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On Relativistic Statistical Thermodynamics

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It is shown that, on the basis of the principle of maximum entropy, statistical thermodynamics can be constructed in as general and unambiguous manner for relativistic system as for non-relativistic systems.

§ 1. Introduction

Since H. Ott[1] criticized the traditional form of relativistic thermodynamics,[2] a large number of mutually contradicting papers[3] have been published concerning this old subject. Most of them are based on phenomenological arguments of classical thermodynamics. Thus, one starts with the second law which relates the change of entropy with the heat supplied. In relativistic thermodynamics, however, the concept of mechanical work and therefore the concept of heat, defined as the energy change subtracted by the mechanical work, are not self-evident. This is why we have such controversy.

No ambiguity can arise, on the other hand, if one properly applies statistical mechanics to relativistic systems. There are two possible ways of deriving thermodynamics. One is similar to the usual derivation of the Maxwell distribution as a particular solution of the Boltzmann kinetic equation. Thus, Balescu[4] has tried to derive thermodynamics from the relativistic kinetic theory developed by himself and Kotera.[5]

If we restrict ourselves to systems in thermal equilibrium, however, we can apply the much more general method of Gibbs. Thus, the system in equilibrium, whether it is relativistic or not, is characterized by the variational principle that the entropy expressed in terms of the phase space distribution function in the classical case, or in terms of the statistical operator (density matrix) in the quantum case shall be maximum under appropriate subsidiary conditions. This principle of maximum entropy is, of course, equivalent to the ergodic theorem and all thermodynamical relations should be derived from it without going into the detail of kinetic theory.

The idea that relativistic thermodynamics should be based on this principle of maximum entropy has been proposed by a number of authors; by Bφrs,[6] Sourian,[7] and Landsberg[8] in classical language and by Eberly and Kujawski[9] in quantum language. In the present note, by summarizing the basic concepts and adding a few thermodynamical arguments, it will be shown that statistical thermodynamics can be constructed on the basis of the principle of maximum entropy in as general and unambiguous a manner for relativistic systems as for non-relativistic systems.

§ 2. The basic formulation

For the sake of logical completeness we start with a brief description of the principle of maximum entropy, although much of the content of this section can be found in preceeding papers.[6)~9)] Let us take a system of quantized fields; the relativistic interaction between particles should be described as field and in quantum theory particles may also be regarded as field. Throughout the present note, we adopt the Heisenberg picture.

Then, our system of quantized fields, interacting with each other in general, is described by a set of linear operators $\phi_\alpha(x)$, each of which is defined at each point $x_\mu=(x, it)$ of the space-time world and operates on state vectors of a Hilbert space. Note that we take $c=1$ and $\hbar=1$. Operators $\phi_\alpha(x)$ obey the field equations, while the statistical distribution of the system is represented by the statistical operator ρ, which is independent of x_μ. The average of any dynamical variable A is given by the well-known formula

$$\langle A\rangle = \mathrm{Tr}(\rho A). \qquad (2\cdot 1)$$

When we go over from one inertial frame of reference to another by an inhomogeneous Lorentz transformation, our system of fields is described by a new set of operators, which are obtained from old operators by a unitary transformation of the Hilbert space of state vectors. In particular, when the transformation is infinitesimal, the generating operators are the energy-momentum vector P_μ and the angular momentum tensor $M_{\mu\nu}$. On the other hand, in the Heisenberg picture, the statistical operator ρ remains invariant, so that the average (2·1) exhibits the same covariant property as the dynamical variable A under the Lorentz transformation.

Now, let us restrict ourselves to systems in thermal equilibrium. Then we have the variational principle that the entropy defined by

$$S = -\mathrm{Tr}(\rho \log \rho) \qquad (2\cdot 2)$$

shall be maximum. The principle is equivalent to the ergodic theorem, on which statistical thermodynamics of non-relativistic systems is constructed. There is no reason why we should doubt its validity for relativistic systems once we admit the establishment of thermal equilibrium.

Since the entropy (2·2) is obviously invariant, we may apply the principle of maximum entropy in any inertial frame of reference. Note, however, that we have a number of subsidiary conditions, which are usually covariant. Depending on subsidiary conditions we impose on ρ, we obtain various equilibrium distributions which are practically equivalent to each other.

For example, the microcanonical distribution is obtained by imposing the subsidiary condition

$$\mathrm{Tr}(\rho\, I(P_\mu'))=1\,. \tag{2·3}$$

Here P_μ' is a c-number vector and I is the projection operator defined by

$$I(P_\mu')=\int_{-\infty}^{\infty}\frac{d^4\xi}{(2\pi)^4}\exp[i\xi_\mu(P_\mu-P_\mu')]. \tag{2·4}$$

In addition to (2·3), we have the normalization

$$\mathrm{Tr}(\rho)=1 \tag{2·5}$$

which should always be satisfied. Then the entropy is maximum for

$$\rho=\exp(-S)\cdot I(P_\mu'). \tag{2·6}$$

The entropy is determined as a function of P_μ' by inserting (2·6) into (2·5). Thus, P_μ' is the macroscopic energy-momentum.

Mathematically it is more convenient to introduce the canonical distribution by imposing the subsidiary condition

$$\mathrm{Tr}(\rho P_\mu)=P_\mu'. \tag{2·7}$$

With use of Langrange multipliers Ω and β_μ, we thus obtain

$$\rho=\exp[\Omega+\beta_\mu P_\mu]. \tag{2·8}$$

Since ρ is invariant, Ω is a scalar and β_μ is a vector. Inserting (2·8) into (2·5), we have Ω as a function of β_μ, and from (2·7)

$$P_\mu'=\frac{\partial\Omega}{\partial\beta_\mu}. \tag{2·9}$$

On the other hand, inserting (2·8) into (2·2), we get

$$S=-\Omega-\beta_\mu P_\mu'. \tag{2·10}$$

From (2·9) and (2·10),

$$\beta_\mu=-\frac{\partial S}{\partial P_\mu'}. \tag{2·11}$$

§ 3. The choice of thermodynamic variables

Now, in order for the canonical distribution (2·8) to be bounded, the vector β_μ should be time-like. Hence there exists an inertial frame K_0, in which $\beta_\mu = (0, 0, 0, iT_0^{-1})$. We denote the fourth component of P_μ in this frame by iH. Then

$$\rho=\exp[\Omega-(H/T_0)] \tag{3·1}$$

which is the usual canonical distribution. Thus, our system is at rest, in the macroscopic sense, with respect to this frame K_0, its proper temperature is equal to $T_0>0$, and its rest energy is given by

$$E_0=\mathrm{Tr}(\rho H). \tag{3·2}$$

In the inertial frame K, in which the frame K_0 is moving with the velocity $\boldsymbol{v}$,

$$\beta_\mu=\left(\frac{\boldsymbol{v}}{T_0(1-v^2)^{1/2}},\ \frac{i}{T_0(1-v^2)^{1/2}}\right). \tag{3·3}$$

Clearly $\boldsymbol{v}$ is the velocity of the macroscopic translation of our system as a whole.

In accordance with (3·3), we now write $P_\mu'=(\boldsymbol{G}, iE)$, where $\boldsymbol{G}$ is the macroscopic momentum and E is the macroscopic energy. Then (2·11) can be written as

$$\frac{1}{T_G}=\left(\frac{\partial S}{\partial E}\right)_{\boldsymbol{G}},\quad \frac{\boldsymbol{v}}{T_G}=-\left(\frac{\partial S}{\partial \boldsymbol{G}}\right)_E. \tag{3·4}$$

Here T_G is the Planck temperature defined by

$$T_G=T_0(1-v^2)^{1/2}. \tag{3·5}$$

We may also take E, $\boldsymbol{v}$ as thermodynamic variables. For simplicity, let us assume that our system as a whole is isolated. Then

$$E=E_0[1-v^2]^{-1/2}, \tag{3·6}$$

$$\boldsymbol{G}=E_0\boldsymbol{v}[1-v^2]^{-1/2}=E\boldsymbol{v}, \tag{3·7}$$

where E_0 is the rest energy (3·2). We can rewrite (3·4) as

$$T_v dS=dE-d[1-v^2]^{-1/2}. \tag{3·8}$$

Here T_v is the Ott temperature[1] defined by

$$\frac{1}{T_v}=\left(\frac{\partial S}{\partial E}\right)_{\boldsymbol{v}}=\frac{[1-v^2]^{1/2}}{T_0}. \tag{3·9}$$

Finally, since the entropy is invariant, we may calculate it by the use of (3·1). Then S is a function of the rest energy E_0. The variable $\boldsymbol{v}$ might seem to be redundant. We should notice, however, that the change of E_0 is restricted through the energy-momentum conservation law. It is then convenient to regard S as a function of E_0 and $\boldsymbol{v}$, of which we have

$$\frac{1}{T_0}=\left(\frac{\partial S}{\partial E_0}\right)_{\boldsymbol{v}},\quad 0=\left(\frac{\partial S}{\partial \boldsymbol{v}}\right)_{E_0} \tag{3·10}$$

as we can check by expressing dE in (3·8) in terms of E_0 and $\boldsymbol{v}$.

Thus, as far as thermodynamical identities are concerned, it is a matter of choice of thermodynamic variables whichever temperature we use.

§ 4. Thermal contact

In non-relativistic statistical thermodynamics, the concepts of temperature, heat, and heat reservoir are introduced by considering the thermal contact of two macroscopic systems. Let us extend these concepts to relativistic systems.

Suppose that we have a system consisting of two macroscopic subsystems without mutual interaction. Each subsystem is therefore in its own state of equilibrium. Suppose then that we introduce some weak interaction, through which the two subsystems start to exchange energy-momentum with each other. The entire system will eventually reach a new state of equilibrium. We assume that all parameters other than energy-momentum are kept constant in each subsystem. According to the principle of maximum entropy, the total entropy, which is the sum S_a+S_b of entropies of subsystems, cannot decrease

$$\Delta S_a+\Delta S_b\geqq 0\,. \tag{4·1}$$

In addition to this second law, we have the first law in the covariant form

$$\Delta E_a+\Delta E_b=0\,, \tag{4·2}$$

$$\Delta \boldsymbol{G}_a+\Delta \boldsymbol{G}_b=0\,. \tag{4·3}$$

We assume that the change of energy-momentum is small, so that by the use of (3·4) we rewrite (4·1) as

$$\left(\frac{1}{T_{Ga}}-\frac{1}{T_{Gb}}\right)\Delta E_a-\left(\frac{\boldsymbol{v}_a}{T_{Ga}}-\frac{\boldsymbol{v}_b}{T_{Gb}}\right)\Delta \boldsymbol{G}_a\geqq 0\,. \tag{4·4}$$

The equality sign holds when the entire system is in thermal equilibrium from the outset, so that our process is reversible. In general, ΔE_a and $\Delta \boldsymbol{G}_a$ are independent of each other, so that we obtain the conditions for complete equilibrium

$$T_{0a}=T_{0b}\,,\quad \boldsymbol{v}_a=\boldsymbol{v}_b\,. \tag{4·5}$$

Now, in non-relativistic thermodynamics, the sign of the temperature difference indicates the direction of heat flow when two systems are brought into thermal contact. We cannot draw such a conclusion from (4·4), unless we specify the process of exchanging energy-momentum in more detail. In other words, (4·4) covers a much wider range of processes than those which we call thermal contact in non-relativistic thermodynamics. So, let us explicitly state that by thermal contact we mean the process in which each of the two subsystems in contact does not change its macroscopic velocity

$$\Delta \boldsymbol{v}_a=\Delta \boldsymbol{v}_b=0\,. \tag{4·6}$$

Note that this condition is a Lorentz-invariant concept. Note also that under (4·6) the momentum conservation law (4·3) takes the form $\boldsymbol{v}_a\Delta E_a+\boldsymbol{v}_b\Delta E_b=0$ and in conjunction with (4·2) results in

$$\boldsymbol{v}_a=\boldsymbol{v}_b\,. \tag{4·7}$$

Thus one of the two conditions (4·5) is already satisfied; the two subsystems have one and the same rest frame of reference before and after the process. To the observer in this frame, the process is nothing else but the usual heat flow through thermal contact. He will at once write down the second law as

$$\left(\frac{1}{T_{0a}}-\frac{1}{T_{0b}}\right)\Delta E_{0a}\geqq 0\,. \tag{4·8}$$

On the other hand, by the use of (3·8), we can rewrite (4·1) under (4·6) as

$$\left(\frac{1}{T_{va}}-\frac{1}{T_{vb}}\right)\Delta E_{a}\geqq 0\,. \tag{4·9}$$

Thus the direction of energy flow looks determined by the sign of the difference in Ott temperatures. One might be tempted to regard this as one of new relativistic conclusions. Actually the second law written in the form (4·9) does not contain any more information than the classical second law (4·8). Thus, under the condition (4·7), $T_{va}\gtreqless T_{vb}$ is equivalent to $T_{0a}\gtreqless T_{0b}$. Under (4·8), we can even use Planck temperatures because $T_{Ga}\gtreqless T_{Gb}$ is also equivalent to $T_{0a}\gtreqless T_{0b}$.

Now, the difference of Planck temperatures will determine the direction of energy flow, if the exchange of momentum is forbidden with respect to a certain inertial frame of reference

$$\Delta \boldsymbol{G}_a=-\Delta \boldsymbol{G}_b=0\,. \tag{4·10}$$

In non-relativistic thermodynamics, this is equivalent to (4·6), but not so in relativistic theory. First of all, in contrast to (4·6), the condition (4·10) is not a Lorentz-invariant concept. Furthermore, unless the two subsystems are comoving from the outset, at least one of them should change its macroscopic velocity through the energy change under (4·10). In other words, the energy change in this case is not entirely connected with random motion.

As an example of (4·10), let us take a gedanken experiment proposed by Ott.[1] Suppose that a small body a is moving in a big cavity with the velocity $\boldsymbol{v}$ relative to the cavity wall b. The wall emits two photons simultaneously, which have opposite momenta with respect to the rest frame K_b of the wall. The body a absorbs the photons and is thereby decelerated. In order for the energy to flow from b to a in this manner, we should have $T_{Ga}\leqq T_{Gb}$, i.e.

$$T_{0a}(1-v^2)^{1/2}\leqq T_{0b}\,. \tag{4·11}$$

Ott[1] interpreted this as saying $T_{0a}<T_{vb}$, but this is only a matter of taste.

To see the unrealistic nature of (4·10), consider the reverse process, in which two photons are emitted by a and absorbed by b. To satisfy (4·10), these two photons should have opposite momenta with respect to K_b. The assumption is rather unrealistic.

§ 5. Significance of the Ott temperature

We have defined the familiar concept of thermal contact in a Lorentz-invariant manner. We define the heat transferred through the thermal contact by ΔE_a which appears in (4·9). Under (4·6), it obeys the Ott transformation

$$\Delta E_a = \Delta E_{0a}(1-v^2)^{-1/2}. \tag{5·1}$$

In addition to the heat flow between comoving systems, another obvious concept is the adiabatic, reversible change of the macroscopic velocity of a system. This is the macroscopic acceleration of the system as a whole without changing its rest energy E_0. As we see from (3·10), the entropy then remains constant.

Now, in relativistic thermodynamics, we often need to compare temperatures of two systems in relative motion with each other. The basic role is then played by the Ott cycle,[1] which consists of the following four processes: 1° adiabatic, reversible acceleration of a system from a certain velocity $\boldsymbol{v}_0$ to another velocity $\boldsymbol{v}$, 2° reversible heat transfer from a heat reservoir A moving with the same velocity $\boldsymbol{v}$, 3° adiabatic, reversible deceleration from $\boldsymbol{v}$ to $\boldsymbol{v}_0$, 4° reversible heat transfer to a reservoir C moving with the velocity $\boldsymbol{v}_0$, and the system returns to its original state. Obviously, the changes in the rest energy of the system during processes 2° and 4° have the same magnitude and opposite signs, so that we call them $\pm\Delta E_{0a}$. From the second law, we see that the proper temperatures of the two reservoirs should be equal to each other:

$$-(\Delta E_{0a}/T_{0a}) + (\Delta E_{0a}/T_{0c}) = 0\,.$$

The heat absorbed from the reservoir A is given by (5·1) and the difference

$$W = \Delta E_{0a}(1-v_0^2)^{-1/2} - \Delta E_{0a}(1-v^2)^{-1/2} \tag{5·2}$$

should be the mechanical work done by external force.

Now, let us suppose that after completing one Ott cycle we apply the Carnot cycle to extract the heat $\Delta E_{0a}(1-v_0^2)^{-1/2}$ from the reservoir C and to supply another amount of heat, $\Delta E_{0b}(1-v_0^2)^{-1/2}$ to the third reservoir D, which is moving with the same velocity $\boldsymbol{v}_0$.

$$W' = (\Delta E_{0b} - \Delta E_{0a})(1-v_0^2)^{-1/2} \tag{5·3}$$

should be the mechanical work done by external force. Finally, let us suppose, that by the Ott cycle again, but in the reverse direction this time we extract the heat $\Delta E_{0b}(1-v_0^2)^{-1/2}$ from the reservoir D and supply the heat

$$\Delta E_b = \Delta E_{0b}(1-v'^2)^{-1/2} \tag{5·4}$$

to the fourth reservoir B which has the same proper temperature as D, but is moving with a different velocity $\boldsymbol{v}'$. The mechanical work done by external force is given by

$$W'' = \Delta E_{0b}(1-v'^2)^{-1/2} - \Delta E_{0b}(1-v_0^2)^{-1/2}, \tag{5·5}$$

We choose ΔE_{0b} such that the net mechanical work $W+W'+W''$ vanishes. Then

$$\Delta E_{0a}(1-v^2)^{-1/2}=\Delta E_{0b}(1-v'^2)^{-1/2}. \tag{5·6}$$

The second law applied to the Carnot cycle gives us the well known relation $(T_{0a}/T_{0b})=(\Delta E_{0a}/\Delta E_{0b})$. Hence

$$T_{0a}(1-v^2)^{-1/2}=T_{0b}(1-v'^2)^{-1/2}. \tag{5·7}$$

Thus, when two systems have the same Ott temperature, it is possible to transfer energy from one to the other reversibly without the expenditure of mechanical work. This is the real significance of the Ott temperature. The proof described above is due to Ott[1] and we regard it as his most important contribution to our understanding relativistic thermodynamics.

§ 6. Black body radiation

As a simple example of the formalism described in § 2, we take a system of free, neutral, scalar particles with the vanishing rest mass. This is a simpler version of the problem of black body radiation, which was discussed by Arzelies, Gamba, Kibble,[3] and in particular by Eberly and Kujawski.[9] We only add a few remarks.

Suppose that our system is enclosed in a finite volume, so that we write

$$\beta_\mu P_\mu=\sum_{\boldsymbol{k}}\frac{(\boldsymbol{v}\cdot\boldsymbol{k}-k)}{T_0(1-v^2)^{1/2}}n(\boldsymbol{k}). \tag{6·1}$$

The occupation number $n(\boldsymbol{k})$ and its average are scalars. The canonical distribution (2·8) gives us

$$\langle n(\boldsymbol{k})\rangle=\left[\exp\left(\frac{k-\boldsymbol{k}\cdot\boldsymbol{v}}{T_0(1-v^2)^{1/2}}\right)-1\right]^{-1}. \tag{6·2}$$

This is the Planck distribution with the temperature T_0 and the Doppler shifted frequency, in accordance with what one expects intuitively. From the anisotropy of this distribution,[10] one might be able to determine earth's velocity relative to the 3°K cosmic radiation, for example.

From the standard field theoretic expression for the energy-momentum tensor $T_{\mu\nu}$, we see that

$$\begin{aligned}\langle T_{11}\rangle&=\int d^3\boldsymbol{k}\,(k_x^2/k)\langle n(\boldsymbol{k})\rangle\,,\\ \langle T_{44}\rangle&=-\int d^3\boldsymbol{k}\,k\langle n(\boldsymbol{k})\rangle\,,\\ \langle T_{14}\rangle&=i\int d^3\boldsymbol{k}\,k_x\langle n(\boldsymbol{k})\rangle\,.\end{aligned} \tag{6·3}$$

The integral over $\boldsymbol{k}$ can easily be performed by inserting (6·2) into (6·3).

The same result can, of course, be obtained by applying the Lorentz transformation to the expression which we obtain in the rest frame:

$$\langle T_{11}\rangle = \tfrac{1}{3}\alpha T_0^4 (\equiv p_0),$$
$$-\langle T_{44}\rangle = \alpha T_0^4 (\equiv \varepsilon_0). \tag{6·4}$$

The factor α is one half of the usual Stefan-Boltzmann constant. In terms of p_0, ε_0, we have the well-known expressions

$$\langle T_{11}\rangle = (p_0 + v^2\varepsilon_0)(1-v^2)^{-1},$$
$$-\langle T_{44}\rangle = (\varepsilon_0 + v^2 p_0)(1-v^2)^{-1}, \tag{6·5}$$
$$-i\langle T_{14}\rangle = (\varepsilon_0 + p_0)\, v(1-v^2)^{-1}.$$

Now, the macroscopic energy-momentum is given by

$$P_\mu'[\sigma] = \int_\sigma \langle T_{\mu\nu}\rangle d\sigma_\nu, \tag{6·6}$$

where σ is a three dimensional hypersurface and $d\sigma_\mu$ is its surface element vector parallel to the normal. Since our system is enclosed in a finite volume, in which $\langle T_{\mu\nu}\rangle$ is uniform, the integral (6·6) does depend on the surface σ. For instance, for the observer in the rest frame K_0, it is natural to take the hyperplane $\sigma^{(0)}$ whose normal is parallel to the time-axis of the frame K_0. Then (6·6) gives us the rest energy

$$E_0 = V_0 \varepsilon_0, \tag{6·7}$$

where V_0 is the proper volume.

Let us now go over to the frame K, in which the frame K_0 is moving with the velocity $\boldsymbol{v}$. The energy-momentum (6·6) obeys the usual Lorentz transformation (3·6) and (3·7), only when we take the same hyperplane $\sigma^{(0)}$ to define the energy-momentum by (6·6). Indeed, in the frame K, we have

$$d\sigma_\mu^{(0)} = (-v(1-v^2)^{-1/2} dV_0, \quad -i(1-v^2)^{-1/2} dV_0),$$

where dV_0 is the element of the proper volume. In conjunction with (6·5), we can easily check that (6·6) gives us (3·6) and (3·7). Therefore, our thermodynamical arguments given in the preceding sections can be applied, provided that we stick to $\sigma^{(0)}$ in (6·6).

On the other hand, to the observer in the frame K, it is more natural to define the energy-momentum with reference to the hyperplane σ,$^{(1)}$ whose normal is parallel to the time-axis of the frame K itself. Then, the fourth component of (6·6) gives us the energy

$$E^* = -V\langle T_{44}\rangle = V_0(\varepsilon_0 + v^2 p_0)(1-v^2)^{-1/2}. \tag{6·8}$$

Here $V = V_0(1-v^2)^{1/2}$ is the Lorentz-contracted volume. The difference between

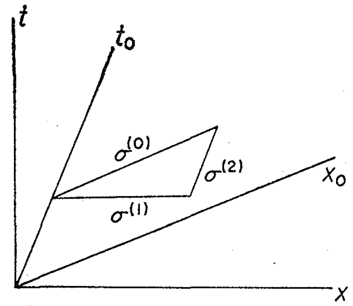

Fig. 1.

E^* and $E = E_0(1-v^2)^{-1/2}$ is the integral (6·6) taken over the hyperplane $\sigma^{(2)}$ which is defined in Fig. 1:

$$P_\mu'[\sigma^{(0)}] = P_\mu'[\sigma^{(1)}] + P_\mu'[\sigma^{(2)}]. \qquad (6\cdot 9)$$

The physical significance is made clear by considering the change of the difference, when T_0 and p_0 are varied, V_0, $\boldsymbol{v}$ being kept constant. Then, from (3·6), (6·4), and (6·8), we have

$$dE^* - dE = V_0\{v^2(1-v^2)^{-1/2}\}dp_0\,. \qquad (6\cdot 10)$$

As was explicitly calculated by Møller,[3] the expression on the right is equal to the mechanical work done by external force which is applied to keep the proper volume constant. When the pressure p_0 is varying with time, forces acting upon opposite faces of the container do not balance each other in the frame K, even if they do in the rest frame K_0. That is why we have the mechanical work (6·10). Thus, when we regard E^* as the energy observed in the frame K, as we usually do, we should also take this mechanical work into account. Then the heat is given by dE and there is no need of modifying our previous arguments. We may as well deal solely with E without introducing E^*. Then E is a sort of heat function, or enthalpy, whose change directly gives us the heat.

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