathfrak{so}(1, 1))$. Let $\mathfrak{g}^d$ be a real form of $\mathfrak{g}_{\boldsymbol{C}} = \mathfrak{sl}(2, \boldsymbol{C})$ such that $(\mathfrak{g}^d, \theta)$ is the Berger dual symmetric pair of $(\mathfrak{g}, \eta)$ (cf. Berger [1, p. 111]), i.e.,

$$\mathfrak{g}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{k} \cap \mathfrak{m}) \oplus i(\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{m})\,,$$

where $\mathfrak{k}$ and $\mathfrak{p}$ (resp. $\mathfrak{h}$ and $\mathfrak{m}$) denote the $+1$ and -1-eigenspaces of θ (resp. η) in $\mathfrak{g}$, respectively. Here, it follows that $\mathfrak{g}^d = \mathfrak{su}(1, 1)$. An element iS belongs to $\mathfrak{g}^d$, and $(\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$ is a para-Hermitian symmetric pair $(\mathfrak{su}(1, 1), \mathfrak{so}(1, 1))$, where $\mathrm{ad}_{\mathfrak{g}^d}\, iS$ induces a para-complex structure of $(\mathfrak{su}(1, 1), \mathfrak{so}(1, 1)) = (\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$. Therefore, a (pseudo-)Hermitian symmetric pair $(\mathfrak{sl}(2, \boldsymbol{R}), \mathfrak{so}(2))$ brings about a para-Hermitian symmetric pair $(\mathfrak{su}(1, 1), \mathfrak{so}(1, 1))$. This poses us the following problem: "Does there exist relation between pseudo-Hermitian symmetric pairs and para-Hermitian symmetric ones?"

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The main purpose of this paper is to demonstrate the following Theorem 1.1 which partially clarifies relation between simple pseudo-Hermitian symmetric pairs and simple para-Hermitian symmetric ones:

THEOREM 1.1. *Let $\mathfrak{g}_C$ be a complex simple Lie algebra. Then, the following two items* (I) *and* (II) *hold*:

(I) *For any real form $\mathfrak{g}$ of $\mathfrak{g}_C$ and pseudo-Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{r})$ with complex structure J, there exist an elliptic element $S \in \mathfrak{g}$, a Cartan involution θ of $\mathfrak{g}$, and an involutive automorphism η of $\mathfrak{g}$ such that*

(1) $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(S))$, *and J is induced by* $\mathrm{ad}_{\mathfrak{g}} S$;

(2) $\theta(S) = S$, $\eta(S) = -S$, *and* $\eta \circ \theta = \theta \circ \eta$;

(3) $(\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$ *is a para-Hermitian symmetric pair with para-complex structure induced by* $\mathrm{ad}_{\mathfrak{g}^d} iS$.

Here, $(\mathfrak{g}^d, \theta)$ *is the Berger dual symmetric pair of* $(\mathfrak{g}, \eta)$.

(II) *For any real form $\bar{\mathfrak{g}}$ of $\mathfrak{g}_C$ and para-Hermitian symmetric pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ with para-complex structure $\bar{I}$, there exist a real form $\mathfrak{g}$ of $\mathfrak{g}_C$, an elliptic element $S \in \mathfrak{g}$, a Cartan involution θ of $\mathfrak{g}$, and an involutive automorphism η of $\mathfrak{g}$ such that*

(1) $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(S))$ *is a pseudo-Hermitian symmetric pair with complex structure induced by* $\mathrm{ad}_{\mathfrak{g}} S$;

(2) $\theta(S) = S$, $\eta(S) = -S$, *and* $\eta \circ \theta = \theta \circ \eta$;

(3) $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}}) = (\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$, *and $\bar{I}$ is induced by* $\mathrm{ad}_{\mathfrak{g}^d} iS$.

Here, $(\mathfrak{g}^d, \theta)$ *is the Berger dual symmetric pair of* $(\mathfrak{g}, \eta)$.

As an application, we actually determine the para-Hermitian symmetric pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ which a (pseudo-)Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{r})$ brings about by means of Theorem 1.1-(I), by using the result in Leung [10, p. 182] which determines Lagrangian reflective submanifolds of irreducible Hermitian symmetric spaces (see Theorem 4.6, also see Remark 4.4).

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2. Preliminaries. This section consists of four subsections. In Subsection 2.1, we recall the notion of para-Hermitian symmetric pair, hyperbolic element and so forth. In Subsection 2.2, we introduce Murakami's setting utilized in [11], and we confirm two Lemmas 2.7 and 2.8. Subsection 2.3 studies relation among pseudo-Hermitian symmetric pairs, elliptic elements and involutions (cf. Proposition 2.10). Finally in Subsection 2.4, we refer to a result of Kaneyuki [3] which investigates relation among para-Hermitian symmetric pairs, hyperbolic elements and involutions (cf. Proposition 2.12).

2.1. Definitions and notation. We will first recall the notion of para-Hermitian symmetric pair and pseudo-Hermitian symmetric pair, and we will next recall the notion of hyperbolic element and elliptic element.

DEFINITION 2.1 (Kaneyuki-Kozai [4, p. 88]). Let $(\mathfrak{l}, \mathfrak{b})$ be the semisimple symmetric pair by involution σ, and let $\mathfrak{n}$ denote the -1-eigenspace of σ in $\mathfrak{l}$. Then, $(\mathfrak{l}, \mathfrak{b})$ is called *para-Hermitian*, if there exist an $\mathrm{ad}_{\mathfrak{l}}\,\mathfrak{b}$-invariant para-complex structure I of $\mathfrak{n}$ and an $\mathrm{ad}_{\mathfrak{l}}\,\mathfrak{b}$-invariant para-Hermitian form $\langle\cdot, \cdot\rangle$ with respect to I on $\mathfrak{n}$, i.e., I is a linear endomorphism of $\mathfrak{n}$ and $\langle\cdot, \cdot\rangle$ is a non-degenerate symmetric bilinear form on $\mathfrak{n}$ such that

(1) $I^2 = \mathrm{id}$ and $I \neq \mathrm{id}$,

(2) $[X, I(Y)] = I([X, Y])$ for any $X \in \mathfrak{b}$ and $Y \in \mathfrak{n}$,

(3) $\langle I(Y_1), Y_2\rangle + \langle Y_1, I(Y_2)\rangle = 0$ for any $Y_1, Y_2 \in \mathfrak{n}$,

(4) $\langle [X, Y_1], Y_2\rangle + \langle Y_1, [X, Y_2]\rangle = 0$ for any $X \in \mathfrak{b}$ and $Y_1, Y_2 \in \mathfrak{n}$.

DEFINITION 2.2 (Berger [1, p. 94]). Let $(\mathfrak{l}, \mathfrak{r})$ be the semisimple symmetric pair by involution ρ, and let $\mathfrak{q}$ denote the -1-eigenspace of ρ in $\mathfrak{l}$. Then, $(\mathfrak{l}, \mathfrak{r})$ is called *pseudo-Hermitian*, if there exist an $\mathrm{ad}_{\mathfrak{l}}\,\mathfrak{r}$-invariant complex structure J of $\mathfrak{q}$ and an $\mathrm{ad}_{\mathfrak{l}}\,\mathfrak{r}$-invariant pseudo-Hermitian form $\langle\cdot, \cdot\rangle$ with respect to J on $\mathfrak{q}$.

DEFINITION 2.3 (Kobayashi [6, p. 5–6]). Let $\mathfrak{l}$ be a real semisimple Lie algebra. An element $X \in \mathfrak{l}$ is called *semisimple*, if the endomorphism $\mathrm{ad}_{\mathfrak{l}}\,X$ of $\mathfrak{l}$ is semisimple. A semisimple element $Z \in \mathfrak{l}$ (resp. $S \in \mathfrak{l}$) is said to be *hyperbolic* (resp. *elliptic*), if all the eigenvalues of $\mathrm{ad}_{\mathfrak{l}}\,Z$ (resp. $\mathrm{ad}_{\mathfrak{l}}\,S$) are real (resp. purely imaginary).

NOTATION. Throughout this paper, we use the following notation:

(n1) $\mathrm{ad}_{\mathfrak{a}}$: the adjoint representation of a Lie algebra $\mathfrak{a}$.

(n2) $B_{\mathfrak{a}}$: the Killing form of a Lie algebra $\mathfrak{a}$.

(n3) $\mathfrak{c}_{\mathfrak{a}}(X)$: the centralizer of X in a Lie algebra $\mathfrak{a}$, for an element $X \in \mathfrak{a}$.

(n4) $\mathfrak{m} \oplus \mathfrak{n}$: the direct sum of vector spaces $\mathfrak{m}$ and $\mathfrak{n}$.

(n5) $f|_A$: the restriction of a mapping f to a set A.

(n6) $\mathfrak{d}_{\mathrm{ss}}$: the semisimple part of a reductive Lie algebra $\mathfrak{d}$, namely $\mathfrak{d}_{\mathrm{ss}} = [\mathfrak{d}, \mathfrak{d}]$.

2.2. Root-space decomposition and Cartan decomposition. From the results of Murakami [11], we will afterward deduce Lemma 2.7, Lemma 2.9, etc. So, we want to introduce Murakami's setting utilized in [11].

Let $\mathfrak{l}_{\boldsymbol{C}}$ be a complex semisimple Lie algebra, let $\mathfrak{h}_{\boldsymbol{C}}$ be a Cartan subalgebra of $\mathfrak{l}_{\boldsymbol{C}}$, and let $\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ denote the set of non-zero roots of $\mathfrak{l}_{\boldsymbol{C}}$ with respect to $\mathfrak{h}_{\boldsymbol{C}}$. Then, there exists a Weyl basis $\{X_\alpha\,;\ \alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})\}$ of $\mathfrak{l}_{\boldsymbol{C}}$ such that, for all $\alpha, \beta \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$,

$$[X_\alpha, X_{-\alpha}] = H_\alpha\,, \quad [H, X_\alpha] = \alpha(H)\cdot X_\alpha \quad \text{for } H \in \mathfrak{h}_{\boldsymbol{C}}\,;$$
$$[X_\alpha, X_\beta] = 0 \quad \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})\,;$$
$$[X_\alpha, X_\beta] = N_{\alpha,\beta}\cdot X_{\alpha+\beta} \quad \text{if } \alpha+\beta \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})\,,$$

where the real constants $N_{\alpha,\beta}$ satisfy $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ (cf. Helgason [2, Theorem 5.5, p. 176]). Here for $\alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$, one defines the element $H_\alpha \in \mathfrak{h}_{\boldsymbol{C}}$ by $B_{\mathfrak{l}_{\boldsymbol{C}}}(H, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}_{\boldsymbol{C}}$, where $B_{\mathfrak{l}_{\boldsymbol{C}}}$ denotes the Killing form of $\mathfrak{l}_{\boldsymbol{C}}$. By using this Weyl basis, we give a compact real form $\mathfrak{l}_u$ of $\mathfrak{l}_{\boldsymbol{C}}$ as follows:

$$\mathfrak{l}_u = i\mathfrak{h}_{\boldsymbol{R}} \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})} \mathrm{span}_{\boldsymbol{R}}\{X_\alpha - X_{-\alpha}\} \oplus \mathrm{span}_{\boldsymbol{R}}\{i(X_\alpha + X_{-\alpha})\} \tag{2.2.1}$$

(see the proof of Theorem 6.3 in Helgason [2, p. 181]), where $\mathfrak{h}_{\boldsymbol{R}}$ is a real vector subspace of $\mathfrak{h}_{\boldsymbol{C}}$ determined by

$$\begin{aligned}\mathfrak{h}_{\boldsymbol{R}} &:= \operatorname{span}_{\boldsymbol{R}}\{H_\alpha ; \alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})\} \\ &\quad (= \{H \in \mathfrak{h}_{\boldsymbol{C}} ; \alpha(H) \in \boldsymbol{R} \text{ for all } \alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})\}) .\end{aligned}$$

Now, let $\Pi_{\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})}$ denote the set of simple roots in $\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$, and let θ be an involutive automorphism of $\mathfrak{l}_{\boldsymbol{C}}$ satisfying three conditions

$$\text{(c1)} \quad \theta(\mathfrak{l}_u) \subset \mathfrak{l}_u , \quad \text{(c2)} \quad \theta(\mathfrak{h}_{\boldsymbol{C}}) \subset \mathfrak{h}_{\boldsymbol{C}} , \quad \text{(c3)} \quad {}^t\theta(\Pi_{\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})}) = \Pi_{\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})} .$$

Denote by $\mathfrak{k}$ and $\mathfrak{p}$ the $+1$ and -1-eigenspaces of θ in $\mathfrak{l}_u$, respectively. One has the following decomposition:

$$\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p} .$$

Then, we define a real form $\mathfrak{l}$ of $\mathfrak{l}_{\boldsymbol{C}}$ by setting

$$\mathfrak{l} := \mathfrak{k} \oplus i\mathfrak{p} .$$

REMARK 2.4. (1) θ is a Cartan involution of $\mathfrak{l}$, and $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ is its Cartan decomposition. (2) $\mathfrak{k} \cap i\mathfrak{h}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of $\mathfrak{k}$, because it follows from ${}^t\theta(\Pi_{\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})}) = \Pi_{\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})}$ that θ leaves fixed a regular element of $\mathfrak{l}_{\boldsymbol{C}}$ contained in $\mathfrak{h}_{\boldsymbol{C}}$ (see Murakami [12, Proposition 1, p. 106]). (3) Every real semisimple Lie algebra can be, up to isomorphism, given by the above fashion (cf. Murakami [13]). Henceforth in Section 2, we assume that a real semisimple Lie algebra $\mathfrak{l}$ is given by the above fashion, and we identify $\operatorname{Aut}(\mathfrak{l})$ and $\operatorname{Aut}(\mathfrak{l}_u)$ with $\{\phi \in \operatorname{Aut}(\mathfrak{l}_{\boldsymbol{C}}) ; \phi(\mathfrak{l}) \subset \mathfrak{l}\}$ and $\{\psi \in \operatorname{Aut}(\mathfrak{l}_{\boldsymbol{C}}) ; \psi(\mathfrak{l}_u) \subset \mathfrak{l}_u\}$, respectively.

In the above setting, Murakami [11, Theorem 3] and its proof allow us to assert the following:

PROPOSITION 2.5 (Murakami [11, p. 118–121]). *Let ψ be an automorphism of $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$. Suppose that it satisfies two conditions*

(a) $\psi(i\mathfrak{h}_{\boldsymbol{R}}) \subset i\mathfrak{h}_{\boldsymbol{R}}$, *and* $\psi \circ \theta = \theta \circ \psi$ *on* $i\mathfrak{h}_{\boldsymbol{R}}$;

(b) ${}^t\psi(\Delta_1(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}} : \theta)) = \Delta_1(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}} : \theta)$,

where $\Delta_1(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}} : \theta) := \{\beta \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}}) ; {}^t\theta(\beta) = \beta \text{ and } \theta(X_\beta) = X_\beta\}$. *Then, there exists an element* $H \in \mathfrak{h}_{\boldsymbol{R}}$ *such that* $\psi \circ \exp \operatorname{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH \in \operatorname{Aut}(\mathfrak{l}) \cap \operatorname{Aut}(\mathfrak{l}_u)$.

In the same setting, Murakami [11] has proved

PROPOSITION 2.6 (Murakami [11, p. 106]). *For an automorphism ψ of $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$, the following three conditions* (i), (ii) *and* (iii) *are mutually equivalent*:

$$\text{(i)} \quad \psi \circ \theta = \theta \circ \psi , \quad \text{(ii)} \quad \psi \in \operatorname{Aut}(\mathfrak{l}) \cap \operatorname{Aut}(\mathfrak{l}_u) , \quad \text{(iii)} \quad \psi(\mathfrak{k}) \subset \mathfrak{k} .$$

Here, θ is the Cartan involution of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$.

We confirm two Lemmas 2.7 and 2.8, and finish this subsection. Here, we are pointed out by the referee that Lemma 2.7 is a special case of a more general statement in Helgason [2, p. 277], and that Nagano-Sekiguchi [14, p. 320] has already asserted Lemma 2.7.

LEMMA 2.7. *Let σ_1 and σ_2 be two involutive automorphisms of a real semisimple Lie algebra $\mathfrak{l}$ such that σ_1 is commutative with σ_2. Then, there exists a Cartan involution τ of $\mathfrak{l}$ such that both σ_1 and σ_2 are commutative with τ.*

PROOF. We will devote ourselves to verifying that there exists an inner automorphism ϕ of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ such that both $\phi \circ \sigma_1 \circ \phi^{-1}$ and $\phi \circ \sigma_2 \circ \phi^{-1}$ are commutative with Cartan involution θ (recall Remark 2.4 for θ and for later). In this case, $\tau := \phi^{-1} \circ \theta \circ \phi$ is a Cartan involution of $\mathfrak{l}$ which is commutative with σ_1 and σ_2.

By Theorem 1 in Murakami [11, p. 108], there exist a unique element $\eta_1 \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)$ and a unique element $X_1 \in \mathfrak{p}$ which satisfy

$$\sigma_1 = \eta_1 \circ \exp \mathrm{ad}_{\mathfrak{l}}\, iX_1\,.$$

Since σ_1 is involutive, one obtains $\eta_1(X_1) = -X_1$ (see the proof of Lemma 10.2 in Berger [1, p. 100]). Define an inner automorphism ϕ_1 of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ by

$$\phi_1 := \exp \mathrm{ad}_{\mathfrak{l}}(i/2)X_1\,.$$

Then, it is clear that $\phi_1 \circ \sigma_1 \circ \phi_1^{-1} = \eta_1 \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)$, and this shows $(\eta_1)^2 = \mathrm{id}$. By use of ϕ_1 and σ_2, let us define an involutive automorphism σ_2' of $\mathfrak{l}$ as follows:

$$\sigma_2' := \phi_1 \circ \sigma_2 \circ \phi_1^{-1}\,.$$

The hypothesis "$\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1$" enables us to see that σ_2' is commutative with η_1 ($= \phi_1 \circ \sigma_1 \circ \phi_1^{-1}$). By arguments similar to those mentioned above, we can deduce that there exist a unique element $\eta_2' \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)$ and a unique element $X_2' \in \mathfrak{p}$ which satisfy

$$\sigma_2' = \eta_2' \circ \exp \mathrm{ad}_{\mathfrak{l}}\, iX_2'\,,$$

and that $\eta_2'(X_2') = -X_2'$. Define an inner automorphism ϕ_2' of $\mathfrak{l}$ by

$$\phi_2' := \exp \mathrm{ad}_{\mathfrak{l}}(i/2)X_2'\,.$$

Then, it follows that $(\phi_2' \circ \phi_1) \circ \sigma_2 \circ (\phi_2' \circ \phi_1)^{-1} = \phi_2' \circ \sigma_2' \circ \phi_2'^{-1} = \eta_2' \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)$. Consequently, $\phi := \phi_2' \circ \phi_1$ is an inner automorphism of $\mathfrak{l}$ such that $\phi \circ \sigma_2 \circ \phi^{-1}$ ($= \eta_2'$) is commutative with θ (cf. Proposition 2.6). So, the rest of proof is to verify that $\phi \circ \sigma_1 \circ \phi^{-1}$ is also commutative with θ. In order to do so, we want to show

$$\eta_1(X_2') = X_2'\,. \tag{2.2.2}$$

Since σ_2' is commutative with η_1 ($= \phi_1 \circ \sigma_1 \circ \phi_1^{-1}$), and since $(\eta_1)^2 = \mathrm{id}$, one perceives that

$$\begin{aligned} \eta_1 \circ \eta_2' \circ \exp \mathrm{ad}_{\mathfrak{l}}\, iX_2' &= \eta_1 \circ \sigma_2' = \sigma_2' \circ \eta_1 \\ &= \eta_2' \circ \exp \mathrm{ad}_{\mathfrak{l}}\, iX_2' \circ \eta_1 \\ &= \eta_2' \circ \eta_1 \circ \eta_1 \circ \exp \mathrm{ad}_{\mathfrak{l}}\, iX_2' \circ \eta_1 \\ &= \eta_2' \circ \eta_1 \circ \exp \mathrm{ad}_{\mathfrak{l}}\, i\eta_1(X_2')\,. \end{aligned} \tag{2.2.3}$$

Proposition 2.6, together with $\eta_1 \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)$, means that $\eta_1 \circ \theta = \theta \circ \eta_1$; so that one has $\eta_1(X_2') \in \mathfrak{p}$, since $X_2' \in \mathfrak{p}$. Therefore, we conclude that $\eta_1 \circ \eta_2', \eta_2' \circ \eta_1 \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)$

and $\exp \mathrm{ad}_{\mathfrak{l}}\, iX_2'$, $\exp \mathrm{ad}_{\mathfrak{l}}\, i\eta_1(X_2') \in \exp \mathrm{ad}_{\mathfrak{l}}\, i\mathfrak{p}$. Accordingly, it follows from (2.2.3) that

$$\eta_1 \circ \eta_2' = \eta_2' \circ \eta_1 \quad \text{and} \quad \exp \mathrm{ad}_{\mathfrak{l}}\, iX_2' = \exp \mathrm{ad}_{\mathfrak{l}}\, i\eta_1(X_2'),$$

because $\mathrm{Aut}(\mathfrak{l}) = (\mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u)) \cdot \exp \mathrm{ad}_{\mathfrak{l}}\, i\mathfrak{p}$ is the direct sum (cf. Theorem 1 in Murakami [11, p. 108]). A mapping $\mathrm{ad}_{\mathfrak{l}}\, iX \mapsto \exp \mathrm{ad}_{\mathfrak{l}}\, iX$, for $X \in \mathfrak{p}$, is injective, and $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ is semisimple; and hence $X_2' = \eta_1(X_2')$. Thus we get (2.2.2). Direct computation and (2.2.2) give us

$$\begin{aligned} \phi \circ \sigma_1 \circ \phi^{-1} &= (\phi_2' \circ \phi_1) \circ \sigma_1 \circ (\phi_2' \circ \phi_1)^{-1} \\ &= \phi_2' \circ \eta_1 \circ \phi_2'^{-1} \\ &= \exp \mathrm{ad}_{\mathfrak{l}}(i/2)X_2' \circ \eta_1 \circ \exp \mathrm{ad}_{\mathfrak{l}}(-i/2)X_2' \\ &= \exp \mathrm{ad}_{\mathfrak{l}}(i/2)X_2' \circ \exp \mathrm{ad}_{\mathfrak{l}}\, \eta_1((-i/2)X_2') \circ \eta_1 \\ &= \eta_1 \in \mathrm{Aut}(\mathfrak{l}) \cap \mathrm{Aut}(\mathfrak{l}_u). \end{aligned}$$

This implies that $\phi \circ \sigma_1 \circ \phi^{-1}$ $(= \eta_1)$ is commutative with θ (cf. Proposition 2.6). □

The following lemma will be helpful to complete the proof of Theorem 1.1:

LEMMA 2.8. *Let $\mathfrak{l}$ be a real semisimple Lie algebra. Then, the following two items* (a) *and* (b) *hold*:

(a) *If S is a non-zero semisimple element of $\mathfrak{l}$ and the eigenvalue of* $\mathrm{ad}_{\mathfrak{l}}\, S$ *is $\pm i$ or zero, then* $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ *is a pseudo-Hermitian symmetric pair with complex structure induced by* $\mathrm{ad}_{\mathfrak{l}}\, S$.

(b) *If Z is a non-zero semisimple element of $\mathfrak{l}$ and the eigenvalue of* $\mathrm{ad}_{\mathfrak{l}}\, Z$ *is ± 1 or zero, then* $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(Z))$ *is a para-Hermitian symmetric pair with para-complex structure induced by* $\mathrm{ad}_{\mathfrak{l}}\, Z$.

PROOF. (a): Since S is semisimple, $\mathfrak{l}$ is decomposed as

$$\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(S) \oplus [S, \mathfrak{l}].$$

One has $(\mathrm{ad}_{\mathfrak{l}}\, S)^2(Y) = -Y$ for any $Y \in [S, \mathfrak{l}]$, because the eigenvalue of $\mathrm{ad}_{\mathfrak{l}}\, S$ is $\pm i$ or zero. Now, let us verify that there exists an involutive automorphism ρ of $\mathfrak{l}$ whose $+1$-eigenspace (resp. -1-eigenspace) coincides with $\mathfrak{c}_{\mathfrak{l}}(S)$ (resp. $[S, \mathfrak{l}]$). Define an inner automorphism ρ of $\mathfrak{l}$ by

$$\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{l}}\, S.$$

Then, since $(\mathrm{ad}_{\mathfrak{l}}\, S)^2(Y) = -Y$ for any $Y \in [S, \mathfrak{l}]$, we obtain

$$\begin{aligned} \rho(Y) = \exp \pi\, \mathrm{ad}_{\mathfrak{l}}\, S(Y) &= \sum_{l \geq 0} \frac{1}{l!} (\pi\, \mathrm{ad}_{\mathfrak{l}}\, S)^l(Y) \\ &= \sum_{m \geq 0} \frac{1}{2m!} (\pi\, \mathrm{ad}_{\mathfrak{l}}\, S)^{2m}(Y) + \sum_{n \geq 0} \frac{1}{(2n+1)!} (\pi\, \mathrm{ad}_{\mathfrak{l}}\, S)^{2n+1}(Y) \\ &= \sum_{m \geq 0} (-1)^m \cdot \frac{\pi^{2m}}{2m!} \cdot Y + \sum_{n \geq 0} (-1)^n \cdot \frac{\pi^{2n+1}}{(2n+1)!} \cdot [S, Y] \\ &= \cos \pi \cdot Y + \sin \pi \cdot [S, Y] = -Y. \end{aligned}$$

On the other hand; it follows that $\rho(X) = \exp \pi \operatorname{ad}_{\mathfrak{l}} S(X) = X$ for every $X \in \mathfrak{c}_{\mathfrak{l}}(S)$. Therefore, ρ is an involutive automorphism of $\mathfrak{l}$ such that the $+1$-eigenspace (resp. -1-eigenspace) of ρ in $\mathfrak{l}$ coincides with $\mathfrak{c}_{\mathfrak{l}}(S)$ (resp. $[S, \mathfrak{l}]$). Hence, $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ is the symmetric pair by involution ρ, and $\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(S) \oplus [S, \mathfrak{l}]$ is the canonical decomposition of $\mathfrak{l}$ with respect to ρ. Furthermore, $J := \operatorname{ad}_{\mathfrak{l}} S$ is a complex structure of the vector space $[S, \mathfrak{l}]$, and $B_{\mathfrak{l}}$ is a pseudo-Hermitian form with respect to J on $[S, \mathfrak{l}]$, where we denote by $B_{\mathfrak{l}}$ the Killing form of $\mathfrak{l}$. Hence, $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(S))$ is a pseudo-Hermitian symmetric pair with complex structure induced by $\operatorname{ad}_{\mathfrak{l}} S$.

(b): Since $Z \in \mathfrak{l}$ is non-zero semisimple and the eigenvalue of $\operatorname{ad}_{\mathfrak{l}} Z$ is ± 1 or zero, $\mathfrak{l}$ is decomposed as follows:

$$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_{+1}\,,$$

where $\mathfrak{l}_0 := \mathfrak{c}_{\mathfrak{l}}(Z)$ and $\mathfrak{l}_{\pm 1}$ denote the ± 1-eigenspaces of $\operatorname{ad}_{\mathfrak{l}} Z$ in $\mathfrak{l}$. Define an inner automorphism σ of $\mathfrak{l}_{\boldsymbol{C}}$ by

$$\sigma := \exp \pi \operatorname{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iZ\,,$$

where $\mathfrak{l}_{\boldsymbol{C}}$ denotes the complexification of $\mathfrak{l}$. It is obvious that $\sigma = \mathrm{id}$ on $\mathfrak{c}_{\mathfrak{l}}(X) = \mathfrak{l}_0$, $\sigma = -\mathrm{id}$ on $\mathfrak{l}_{-1} \oplus \mathfrak{l}_{+1}$, and $\sigma(\mathfrak{l}) \subset \mathfrak{l}$. Accordingly, σ is an involutive automorphism of $\mathfrak{l}$ such that its $+1$ and -1-eigenspaces are $\mathfrak{c}_{\mathfrak{l}}(X)$ and $\mathfrak{l}_{-1} \oplus \mathfrak{l}_{+1}$, respectively. So, $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(Z))$ is the symmetric pair by involution σ, and $\mathfrak{l} = \mathfrak{c}_{\mathfrak{l}}(Z) \oplus (\mathfrak{l}_{-1} \oplus \mathfrak{l}_{+1})$ is the canonical decomposition of $\mathfrak{l}$ with respect to σ. Since $(\operatorname{ad}_{\mathfrak{l}} Z)^2(Y) = Y$ for any $Y \in \mathfrak{l}_{-1} \oplus \mathfrak{l}_{+1}$, one sees that $I := \operatorname{ad}_{\mathfrak{l}} Z$ is a para-complex structure of the vector space $\mathfrak{l}_{-1} \oplus \mathfrak{l}_{+1}$. In addition, $B_{\mathfrak{l}}$ is a para-Hermitian form (with respect to I) on $\mathfrak{l}_{-1} \oplus \mathfrak{l}_{+1}$. Thus, $(\mathfrak{l}, \mathfrak{c}_{\mathfrak{l}}(Z))$ is a para-Hermitian symmetric pair with para-complex structure induced by $\operatorname{ad}_{\mathfrak{l}} Z$. □

2.3. Pseudo-Hermitian symmetric pairs, elliptic elements and involutions. Our aim in this subsection is to prove Proposition 2.10. For the aim, we first prove the following:

LEMMA 2.9. *Let $\mathfrak{l}$ be a real semisimple Lie algebra. Then, for any elliptic element $S \in \mathfrak{l}$, there exists an involutive automorphism η of $\mathfrak{l}$ satisfying $\eta(S) = -S$.*

PROOF. Since S is elliptic, there exists a maximal compact subalgebra $\mathfrak{k}'$ of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ such that $S \in \mathfrak{k}'$. Theorem 7.2 in Helgason [2, p. 183] assures that there exists an inner automorphism ϕ' of $\mathfrak{l}$ satisfying $\phi'(\mathfrak{k}') = \mathfrak{k}$; and thus $\phi'(S) \in \mathfrak{k}$. Moreover, there exists an element $K \in \mathfrak{k}$ such that $\exp \operatorname{ad}_{\mathfrak{l}} K(\phi'(S)) \in \mathfrak{k} \cap i\mathfrak{h}_{\boldsymbol{R}}$, because $\mathfrak{k}$ is a compact Lie algebra and $\mathfrak{k} \cap i\mathfrak{h}_{\boldsymbol{R}}$ is a maximal abelian subalgebra of $\mathfrak{k}$ (cf. Remark 2.4). Hence, there exists an inner automorphism ϕ of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$ such that $\phi(S) \in \mathfrak{k} \cap i\mathfrak{h}_{\boldsymbol{R}}$. We denote $\phi(S)$ by S'. Needless to say, $S' \in \mathfrak{k} \cap i\mathfrak{h}_{\boldsymbol{R}}$.

First, let us construct an involutive automorphism η' of $\mathfrak{l}_{\boldsymbol{C}}$ such that $\eta'(S') = -S'$. Let $\mathfrak{l}_n$ denote a normal real form of $\mathfrak{l}_{\boldsymbol{C}}$ given by

$$\mathfrak{l}_n = \mathfrak{h}_{\boldsymbol{R}} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})} \operatorname{span}_{\boldsymbol{R}}\{X_\alpha\}$$

(see the proof of Theorem 5.10 in Helgason [2, p. 426]), and let $\tilde{\nu}$ denote the conjugation of $\mathfrak{l}_{\boldsymbol{C}}$ with respect to $\mathfrak{l}_n$;

$$\tilde{\nu} : X + iY \mapsto X - iY \quad \text{for} \quad X + iY \in \mathfrak{l}_{\boldsymbol{C}} \ (= \mathfrak{l}_n \oplus i\mathfrak{l}_n).$$

Then, it is natural that $\tilde{\nu}(X_\alpha) = X_\alpha$ for each $\alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$, and $\tilde{\nu} = -\,\mathrm{id}$ on $i\mathfrak{h}_{\boldsymbol{R}}$. Hence, $\tilde{\nu}(\mathfrak{l}_u) \subset \mathfrak{l}_u$ comes from (2.2.1), and therefore

$$\tilde{\tau} \circ \tilde{\nu} = \tilde{\nu} \circ \tilde{\tau}\,,$$

where $\tilde{\tau}$ denotes the conjugation of $\mathfrak{l}_{\boldsymbol{C}}$ with respect to $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$;

$$\tilde{\tau} : Z + iW \mapsto Z - iW \quad \text{for} \quad Z + iW \in \mathfrak{l}_{\boldsymbol{C}} \ (= \mathfrak{l}_u \oplus i\mathfrak{l}_u).$$

Consequently, $\eta' := \tilde{\tau} \circ \tilde{\nu}$ is an involutive automorphism of $\mathfrak{l}_{\boldsymbol{C}}$, and it satisfies $\eta'(S') = -S'$ because $S' \in i\mathfrak{h}_{\boldsymbol{R}}$, $\tilde{\nu} = -\,\mathrm{id}$ on $i\mathfrak{h}_{\boldsymbol{R}}$ and $\tilde{\tau} = \mathrm{id}$ on $i\mathfrak{h}_{\boldsymbol{R}}$.

Next, we want to deduce that the involution η' satisfies the two conditions (a) and (b) in Proposition 2.5. From $\tilde{\nu}(\mathfrak{l}_u) \subset \mathfrak{l}_u$ and $\tilde{\tau} = \mathrm{id}$ on $\mathfrak{l}_u$, it is obvious that $\eta'(\mathfrak{l}_u) \subset \mathfrak{l}_u$, i.e., η' is an automorphism of $\mathfrak{l}_u = \mathfrak{k} \oplus \mathfrak{p}$. By virtue of $\theta(i\mathfrak{h}_{\boldsymbol{R}}) \subset i\mathfrak{h}_{\boldsymbol{R}}$ and $\eta' = -\,\mathrm{id}$ on $i\mathfrak{h}_{\boldsymbol{R}}$, the involution η' satisfies the condition (a);

$$\text{(2.3.1)} \qquad \eta'(i\mathfrak{h}_{\boldsymbol{R}}) \subset i\mathfrak{h}_{\boldsymbol{R}}, \quad \text{and} \quad \eta' \circ \theta = \theta \circ \eta' \quad \text{on } i\mathfrak{h}_{\boldsymbol{R}}.$$

Now, we verify that η' satisfies also the condition (b). For every root $\alpha \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$, one obtains ${}^t\eta'(\alpha) = -\alpha$ because $\eta' = -\,\mathrm{id}$ on $\mathfrak{h}_{\boldsymbol{C}} = \mathfrak{h}_{\boldsymbol{R}} \oplus i\mathfrak{h}_{\boldsymbol{R}}$. Therefore, it follows that ${}^t\eta'(\Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})) = \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$. Take any root $\beta \in \Delta(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}})$ such that ${}^t\theta(\beta) = \beta$ and $\theta(X_\beta) = X_\beta$. Since $\theta(X_{-\beta}) = X_{-\beta}$ (cf. Murakami [11, p. 113]), we have

$$\begin{cases} {}^t\theta({}^t\eta'(\beta)) = -{}^t\theta(\beta) = -\beta = {}^t\eta'(\beta)\,, \\ \theta(X_{{}^t\eta'(\beta)}) = \theta(X_{-\beta}) = X_{-\beta} = X_{{}^t\eta'(\beta)}\,. \end{cases}$$

So, the involution η' also satisfies the condition (b);

$$\text{(2.3.2)} \qquad {}^t\eta'(\Delta_1(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}} : \theta)) = \Delta_1(\mathfrak{l}_{\boldsymbol{C}}, \mathfrak{h}_{\boldsymbol{C}} : \theta)\,.$$

Accordingly, by (2.3.1), (2.3.2) and Proposition 2.5, there exists an element $H \in \mathfrak{h}_{\boldsymbol{R}}$ such that $\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH$ is an automorphism of $\mathfrak{l} = \mathfrak{k} \oplus i\mathfrak{p}$. Since $iH, S' \in i\mathfrak{h}_{\boldsymbol{R}}$, one has $[iH, S'] = 0$. This, together with $\eta'(S') = -S'$, shows that

$$(\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH)(S') = -S'\,.$$

Moreover, $\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH$ is involutive. Indeed, it follows from $iH \in i\mathfrak{h}_{\boldsymbol{R}}$ that $\eta'(iH) = -iH$. Therefore, we confirm that

$$\begin{aligned}(\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH) \circ (\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH) &= \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} \eta'(iH) \circ \eta' \circ \eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH \\ &= \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} \eta'(iH) \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH \\ &= \mathrm{id}\end{aligned}$$

since $(\eta')^2 = \mathrm{id}$. Hence, $\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH$ is an involutive automorphism of $\mathfrak{l}$ such that $(\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH)(S') = -S'$. Consequently, $\eta := \phi^{-1} \circ (\eta' \circ \exp \mathrm{ad}_{\mathfrak{l}_{\boldsymbol{C}}} iH) \circ \phi$ is an involutive automorphism of $\mathfrak{l}$ which satisfies $\eta(S) = -S$. □

Now, we are in a position to prove Proposition 2.10.

PROPOSITION 2.10. *Let $\mathfrak{g}_{\boldsymbol{C}}$ be a complex simple Lie algebra. Then, for any real form $\mathfrak{g}$ of $\mathfrak{g}_{\boldsymbol{C}}$ and pseudo-Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{r})$ with complex structure J, there exist an elliptic element $S \in \mathfrak{g}$, a Cartan involution θ of $\mathfrak{g}$ and an involutive automorphism η of $\mathfrak{g}$ such that*

(i) $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(S)$,
(ii) *J is induced by* $\mathrm{ad}_{\mathfrak{g}} S$,
(iii) $\theta(S) = S$, $\eta(S) = -S$ *and* $\eta \circ \theta = \theta \circ \eta$.

PROOF. By the results of Shapiro [16, p. 533–534], one knows that there exists an elliptic element $S \in \mathfrak{g}$ such that (i) $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(S)$ and (ii) J is induced by $\mathrm{ad}_{\mathfrak{g}} S$; in addition, one also knows that $\rho := \exp \pi\, \mathrm{ad}_{\mathfrak{g}} S$ is an involutive automorphism of $\mathfrak{g}$, and $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(S)$ is the $+1$-eigenspace of ρ in $\mathfrak{g}$. There exists an involutive automorphism η of $\mathfrak{g}$ which satisfies $\eta(S) = -S$ by Lemma 2.9. Since $\rho = \exp \pi\, \mathrm{ad}_{\mathfrak{g}} S$ is involutive and $\eta(S) = -S$, we perceive that ρ is commutative with η. So, Lemma 2.7 allows us to get a Cartan involution θ of $\mathfrak{g}$ satisfying $\theta \circ \rho = \rho \circ \theta$ and $\eta \circ \theta = \theta \circ \eta$.

The rest of proof is to show that $\theta(S) = S$. Henceforth, we will devote ourselves to showing that $\theta(S) = S$. From $\theta \circ \rho = \rho \circ \theta$ and $\mathfrak{c}_{\mathfrak{g}}(S)$ being the $+1$-eigenspace of ρ, it follows that $\theta(\mathfrak{c}_{\mathfrak{g}}(S)) = \mathfrak{c}_{\mathfrak{g}}(S)$, and hence

$$\theta(\mathfrak{c}_{\mathfrak{g}}(S)_{\mathrm{z}}) = \mathfrak{c}_{\mathfrak{g}}(S)_{\mathrm{z}}\,.$$

Here, $\mathfrak{c}_{\mathfrak{g}}(S)_{\mathrm{z}}$ denotes the center of $\mathfrak{c}_{\mathfrak{g}}(S)$. Accordingly, there exists a non-zero number $\lambda \in \boldsymbol{R}$ satisfying

$$\theta(S) = \lambda \cdot S$$

because $\dim_{\boldsymbol{R}} \mathfrak{c}_{\mathfrak{g}}(S)_{\mathrm{z}} = 1$ (cf. Corollary 2.3 in Shapiro [16, p. 532]). Since $\theta^2 = \mathrm{id}$ and $S \neq 0$, one has $\lambda = 1$ or -1. This yields $\theta(S) = S$ or $-S$. Hence, we deduce that $\theta(S) = S$, because θ is a Cartan involution of $\mathfrak{g}$ and S is a non-zero elliptic element of $\mathfrak{g}$. □

REMARK 2.11. The element S in Proposition 2.10 is a non-zero, semisimple element of $\mathfrak{g}$ such that the eigenvalue of $\mathrm{ad}_{\mathfrak{g}} S$ is $\pm i$ or zero.

2.4. Para-Hermitian symmetric pairs, hyperbolic elements and involutions. Lemma 2.1 in Kaneyuki [3] and its proof enable us to get the following proposition which we need later.

PROPOSITION 2.12 (Kaneyuki [3, p. 477–478]). *Let $\mathfrak{g}_{\boldsymbol{C}}$ be a complex simple Lie algebra. Then, for any real form $\mathfrak{g}$ of $\mathfrak{g}_{\boldsymbol{C}}$ and para-Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{h})$ with para-complex structure I, there exist a hyperbolic element $Z \in \mathfrak{g}$, a Cartan involution τ of $\mathfrak{g}$ and an involutive automorphism σ of $\mathfrak{g}$ such that*

(i) $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(Z)$,
(ii) *I is induced by* $\mathrm{ad}_{\mathfrak{g}} Z$,
(iii) $\tau(Z) = -Z$, $\sigma(Z) = Z$ *and* $\sigma \circ \tau = \tau \circ \sigma$.

REMARK 2.13. The element Z in Proposition 2.12 is a non-zero semisimple element of $\mathfrak{g}$ such that the eigenvalue of $\mathrm{ad}_{\mathfrak{g}} Z$ is ± 1 or zero.

3. Proof of Theorem 1.1. In this section, we will demonstrate Theorem 1.1 in Section 1. In order to do so, we show the following:

PROPOSITION 3.1. *Let $\mathfrak{g}_{\boldsymbol{C}}$ be a complex simple Lie algebra, let $\mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}}$ denote the set of quartets $(\mathfrak{g}, S, \theta, \eta)$ such that*

(1) *$\mathfrak{g}$ is a real form of $\mathfrak{g}_{\boldsymbol{C}}$,*

(2) *S is a non-zero semisimple element of $\mathfrak{g}$ such that the eigenvalue of $\mathrm{ad}_{\mathfrak{g}} S$ is $\pm i$ or zero,*

(3) *θ is a Cartan involution of $\mathfrak{g}$ which satisfies $\theta(S) = S$,*

(4) *η is an involutive automorphism of $\mathfrak{g}$ such that $\eta(S) = -S$ and $\eta \circ \theta = \theta \circ \eta$; and let $\mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$ denote the set of quartets $(\bar{\mathfrak{g}}, \bar{Z}, \bar{\tau}, \bar{\sigma})$ such that*

(i) *$\bar{\mathfrak{g}}$ is a real form of $\mathfrak{g}_{\boldsymbol{C}}$,*

(ii) *$\bar{Z}$ is a non-zero semisimple element of $\bar{\mathfrak{g}}$ such that the eigenvalue of $\mathrm{ad}_{\bar{\mathfrak{g}}} \bar{Z}$ is ± 1 or zero,*

(iii) *$\bar{\tau}$ is a Cartan involution of $\bar{\mathfrak{g}}$ which satisfies $\bar{\tau}(\bar{Z}) = -\bar{Z}$,*

(iv) *$\bar{\sigma}$ is an involutive automorphism of $\bar{\mathfrak{g}}$ such that $\bar{\sigma}(\bar{Z}) = \bar{Z}$ and $\bar{\sigma} \circ \bar{\tau} = \bar{\tau} \circ \bar{\sigma}$.*

Then, the following mapping F is a bijection of $\mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}}$ onto $\mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$:

$$F: \begin{array}{ccc} \mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}} & \longrightarrow & \mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}} \\ \Downarrow & & \Downarrow \\ (\mathfrak{g}, S, \theta, \eta) & \mapsto & (\mathfrak{g}^d, iS, \eta, \theta)\,. \end{array} \quad \text{(bijective)}$$

Here, $(\mathfrak{g}^d, \theta)$ is the Berger dual symmetric pair of $(\mathfrak{g}, \eta)$.

PROOF. First, let us confirm that, for any $(\mathfrak{g}, S, \theta, \eta) \in \mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}}$, the quartet $(\mathfrak{g}^d, iS, \eta, \theta)$ belongs to $\mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ (resp. $\mathfrak{h}$ and $\mathfrak{m}$) denote the $+1$ and -1-eigenspaces of θ (resp. η) in $\mathfrak{g}$, respectively. Then, $\mathfrak{g}^d$ is a real form of $\mathfrak{g}_{\boldsymbol{C}}$ given by

$$\mathfrak{g}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{k} \cap \mathfrak{m}) \oplus i(\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{m})\,,$$

because $(\mathfrak{g}^d, \theta)$ is the Berger dual symmetric pair of $(\mathfrak{g}, \eta)$ (cf. Oshima-Sekiguchi [15, p. 435–436]). Notice that η is a Cartan involution of $\mathfrak{g}^d$ (cf. Oshima-Sekiguchi [15, p. 435]), where η is extended to $\mathfrak{g}_{\boldsymbol{C}}$ as $\boldsymbol{C}$-linear involution. From $\theta(S) = S$ and $\eta(S) = -S$, we have $iS \in i(\mathfrak{k} \cap \mathfrak{m}) \subset \mathfrak{g}^d$. Naturally, iS is a non-zero semisimple element of $\mathfrak{g}^d$ such that the eigenvalue of $\mathrm{ad}_{\mathfrak{g}^d} iS$ is ± 1 or zero. It is obvious that $\eta(iS) = -iS$ and $\theta(iS) = iS$, where θ is also extended to $\mathfrak{g}_{\boldsymbol{C}}$ as $\boldsymbol{C}$-linear involution. Consequently, by virtue of $\eta \circ \theta = \theta \circ \eta$ we deduce that the quartet $(\mathfrak{g}^d, iS, \eta, \theta)$ belongs to $\mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$. This means that $F((\mathfrak{g}, S, \theta, \eta)) \in \mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$ for every $(\mathfrak{g}, S, \theta, \eta) \in \mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}}$.

In a similar way, we can see that, for any $(\bar{\mathfrak{g}}, \bar{Z}, \bar{\tau}, \bar{\sigma}) \in \mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$, a quartet $(\bar{\mathfrak{g}}^d, -i\bar{Z}, \bar{\sigma}, \bar{\tau})$ belongs to $\mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}}$. Here, $\bar{\mathfrak{g}}^d$ denotes a real form of $\mathfrak{g}_{\boldsymbol{C}}$ such that $(\bar{\mathfrak{g}}^d, \bar{\tau})$ is the Berger dual symmetric pair of $(\bar{\mathfrak{g}}, \bar{\sigma})$. Accordingly, one gets a mapping F' of $\mathcal{H}_{\mathfrak{g}_{\boldsymbol{C}}}$ into $\mathcal{E}_{\mathfrak{g}_{\boldsymbol{C}}}$ defined by

$F' : (\bar{\mathfrak{g}}, \bar{Z}, \bar{\tau}, \bar{\sigma}) \mapsto (\bar{\mathfrak{g}}^d, -i\bar{Z}, \bar{\sigma}, \bar{\tau})$. It is natural that $F \circ F' = \mathrm{id}_{\mathcal{H}_{\mathfrak{g}_C}}$ and $F' \circ F = \mathrm{id}_{\mathcal{E}_{\mathfrak{g}_C}}$. Hence, F is a bijection of $\mathcal{E}_{\mathfrak{g}_C}$ onto $\mathcal{H}_{\mathfrak{g}_C}$. □

From now on, let us demonstrate Theorem 1.1.

PROOF OF THEOREM 1.1. (I): Let us prove the first item (I). Let $\mathfrak{g}$ be a real form $\mathfrak{g}_C$, and let $(\mathfrak{g}, \mathfrak{r})$ be a pseudo-Hermitian symmetric pair with complex structure J. Proposition 2.10 assures that there exist an elliptic element $S \in \mathfrak{g}$, a Cartan involution θ of $\mathfrak{g}$ and an involutive automorphism η of $\mathfrak{g}$ such that

(i) $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(S)$,
(ii) J is induced by $\mathrm{ad}_{\mathfrak{g}} S$,
(iii) $\theta(S) = S$, $\eta(S) = -S$ and $\eta \circ \theta = \theta \circ \eta$.

Therefore, it suffices to deduce that $(\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$ is a para-Hermitian symmetric pair with para-complex structure induced by $\mathrm{ad}_{\mathfrak{g}^d} iS$. Here, $(\mathfrak{g}^d, \theta)$ is the Berger dual symmetric pair of $(\mathfrak{g}, \eta)$;

$$\mathfrak{g}^d = (\mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{k} \cap \mathfrak{m}) \oplus i(\mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{m}),$$

where $\mathfrak{k}$ and $\mathfrak{p}$ (resp. $\mathfrak{h}$ and $\mathfrak{m}$) denote the $+1$ and -1-eigenspaces of θ (resp. η) in $\mathfrak{g}$, respectively. It is clear that $iS \in i(\mathfrak{k} \cap \mathfrak{m}) \subset \mathfrak{g}^d$. Besides, by Remark 2.11, iS is a non-zero semisimple element of $\mathfrak{g}^d$ such that the eigenvalue of $\mathrm{ad}_{\mathfrak{g}^d} iS$ is ± 1 or zero. Consequently, $(\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$ is a para-Hermitian symmetric pair with para-complex structure induced by $\mathrm{ad}_{\mathfrak{g}^d} iS$ (cf. Lemma 2.8-(b)).

(II): Let $\bar{\mathfrak{g}}$ be a real form $\mathfrak{g}_C$, and let $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ be a para-Hermitian symmetric pair with para-complex structure $\bar{I}$. Then, Proposition 2.12 implies that there exist a hyperbolic element $\bar{Z} \in \bar{\mathfrak{g}}$, a Cartan involution $\bar{\tau}$ of $\bar{\mathfrak{g}}$, and an involutive automorphism $\bar{\sigma}$ of $\bar{\mathfrak{g}}$ such that

(i) $\bar{\mathfrak{b}} = \mathfrak{c}_{\bar{\mathfrak{g}}}(\bar{Z})$,
(ii) $\bar{I}$ is induced by $\mathrm{ad}_{\bar{\mathfrak{g}}} \bar{Z}$,
(iii) $\bar{\tau}(\bar{Z}) = -\bar{Z}$, $\bar{\sigma}(\bar{Z}) = \bar{Z}$ and $\bar{\sigma} \circ \bar{\tau} = \bar{\tau} \circ \bar{\sigma}$.

Thus by Remark 2.13 and Proposition 3.1 for $\mathcal{H}_{\mathfrak{g}_C}$, we deduce that the quartet $(\bar{\mathfrak{g}}, \bar{Z}, \bar{\tau}, \bar{\sigma})$ belongs to $\mathcal{H}_{\mathfrak{g}_C}$. Proposition 3.1 enables us to obtain an element $(\mathfrak{g}, S, \theta, \eta) \in \mathcal{E}_{\mathfrak{g}_C}$ such that $(\mathfrak{g}^d, iS, \eta, \theta) = (\bar{\mathfrak{g}}, \bar{Z}, \bar{\tau}, \bar{\sigma})$. Here, $(\mathfrak{g}^d, \theta)$ is the Berger dual symmetric pair of $(\mathfrak{g}, \eta)$. From the definition of $\mathcal{E}_{\mathfrak{g}_C}$, it follows that (1) $\mathfrak{g}$ is a real form of $\mathfrak{g}_C$, (2) S is an elliptic element of $\mathfrak{g}$, (3) θ is a Cartan involution of $\mathfrak{g}$ which satisfies $\theta(S) = S$ and (4) η is an involutive automorphism of $\mathfrak{g}$ which satisfies $\eta(S) = -S$ and $\eta \circ \theta = \theta \circ \eta$. Since $(\bar{\mathfrak{g}}, \bar{Z}) = (\mathfrak{g}^d, iS)$, the rest of proof is to confirm that $(\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(S))$ is a pseudo-Hermitian symmetric pair with complex structure induced by $\mathrm{ad}_{\mathfrak{g}} S$. However, that is confirmed, because the element S is a non-zero semisimple element of $\mathfrak{g}$ and the eigenvalue of $\mathrm{ad}_{\mathfrak{g}} S$ is $\pm i$ or zero (see Lemma 2.8-(a)). Hence the second item (II) holds, too. □

4. Application. In 1979, Leung [10, p. 182] has determined Lagrangian reflective submanifolds of irreducible Hermitian symmetric spaces. By use of his results, we will determine the para-Hermitian symmetric pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ which a (pseudo-)Hermitian symmetric pair

$(\mathfrak{g}, \mathfrak{r})$ brings about by means of Theorem 1.1-(I) (see Theorem 4.6 and Remark 4.4). For the goal, we first prove the following:

LEMMA 4.1. *Let* $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}}) = (\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$ *be the para-Hermitian symmetric pair which a pseudo-Hermitian symmetric pair* $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{g}, \mathfrak{c}_{\mathfrak{g}}(S))$ *and two involutions* $\theta, \eta \in \mathrm{Aut}(\mathfrak{g})$ *bring about by means of Theorem* 1.1-(I). *Then,* $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ *is given as follows*:

(i) $(\bar{\mathfrak{g}}, \theta)$ *is the Berger dual symmetric pair of* $(\mathfrak{g}, \eta)$;

(ii) $\bar{\mathfrak{b}} = (\mathfrak{r}_{\mathrm{ss}})^d \oplus \boldsymbol{R}$, *where* $((\mathfrak{r}_{\mathrm{ss}})^d, \theta')$ *is the Berger dual symmetric pair of* $(\mathfrak{r}_{\mathrm{ss}}, \eta')$.

Here, $\mathfrak{r}_{\mathrm{ss}}$ *denotes the semisimple part of* $\mathfrak{r}$, *and* $\theta' := \theta|_{\mathfrak{r}_{\mathrm{ss}}}$ *(resp.* $\eta' := \eta|_{\mathfrak{r}_{\mathrm{ss}}}$*).*

REMARK 4.2. Let $\mathfrak{h}$ denote the $+1$-eigenspace of η in $\mathfrak{g}$. By Lemma 4.1, we can completely determine $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ by using three structures of $(\mathfrak{g}, \mathfrak{r})$, $\mathfrak{h}$ and $\mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{h}$. Indeed, $\bar{\mathfrak{g}}$ is determined by the Berger dual symmetric pair of $(\mathfrak{g}, \mathfrak{h})$. Furthermore, $(\mathfrak{r}_{\mathrm{ss}})^d$ is determined by the Berger dual symmetric pair of $(\mathfrak{r}_{\mathrm{ss}}, \mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{h})$, and $\bar{\mathfrak{b}}$ is given by $\bar{\mathfrak{b}} = (\mathfrak{r}_{\mathrm{ss}})^d \oplus \boldsymbol{R}$. Here, we remark that Oshima-Sekiguchi [15] tables Berger's dual symmetric pairs, where there are some minor misprints in [15] (cf. [5, p. 660]).

PROOF OF LEMMA 4.1. The first item (i) is obvious (see Theorem 1.1-(I)). So, we only show the second item (ii). Since $\bar{\mathfrak{b}}$ is reductive, it is decomposed as follows:

$$\bar{\mathfrak{b}} = \bar{\mathfrak{b}}_{\mathrm{ss}} \oplus \bar{\mathfrak{b}}_{\mathrm{z}},$$

where $\bar{\mathfrak{b}}_{\mathrm{ss}}$ and $\bar{\mathfrak{b}}_{\mathrm{z}}$ denote the semisimple part and the center of $\bar{\mathfrak{b}}$, respectively. Since $\bar{\mathfrak{g}}$ is a real form of $\mathfrak{g}_{\boldsymbol{C}}$ and $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}}) = (\mathfrak{g}^d, \mathfrak{c}_{\mathfrak{g}^d}(iS))$ is para-Hermitian, Koh [7, p. 304 Lemma I and p. 306 Theorem 6] allows us to have

$$\bar{\mathfrak{b}}_{\mathrm{z}} = \boldsymbol{R}.$$

Therefore, the rest of proof is to deduce that $\bar{\mathfrak{b}}_{\mathrm{ss}} = (\mathfrak{r}_{\mathrm{ss}})^d$. From $\theta(S) = S$, $\eta(S) = -S$ and $\mathfrak{r} = \mathfrak{c}_{\mathfrak{g}}(S)$, it follows that $\theta(\mathfrak{r}) \subset \mathfrak{r}$ and $\eta(\mathfrak{r}) \subset \mathfrak{r}$. This, combined with $\mathfrak{r}_{\mathrm{ss}} = [\mathfrak{r}, \mathfrak{r}]$, implies that $\theta(\mathfrak{r}_{\mathrm{ss}}) \subset \mathfrak{r}_{\mathrm{ss}}$ and $\eta(\mathfrak{r}_{\mathrm{ss}}) \subset \mathfrak{r}_{\mathrm{ss}}$. Thus, $\theta' = \theta|_{\mathfrak{r}_{\mathrm{ss}}}$ is a Cartan involution of $\mathfrak{r}_{\mathrm{ss}}$ and $\eta' = \eta|_{\mathfrak{r}_{\mathrm{ss}}}$ is an involutive automorphism of $\mathfrak{r}_{\mathrm{ss}}$. Naturally, $\eta' \circ \theta' = \theta' \circ \eta'$ comes from $\eta \circ \theta = \theta \circ \eta$. Now, let us consider the semisimple Lie algebra $(\mathfrak{r}_{\mathrm{ss}})^d$. Let $\mathfrak{k}$ and $\mathfrak{p}$ (resp. $\mathfrak{h}$ and $\mathfrak{m}$) denote the $+1$ and -1-eigenspaces of θ (resp. η) in $\mathfrak{g}$, respectively. Then, one has

$$\begin{aligned}(\mathfrak{r}_{\mathrm{ss}})^d =& (\mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{k} \cap \mathfrak{m}) \oplus i(\mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{p} \cap \mathfrak{m}) \\ =& ([\mathfrak{c}_{\mathfrak{g}}(S), \mathfrak{c}_{\mathfrak{g}}(S)] \cap \mathfrak{k} \cap \mathfrak{h}) \oplus i([\mathfrak{c}_{\mathfrak{g}}(S), \mathfrak{c}_{\mathfrak{g}}(S)] \cap \mathfrak{k} \cap \mathfrak{m}) \\ & \oplus i([\mathfrak{c}_{\mathfrak{g}}(S), \mathfrak{c}_{\mathfrak{g}}(S)] \cap \mathfrak{p} \cap \mathfrak{h}) \oplus ([\mathfrak{c}_{\mathfrak{g}}(S), \mathfrak{c}_{\mathfrak{g}}(S)] \cap \mathfrak{p} \cap \mathfrak{m}) \\ =& [\mathfrak{c}_{\mathfrak{g}^d}(iS), \mathfrak{c}_{\mathfrak{g}^d}(iS)] \\ =& \bar{\mathfrak{b}}_{\mathrm{ss}},\end{aligned}$$

because $((\mathfrak{r}_{\mathrm{ss}})^d, \theta')$ is the Berger dual symmetric pair of $(\mathfrak{r}_{\mathrm{ss}}, \eta')$ and $\bar{\mathfrak{b}} = \mathfrak{c}_{\mathfrak{g}^d}(iS) = (\mathfrak{c}_{\mathfrak{g}}(S) \cap \mathfrak{k} \cap \mathfrak{h}) \oplus i(\mathfrak{c}_{\mathfrak{g}}(S) \cap \mathfrak{k} \cap \mathfrak{m}) \oplus i(\mathfrak{c}_{\mathfrak{g}}(S) \cap \mathfrak{p} \cap \mathfrak{h}) \oplus (\mathfrak{c}_{\mathfrak{g}}(S) \cap \mathfrak{p} \cap \mathfrak{m})$. Hence, (ii) is also proved. □

Leung [10, p. 182] determines Lagrangian reflective submanifolds of irreducible Hermitian symmetric spaces by selecting them from reflective submanifolds in his previous papers

[8, 9]. Furthermore, he determines reflective submanifolds in [8, 9], by using Table II in Berger [1, p. 157–161]. Considering Berger's process of getting Table II, we can assert the following:

LEMMA 4.3. *Let G/R be an irreducible Hermitian symmetric space of non-compact type* (*resp. compact type*), *let L be a Lagrangian reflective submanifold of G/R determined by* Leung [10, p. 182], *let θ denote the Cartan involution of $\mathfrak{g}$ such that $\mathfrak{r} = \{X \in \mathfrak{g}\,;\, \theta(X) = X\}$* (*resp. $\theta = \mathrm{id}$*), *and let η denote the involutive automorphism of $\mathfrak{g}$ inducing L, where $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{r} := \mathrm{Lie}(R)$. Then, θ and η satisfy the following two conditions*:

(1) *$\theta(S) = S$, $\eta(S) = -S$ and $\eta \circ \theta = \theta \circ \eta$*;

(2) *T_oL is isomorphic to the coset vector space $\mathfrak{h}/(\mathfrak{r} \cap \mathfrak{h})$.*

Here, we denote by S any central element of $\mathfrak{r}$, denote by $\mathfrak{h}$ the $+1$-eigenspace of η in $\mathfrak{g}$, and denote by T_oL the tangent space of L at the origin.

REMARK 4.4. Theorem 1.1-(I) enables us to obtain a para-Hermitian symmetric pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ by using a pseudo-Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{r})$ and two involutions $\theta, \eta \in \mathrm{Aut}(\mathfrak{g})$. So, both θ and η are required in the determination of $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$. However, Lemma 4.3 implies that L can be substituted for η, and the involution whose $+1$-eigenspace coincides with $\mathfrak{r}$ (resp. the identity mapping) can be substituted for θ, in the case where $(\mathfrak{g}, \mathfrak{r})$ is non-compact (resp. compact) Hermitian. For these reasons, $(\mathfrak{g}, \mathfrak{r})$ and L bring about a para-Hermitian symmetric pair by means of Theorem 1.1-(I), if $(\mathfrak{g}, \mathfrak{r})$ is Hermitian.

Now, let us explain how to determine the para-Hermitian symmetric pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ which a Hermitian symmetric pair $(\mathfrak{g}, \mathfrak{r})$ and L bring about by means of Theorem 1.1-(I). Here, L is a Lagrangian reflective submanifold of G/R determined by Leung [10, p. 182], $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{r} = \mathrm{Lie}(R)$.

EXAMPLE 4.5 (Case $(\mathfrak{g}, \mathfrak{r}) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 \oplus \mathfrak{t})$ and $L = (E_{6(-26)}/F_4) \times \boldsymbol{R}$). Let $(\mathfrak{g}, \mathfrak{r}) := (\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 \oplus \mathfrak{t})$. Leung [10, p. 182] shows that $L := (E_{6(-26)}/F_4) \times \boldsymbol{R}$ is a Lagrangian reflective submanifold of $G/R = E_{7(-25)}/(E_6 \times T)$. We are going to determine the para-Hermitian symmetric pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ which $(\mathfrak{g}, \mathfrak{r})$ and L bring about by means of Theorem 1.1-(I). In terms of $L = (E_{6(-26)}/F_4) \times \boldsymbol{R}$ and Lemma 4.3, one comprehends that

$$\mathfrak{h}/(\mathfrak{r} \cap \mathfrak{h}) = (\mathfrak{e}_{6(-26)}/\mathfrak{f}_4) \oplus \boldsymbol{R}\,. \tag{4.0.1}$$

Here and hereafter, we utilize the same notation in Lemma 4.3. Then, Table II in Berger [1, p. 157–161] enables us to obtain

$$\mathfrak{h} = \mathfrak{e}_{6(-26)} \oplus \boldsymbol{R} \tag{4.0.2}$$

since $(\mathfrak{g}, \mathfrak{h})$ is a symmetric pair and satisfies (4.0.1). Therefore from (4.0.1), it is easy to see that $\mathfrak{r} \cap \mathfrak{h} = \mathfrak{f}_4$. That yields

$$\mathfrak{r}_{\mathrm{ss}} \cap \mathfrak{h} = \mathfrak{f}_4 \tag{4.0.3}$$

since $\mathfrak{r}_{ss} = \mathfrak{e}_6$ and $(\mathfrak{r}_{ss}, \mathfrak{r}_{ss} \cap \mathfrak{h})$ is a symmetric pair. Accordingly, Remark 4.2, together with (4.0.2) and (4.0.3), implies that $(\mathfrak{g}, \mathfrak{r})$ and L bring about a para-Hermitian symmetric pair

$$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}}) = (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \boldsymbol{R})$$

by means of Theorem 1.1-(I) (recall Remark 4.4).

In a similar way, we deduce the following (recall Remark 4.4 again):

THEOREM 4.6. *By means of Theorem* 1.1-(I), *a Hermitian symmetric pair* $(\mathfrak{g}, \mathfrak{r})$ *and* L *bring about the following para-Hermitian symmetric pair* $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$. *Here*, L *denotes a Lagrangian reflective submanifold of* G/R *determined by Leung* [10, p. 182], $\mathfrak{g} = \mathrm{Lie}(G)$ *and* $\mathfrak{r} = \mathrm{Lie}(R)$.

Compact type		
1	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{su}(n+m), \mathfrak{su}(n) \oplus \mathfrak{su}(m) \oplus \mathfrak{t})$, $n \geq m \geq 1$
	L	$SO(n+m)/(SO(n) \times SO(m))$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{sl}(n+m, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{R}) \oplus \mathfrak{sl}(m, \boldsymbol{R}) \oplus \boldsymbol{R})$
2	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{su}(2n+2m), \mathfrak{su}(2n) \oplus \mathfrak{su}(2m) \oplus \mathfrak{t})$, $n \geq m \geq 1$
	L	$Sp(n+m)/(Sp(n) \times Sp(m))$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{su}^*(2n+2m), \mathfrak{su}^*(2n) \oplus \mathfrak{su}^*(2m) \oplus \boldsymbol{R})$
3	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{su}(2p), \mathfrak{su}(p) \oplus \mathfrak{su}(p) \oplus \mathfrak{t})$, $p \geq 2$
	L	$U(p)$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{su}(p, p), \mathfrak{sl}(p, \boldsymbol{C}) \oplus \boldsymbol{R})$
4	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{so}(q+2), \mathfrak{so}(q) \oplus \mathfrak{t})$, $q \geq 3$
	L	$(SO(k+1)/SO(k)) \times (SO(q-k+1)/SO(q-k))$, $1 \leq k \leq [q/2]$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{so}(k+1, q-k+1), \mathfrak{so}(k, q-k) \oplus \boldsymbol{R})$
5	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{so}(p+2), \mathfrak{so}(p) \oplus \mathfrak{t})$, $1 \leq p$ and $p \neq 2$
	L	$SO(p+1)/SO(p)$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{so}(1, p+1), \mathfrak{so}(p) \oplus \boldsymbol{R})$
6	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{so}(2n), \mathfrak{su}(n) \oplus \mathfrak{t})$, $n \geq 3$
	L	$SO(n)$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{so}(n, n), \mathfrak{sl}(n, \boldsymbol{R}) \oplus \boldsymbol{R})$
7	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{so}(4n), \mathfrak{su}(2n) \oplus \mathfrak{t})$, $n \geq 3$
	L	$(SU(2n)/Sp(n)) \times T$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{so}^*(4n), \mathfrak{su}^*(2n) \oplus \boldsymbol{R})$

Compact type		
8	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{sp}(n), \mathfrak{su}(n) \oplus \mathfrak{t}), n \geq 3$
	L	$(SU(n)/SO(n)) \times T$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{sp}(n, \boldsymbol{R}), \mathfrak{sl}(n, \boldsymbol{R}) \oplus \boldsymbol{R})$
9	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{sp}(2m), \mathfrak{su}(2m) \oplus \mathfrak{t}), m \geq 2$
	L	$Sp(m)$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{sp}(m, m), \mathfrak{su}^*(2m) \oplus \boldsymbol{R})$
10	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{e}_6, \mathfrak{so}(10) \oplus \mathfrak{t})$
	L	$F_4/SO(9)$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{e}_{6(-26)}, \mathfrak{so}(1, 9) \oplus \boldsymbol{R})$
11	$(\mathfrak{g}, \mathfrak{r})$	the same as $(\mathfrak{g}, \mathfrak{r})$ in the above 10-th item
	L	$Sp(4)/(Sp(2) \times Sp(2))$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{e}_{6(6)}, \mathfrak{so}(5, 5) \oplus \boldsymbol{R})$
12	$(\mathfrak{g}, \mathfrak{r})$	$(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{t})$
	L	$SU(8)/Sp(4)$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{e}_{7(7)}, \mathfrak{e}_{6(6)} \oplus \boldsymbol{R})$
13	$(\mathfrak{g}, \mathfrak{r})$	the same as $(\mathfrak{g}, \mathfrak{r})$ in the above 12-th item
	L	$(E_6/F_4) \times T$
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \oplus \boldsymbol{R})$
Non-compact type		
$1 \leq j \leq 13$	$(\mathfrak{g}, \mathfrak{r})$	the non-compact dual of $(\mathfrak{g}, \mathfrak{r})$ in the above j-th item
	L	the non-compact dual of L in the above j-th item
	$(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$	the same as $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ in the above j-th item

REMARK 4.7. Theorem 4.6 gives us all para-Hermitian symmetric pairs $(\bar{\mathfrak{g}}, \bar{\mathfrak{b}})$ on the list of Kaneyuki-Kozai [4, p. 97], in the case where $\bar{\mathfrak{g}}$ are real forms of complex simple Lie algebras.

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