

- Keesom, W. H. 1921 *Phys. Z.* **22**, 129.
 Keyes, F. G. and co-workers 1922 *J. Math. Phys. Mass. Inst. Tech.* **1**, 191.
 — 1927 *J. Amer. Chem. Soc.* **49**, 1403.
 Lewis, D. T. 1938 *J. Chem. Soc.* pp. 261, 1056, 1061.
 Mack, E. and Melaven, R. M. 1931 *J. Amer. Chem. Soc.* **54**, 888.
 Sidgwick, N. V. 1932 *Ann. Rep. Chem. Soc.* p. 67.
 — 1933 *The covalent link*. Cornell Univ. Press.
 Whytlaw-Gray, R. and Burt, F. P. 1909 *J. Chem. Soc.* **95**, 1633.
 Woodhead, M. and Whytlaw-Gray, R. 1933 *J. Chem. Soc.* p. 846.

On relativistic wave equations for particles of arbitrary spin in an electromagnetic field

BY M. FIERZ AND W. PAULI

Physikalisches Institut der Eidgenössischen Technischen Hochschule, Zürich

(Communicated by P. A. M. Dirac, F.R.S.—Received 31 May 1939)

1. INTRODUCTION

The investigations of Dirac (1936) on relativistic wave equations for particles with arbitrary spin have recently been followed up by one of us (Fierz, 1939, referred to as (A)) It was there found possible to set up a scheme of second quantization in the absence of an external field, and to derive expressions for the current vector and the energy-momentum tensor. These considerations will be extended in the present paper to the case when there is an external electromagnetic field, but we shall in the first instance disregard the second quantization and confine ourselves to a *c*-number theory.

The difficulty of this problem is illustrated by the fact that the most immediate method of taking into account the effect of the electromagnetic field, proposed by Dirac (1936), leads to inconsistent equations as soon as the spin is greater than 1. To make this clear we consider Dirac's equations for a particle of spin 3/2, which in the force-free case run as follows:

$$\left. \begin{aligned} \kappa b_{\gamma}^{\dot{\alpha}\dot{\beta}} &= p^{\dot{\alpha}\rho} a_{\gamma\rho}^{\dot{\beta}} = p^{\dot{\beta}\rho} a_{\gamma\rho}^{\dot{\alpha}}, \\ \kappa a_{\alpha\beta}^{\dot{\gamma}} &= p_{\alpha\dot{\rho}} b_{\beta}^{\dot{\rho}\dot{\gamma}} = p_{\beta\dot{\rho}} b_{\alpha}^{\dot{\rho}\dot{\gamma}}, \end{aligned} \right\} \quad (1)$$

where $a_{\alpha\beta}^{\dot{\gamma}} = a_{\beta\alpha}^{\dot{\gamma}}$ and $b_{\dot{\gamma}}^{\dot{\alpha}\beta} = b_{\dot{\gamma}}^{\dot{\alpha}\beta}$ are symmetrical spinors. Dirac attempted to take the external electromagnetic field into account by replacing the spinor $p_{\alpha\dot{\rho}}$ by $\Pi_{\alpha\dot{\rho}}$, which arises from it by substituting $-i\partial/\partial x_k - e\phi_k/hc$ for $-i\partial/\partial x_k$ (ϕ_k being the electromagnetic potentials). The $\Pi_{\alpha\dot{\rho}}$ are then non-commuting operators satisfying the relations (cf. Appendix)

$$\Pi_{\dot{\alpha}\beta} \Pi^{\dot{\gamma}\delta} - \Pi^{\dot{\gamma}\delta} \Pi_{\dot{\alpha}\beta} = \delta_{\dot{\alpha}}^{\dot{\gamma}} f_{\beta}^{\delta} + \delta_{\beta}^{\delta} f_{\dot{\alpha}}^{\dot{\gamma}}, \quad (2.1)$$

where $f_{\alpha\beta}$ and $f_{\dot{\alpha}\dot{\beta}}$ are the two symmetrical spinors ($f_{\alpha}^{\alpha} = f_{\dot{\alpha}}^{\dot{\alpha}} = 0$) associated with the antisymmetrical field tensor f_{ik} . By contracting this relation it follows that

$$\Pi_{\alpha\dot{\rho}} \Pi^{\dot{\rho}\beta} - \Pi^{\dot{\rho}\beta} \Pi_{\alpha\dot{\rho}} = 2f_{\alpha}^{\beta}.$$

On the other hand, just as in the force-free case, we have

$$\Pi_{\alpha\dot{\rho}} \Pi^{\dot{\rho}\beta} + \Pi^{\dot{\rho}\beta} \Pi_{\alpha\dot{\rho}} = -2\Pi^2 \delta_{\alpha}^{\beta}, \quad (2.2)$$

and so

$$\Pi_{\alpha\dot{\rho}} \Pi^{\dot{\rho}\beta} = -\Pi^2 \delta_{\alpha}^{\beta} + f_{\alpha}^{\beta}. \quad (2.3)$$

From Dirac's proposed equations

$$\kappa b_{\dot{\gamma}}^{\dot{\alpha}\beta} = \Pi^{\dot{\alpha}\rho} a_{\dot{\gamma}\rho}^{\beta} = \Pi^{\dot{\beta}\rho} a_{\dot{\gamma}\rho}^{\dot{\alpha}}, \quad (3.1)$$

$$\kappa a_{\alpha\beta}^{\dot{\gamma}} = \Pi_{\alpha\dot{\rho}} b_{\dot{\rho}\beta}^{\dot{\gamma}} = \Pi_{\dot{\rho}\beta} b_{\alpha}^{\dot{\gamma}\dot{\rho}}, \quad (3.2)$$

however, it would follow that

$$\Pi_{\alpha\dot{\rho}} \Pi^{\dot{\rho}\sigma} a_{\dot{\rho}\sigma}^{\dot{\gamma}} = \kappa \Pi_{\alpha\dot{\rho}} b_{\dot{\rho}\beta}^{\dot{\gamma}} = \kappa^2 a_{\alpha\beta}^{\dot{\gamma}},$$

and so from (2.3)

$$-\Pi^2 a_{\alpha\beta}^{\dot{\gamma}} + f_{\alpha}^{\sigma} a_{\dot{\rho}\sigma}^{\dot{\gamma}} = \kappa^2 a_{\alpha\beta}^{\dot{\gamma}}.$$

Since the right-hand side is symmetrical in α and β it follows that the subsidiary condition

$$f_{\alpha}^{\sigma} a_{\dot{\rho}\sigma}^{\dot{\gamma}} = f_{\beta}^{\sigma} a_{\dot{\rho}\sigma}^{\dot{\gamma}},$$

or

$$f^{\rho\sigma} a_{\dot{\rho}\sigma}^{\dot{\gamma}} = 0, \quad (3.3)$$

must be satisfied by the spinor field $a_{\alpha\beta}^{\dot{\gamma}}$; but this cannot in general be satisfied simultaneously with the other equations.

One might at first hope to avoid this objection by replacing (3.1), (3.2) by the weaker conditions

$$2\kappa b_{\dot{\gamma}}^{\dot{\alpha}\beta} = \Pi^{\dot{\alpha}\rho} a_{\dot{\gamma}\rho}^{\beta} + \Pi^{\dot{\beta}\rho} a_{\dot{\gamma}\rho}^{\dot{\alpha}}, \quad (4.1)$$

$$2\kappa a_{\alpha\beta}^{\dot{\gamma}} = \Pi_{\alpha\dot{\rho}} b_{\dot{\rho}\beta}^{\dot{\gamma}} + \Pi_{\dot{\rho}\beta} b_{\alpha}^{\dot{\gamma}\dot{\rho}}. \quad (4.2)$$

It is to be remarked, however, that even in the force-free case such a system no longer leads to a wave equation of second order. (A closer discussion shows that these equations describe, besides particles of spin 3/2 and rest-mass κ , also particles of spin 1/2 and rest-mass 2κ). And further, the expression for the total charge turns out to be no longer positive definite, and this makes quantization consistent with the exclusion principle impossible (for quantization consistent with Bose statistics the total energy is, on the other hand, not positive for the case of half-integral spins).

This modification was therefore abandoned and the equations (1) were retained for the force-free case. The problem then arose, besides replacing the $p_{\alpha\beta}$ by the $\Pi_{\alpha\beta}$, of adding to these equations extra terms, depending on the field strengths, in such a way that they remained self-consistent in the presence of an external field.

A completely analogous problem arises for integral spins. For instance, the field equations for spin 2, involving according to (A) a symmetrical tensor A_{ik} , whose trace $\sum_i A_{ii}$ vanishes, are

$$\square A_{ik} = \kappa^2 A_{ik}, \tag{5.1}$$

$$\frac{\partial A_{ik}}{\partial x_i} = 0. \tag{5.2}$$

The second set of conditions is indispensable if the total energy is to be positive definite. In fact if they were omitted those waves with only components of the type A_{4i} would give rise to negative values of the total energy. On the other hand, the equations which arise from (5.1), (5.2) when $\partial/\partial x_k$ is replaced by $\partial/\partial x_k - ie\phi_k/\hbar c$ are not compatible, for the operators $\Pi^2 \equiv \sum_k \Pi_k^2$ and Π_k are not commutative ($\Pi_k = -i\partial/\partial x_k - e\phi_k/\hbar c$).

We shall not attack the problem of deriving such additional terms to make the equations compatible directly but solve it by an artifice. This consists in introducing auxiliary tensors or spinors of lower rank than the original ones (for spin 3/2 they will be simple spinors c_α and $d_{\dot{\alpha}}$; for spin 2 a scalar C) and deriving all equations from a variation principle without having to introduce extra conditions. By suitably choosing the numerical coefficients in the Lagrange function it will follow from the field equations (derived from the variation) that in the absence of an external field the auxiliary quantities vanish and the additional conditions (5.2) or (1) are satisfied automatically (cf. § 2, equations (10), (11)).

That such a procedure is reasonable seems to be shown by the fact that, for vanishing rest-mass, our equations for the case of spin 2 go over into those

of the relativity theory of weak gravitational fields (i.e. $g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}$, neglecting terms of order higher than the first in $\gamma_{\mu\nu}$); the "gauge-transformations" are identical with the changes induced in $\gamma_{\mu\nu}$ by infinitesimal co-ordinate transformations (§ 6).

Although the following only deals in detail with the interaction of the particles and an external electromagnetic field, the interaction with other particles which can be absorbed and emitted could be formulated analogously. For instance, the interaction with a new scalar field ψ could be introduced by extra terms in the Lagrange function which arise from those in the Lagrangian of the following pages containing a factor κ^2 by replacement of κ^2 by ψ . On the other hand, it is important that a one-to-one correspondence should be possible between the states (eigenfunctions) with the external field and without. This is equivalent to saying that the number of conditions which the field and auxiliary variables (and their time-derivatives for integral spins) must satisfy *at a definite time* is not diminished by the presence of an external field. Otherwise, as is illustrated in Appendix I by a special example with particles of spin 1, singularities occur when the external field is made to vanish slowly. In the main text, however, this requirement of the continued existence of subsidiary conditions in an external field is always fulfilled.

This requirement also seems important for the second quantization of the fields, a topic not treated in detail here. It enables one, namely, starting from the commutation rules of (A), to make an expansion of the commutation brackets of all field quantities in powers of the charge e . It is to be remarked that with particles of spin greater than 1 the charge-densities at different points no longer commute. A closer study of this circumstance, which strongly distinguishes the spin values of 0, 1/2, 1 (cf. A, Introduction) is to be desired.

As may be seen from our last section (§ 8), our aim was not so much to set up the most general possible relativistic equations for particles of higher spin but rather to show that, in the present state of the theory, the existence of elementary particles of spin higher than 1 cannot be excluded, although the theory for such particles is considerably more complicated than for smaller spin values. In this connexion it may be mentioned that we have been unable to generalize the field-equations which in the notation of (A) correspond to $k \neq l$, or the current-vectors $s^{(q)}$ for which $q > 1$ (cf. (A), I, II, III and (5·6)).

I. SPIN 2

2. DERIVATION OF THE FORCE-FREE EQUATIONS FROM
A VARIATION PRINCIPLE

As an example of the theory of a wave-field corresponding to particles of spin higher than 1 in interaction with other fields let us first consider the theory for spin 2.

As was shown in (A) such a field is described in the absence of external fields by a symmetrical tensor A_{ik} of second rank, whose trace is zero, satisfying the wave equation

$$\square A_{ik} = \kappa^2 A_{ik}, \tag{5.1}$$

and for which the additional condition

$$\frac{\partial A_{ik}}{\partial x_i} = 0 \tag{5.2}$$

is fulfilled. In this the indices i, k run from 1 to 4. (x_i) stands for (x, y, z, ict) and $\square \equiv \nabla^2 - (1/c^2) \partial^2/\partial t^2$. Summation over indices occurring twice is to be understood. It can be shown that the total energy of the field is positive only if the extra condition (5.2) is satisfied, i.e. if the vector $\partial A_{ik}/\partial x_i$ does not vanish it describes particles of negative energy. If one introduces external fields, this must be done in such a manner that after they have been shut off the condition (5.2) is again fulfilled, so that no new particles of negative energy should be created.

In order to discover a correct generalization of equations (5.1), (5.2) for external forces we shall look for a variation principle

$$\delta \int L d\Omega = 0,$$

from which (5.1) and (5.2) can be derived. At this point it is useful to introduce an auxiliary scalar field C , on which L will be taken to depend and which is to be varied independently of A_{ik} . The introduction of C is an artifice which enables one to derive the additional condition (5.2) from the Lagrange function by variation. For simplicity let us assume that A_{ik} and C are "real" fields, i.e.

$$A_{ik}^* = A_{ik}; \quad C^* = C.$$

(Here the tensor $A_{ik \dots l}^*$, conjugate to the tensor $A_{ik \dots l}$, is equal to $(-)^n \bar{A}_{ik \dots l}$, where n is the number of times 4 appears among the indices i, k, \dots, l , and the bar denotes the complex conjugate. "Real" tensors are those for which $A_{ik \dots l}^* = A_{ik \dots l}$.)

For the function L we make the following choice:

$$L = \kappa^2 A_{ik} A_{ik} + \frac{\partial A_{ik}}{\partial x_l} \frac{\partial A_{ik}}{\partial x_l} + a_1 \frac{\partial A_{rk}}{\partial x_r} \frac{\partial A_{sk}}{\partial x_s} + a_2 \kappa^2 C^2 + a_3 \frac{\partial C}{\partial x_l} \frac{\partial C}{\partial x_l} + \frac{\partial A_{rk}}{\partial x_r} \frac{\partial C}{\partial x_k}. \quad (6)$$

A_{ik} is of course symmetrical and fulfils the trace condition $A_{ii} = 0$, a fact which must be remembered when performing the variation. By varying A_{ik} and C we obtain the following equations:

$$2\kappa^2 A_{ik} - 2\Box A_{ik} - a_1 \left(\frac{\partial^2 A_{rk}}{\partial x_r \partial x_i} + \frac{\partial^2 A_{ri}}{\partial x_r \partial x_k} - \frac{1}{2} \delta_{ik} \frac{\partial^2 A_{rs}}{\partial x_r \partial x_s} \right) - \frac{\partial^2 C}{\partial x_i \partial x_k} + \frac{1}{4} \delta_{ik} \Box C = 0, \quad (7.1)$$

$$2a_2 \kappa^2 C - 2a_3 \Box C - \frac{\partial^2 A_{rk}}{\partial x_r \partial x_k} = 0. \quad (7.2)$$

Let us now determine the three constants a_1 , a_2 , a_3 in such a way that $\partial A_{ik}/\partial x_i$ and C vanish as a consequence of (7.1) and (7.2). For this we differentiate (7.1) with respect to x_i and obtain

$$2\kappa^2 \frac{\partial A_{ik}}{\partial x_i} = 2\Box \frac{\partial A_{ik}}{\partial x_i} + a_1 \left(\Box \frac{\partial A_{ik}}{\partial x_i} + \frac{1}{2} \frac{\partial^3 A_{si}}{\partial x_i \partial x_s \partial x_k} \right) + \frac{3}{4} \Box \frac{\partial C}{\partial x_k}. \quad (8)$$

If we now put $a_1 = -2$, the right-hand side will only contain derivatives of the scalars C and $\partial^2 A_{ik}/\partial x_i \partial x_k$, which latter we shall denote by A for brevity:

$$\frac{\partial^2 A_{ik}}{\partial x_i \partial x_k} \equiv A.$$

The equation (8) then becomes

$$2\kappa^2 \frac{\partial A_{ik}}{\partial x_i} = -\frac{\partial A}{\partial x_k} + \frac{3}{4} \Box \frac{\partial C}{\partial x_k}. \quad (8')$$

This equation means that the vector $\partial A_{ik}/\partial x_i$ is the gradient of a scalar, and can therefore describe only particles of spin zero.

Now let us differentiate (8) with respect to x_k and obtain together with (7.2) the two equations

$$2\kappa^2 A + \Box A - \frac{3}{4} \Box \Box C = 0, \quad (9)$$

$$-A + 2a_2 \kappa^2 C - 2a_3 \Box C = 0. \quad (7.2')$$

This is a linear, homogeneous system of equations for A and C . We now choose a_2 and a_3 in such a manner that the operator determinant of the system shall never vanish; A and C will then vanish, and thus according to (8) also $\partial A_{ik}/\partial x_i$.

One obtains the following expression for the determinant

$$4a_2\kappa^4 + 2\kappa^2(a_2 - 2a_3)\square - (2a_3 + \frac{3}{4})\square\square. \tag{10}$$

If we put $a_3 = -\frac{3}{8}, \quad 2a_3 = a_2 = -\frac{3}{4},$

then the determinant has the value $-3\kappa^4$, and since κ is supposed to be different from zero, it will never be zero. We therefore have

$$A = 0, \quad C = 0, \quad \frac{\partial A_{ik}}{\partial x_i} = 0.$$

We are thus led to the Lagrange function

$$L = \kappa^2 A_{ik} A_{ik} + \frac{\partial A_{ik}}{\partial x_l} \frac{\partial A_{ik}}{\partial x_l} - 2 \frac{\partial A_{rk}}{\partial x_r} \frac{\partial A_{sk}}{\partial x_s} - \frac{3}{4} \kappa^2 C^2 - \frac{3}{8} \frac{\partial C}{\partial x_l} \frac{\partial C}{\partial x_l} + \frac{\partial A_{rk}}{\partial x_r} \frac{\partial C}{\partial x_k}, \tag{11}$$

and the corresponding field equations

$$2\kappa^2 A_{ik} - 2\square A_{ik} + 2 \left(\frac{\partial^2 A_{sk}}{\partial x_s \partial x_i} + \frac{\partial^2 A_{si}}{\partial x_s \partial x_k} - \frac{1}{2} \delta_{ik} \frac{\partial^2 A_{rs}}{\partial x_r \partial x_s} \right) - \frac{\partial^2 C}{\partial x_i \partial x_k} + \frac{1}{4} \delta_{ik} \square C = 0, \tag{12.1}$$

$$-\frac{3}{2} \kappa^2 C + \frac{3}{4} C - \frac{\partial^2 A_{rk}}{\partial x_r \partial x_k} = 0, \tag{12.2}$$

from which one can derive equations (5.1) and (5.2), and also the equation $C = 0$.

As we are interested in the influence of external forces on the field A_{ik} it will be useful to use a notation in which the time is separated from the other co-ordinates. We shall therefore discuss equations (12.1), (12.2) from this point of view.

The field in the example considered belongs to the spin value $f = 2$, and therefore gives $2f + 1 = 5$ states for a given direction and frequency. The differential equations for the fields A_{ik} and C are of second order. At a given time, therefore, one can prescribe the values of 5 components of A_{ik} and their time-derivatives at all points of space. Since the field A_{ik} has altogether $(f + 1)^2 = 9$ components, there remain, together with the one component of C , 5 components and their first time-derivatives which cannot be given at will. That is to say, there must be 10 subsidiary conditions,* containing

* We would like to point out that we use the term "additional conditions" in the sense of equations not following from a variation principle giving the main equations (§2, §4), whereas the term "subsidiary conditions" refers to equations derived from the variation but which have the effect of reducing the number of degrees of freedom.

perhaps higher space-derivatives, but only first derivatives with respect to time, from which, if 5 components of A_{ik} and their time-derivatives are given, one can calculate the remaining ten quantities. These conditions can be derived from equations (12.1) and (12.2).

In the above-defined sense the following equations are to be regarded as subsidiary conditions:

$$C = 0, \quad (13.1)$$

$$\frac{\partial C}{\partial x_4} = 0. \quad (13.2)$$

Let us further consider (12.1) for $i \neq 4$, $k = 4$. To be able to write the equation conveniently in this form we introduce Greek indices α, β, \dots , which run only from 1 to 3. We then get

$$2\kappa^2 A_{\alpha 4} - 2 \frac{\partial^2}{\partial x_\beta^2} A_{\alpha 4} + 2 \frac{\partial^2 A_{\beta 4}}{\partial x_\alpha \partial x_\beta} + 2 \frac{\partial^2 A_{\alpha\beta}}{\partial x_\beta \partial x_4} + 2 \frac{\partial^2 A_{44}}{\partial x_\alpha \partial x_4} - \frac{\partial^2 C}{\partial x_\alpha \partial x_4} = 0, \quad (\alpha = 1, 2, 3). \quad (13.3) \dots (13.5)$$

These constitute three more conditions, since the second time-derivatives of $A_{\alpha 4}$ have dropped out. By adding the (4, 4)-component of (12.1) and (12.2) we get a sixth condition:

$$2\kappa^2 A_{44} - 2 \frac{\partial^2 A_{44}}{\partial x_\beta^2} - \frac{\partial^2 A_{\alpha\beta}}{\partial x_\alpha \partial x_\beta} + \frac{\partial^2 C}{\partial x_\beta^2} - \frac{3}{2} \kappa^2 C = 0. \quad (13.6)$$

Differentiating (12.1) with respect to x_i we obtain

$$2\kappa^2 \frac{\partial A_{ik}}{\partial x_i} + \frac{\partial^3 A_{is}}{\partial x_k \partial x_i \partial x_s} - \frac{3}{4} \frac{\partial}{\partial x_k} \square C = 0.$$

Combining this with (12.2) we obtain

$$2 \frac{\partial A_{ik}}{\partial x_i} - \frac{3}{2} \frac{\partial C}{\partial x_k} = 0. \quad (13.7) \dots (13.10)$$

The above equation holds for $k = 1, \dots, 4$. We have therefore found 10 subsidiary conditions.

3. INTRODUCTION OF INTERACTIONS

The theory as presented up till now is equivalent to the theory in (A). By adding suitable terms to the Lagrangian we can introduce interactions with other fields. One must take care, however, that the subsidiary con-

ditions are not impaired. This would mean that the dimensionality of the manifold of states was altered by "switching on" the forces, and it turns out that these new states give rise to singularities when the field is "switched off", as we shall illustrate by means of an example in the Appendix. We shall here consider the effect of an electromagnetic field.

In this case we must naturally assume that the fields A_{ik} and C are complex. Let ϕ_k be the four-potential of the electromagnetic field, e the charge of the particles. Then we have

$$i\Pi_k = \frac{\partial}{\partial x_k} - \frac{ie}{\hbar c} \phi_k,$$

and

$$f_{ik} \equiv \Pi_i \Pi_k - \Pi_k \Pi_i = \frac{ie}{\hbar c} \left(\frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k} \right).$$

We take the following form for L :

$$L = \kappa^2 A_{ik}^* A_{ik} + \Pi_l A_{ik} \Pi_l^* A_{ik}^* - 2\Pi_r A_{rk} \Pi_s^* A_{sk}^* + f_{ir} A_{rk} A_{ik}^* + \frac{1}{2} \{ \Pi_r A_{rk} \Pi_k^* C^* + \Pi_r^* A_{rk}^* \Pi_k C \} - \frac{3}{4} \kappa^2 C^* C - \frac{3}{8} \Pi_l C \Pi_l^* C^*. \quad (14)$$

The term $f_{ir} A_{rk} A_{ik}^*$ (proportional to the field strengths) has been added because then the derivation of the subsidiary conditions is particularly simple, but it is not necessary. The Lagrange function can also be brought into another form by partial integration, namely,

$$L = \kappa^2 A_{ik}^* A_{ik} + \frac{1}{2} \{ \Pi_l A_{ik} - \Pi_i A_{lk} \} \{ \Pi_l^* A_{ik}^* - \Pi_i^* A_{lk}^* \} - \Pi_r A_{rk} \Pi_s^* A_{sk}^* - \frac{3}{4} \kappa^2 C^* C + \frac{1}{2} \{ \Pi_r A_{rk} \Pi_k^* C^* + \Pi_r^* A_{rk}^* \Pi_k C \} - \frac{3}{8} \Pi_l C \Pi_l^* C^*.$$

Performing the variations with respect to A_{ik}^* and C^* one obtains the equations

$$2\kappa^2 A_{ik} + 2\Pi^2 A_{ik} - 2\{ \Pi_i \Pi_s A_{sk} + \Pi_k \Pi_s A_{si} - \frac{1}{2} \delta_{ik} \Pi_r \Pi_s A_{rs} \} + f_{ir} A_{rk} + f_{kr} A_{ri} + \frac{1}{2} \{ \Pi_i \Pi_k + \Pi_k \Pi_i \} C - \frac{1}{4} \delta_{ik} \Pi^2 C = 0, \quad (15.1)$$

$$-\frac{3}{2} \kappa^2 C - \frac{3}{4} \Pi^2 C + \Pi_r \Pi_s A_{rs} = 0. \quad (15.2)$$

We shall now show that again 10 subsidiary conditions follow from these equations. We obtain three such conditions from (15.1) by putting $i \neq 4$ $k = 4$:

$$2\kappa^2 A_{\alpha 4} + 2\Pi_\beta^2 A_{\alpha 4} - 2\Pi_\alpha \Pi_\beta A_{4\beta} - 2\Pi_\alpha \Pi_4 A_{44} - 2\Pi_4 \Pi_\beta A_{\beta\alpha} + f_{\alpha r} A_{r4} + f_{4\beta} A_{\beta\alpha} + \Pi_\alpha \Pi_4 C + f_{4\alpha} C = 0. \quad (16.3) \dots (16.5)$$

$$(\alpha, \beta = 1, 2, 3; \quad r = 1, 2, 3, 4).$$

Adding the (4, 4)-component of (15.1) to (15.2) gives another:

$$2\kappa^2 A_{44} + 2\Pi_\beta^2 A_{44} + 2\Pi_\alpha \Pi_\beta A_{\alpha\beta} - \Pi_\alpha^2 C - \frac{3}{4} \kappa^2 C = 0. \quad (16.6)$$

Applying the operator Π_k to (15.1) and using (15.2) gives four more:

$$2\kappa^2 \Pi_i A_{ik} + 3f_{iu} \Pi_l A_{ik} + \frac{1}{i} \frac{\partial f_{il}}{\partial x_l} A_{ik} - 3f_{ik} \Pi_s A_{st} - \frac{1}{i} \frac{\partial f_{rk}}{\partial x_i} A_{ri} - \frac{3}{2} \kappa^2 \Pi_k C + \frac{3}{2} f_{lk} \Pi_l C + \frac{1}{i} \frac{\partial f_{lk}}{\partial x_l} C = 0, \quad (16.7) \dots (16.10)$$

$$(k = 1, 2, 3, 4).$$

Applying Π_k to these equations once more gives

$$2\kappa^2 \Pi_i \Pi_k A_{ik} - \frac{3}{2} \kappa^2 \Pi^2 C + \frac{3}{i} \frac{\partial f_{iu}}{\partial x_k} \Pi_l A_{ik} - \frac{3}{i} \frac{\partial f_{ik}}{\partial x_k} \Pi_l A_{li} - 3f_{ik} f_{kl} A_{li} + \Pi_k \frac{1}{i} \frac{\partial f_{il}}{\partial x_l} A_{ik} - \Pi_k \frac{1}{i} \frac{\partial f_{rk}}{\partial x_l} A_{rl} + \frac{3}{2i} \frac{\partial f_{lk}}{\partial x_k} \Pi_l C + \frac{3}{4} f_{lk} f_{kl} C + \Pi_k \frac{1}{i} \frac{\partial f_{lk}}{\partial x_l} C = 0.$$

Using (15.2) we obtain from this the subsidiary condition

$$3\kappa^4 C + \frac{2}{i} \frac{\partial f_{il}}{\partial x_k} \Pi_l A_{ik} - \frac{2}{i} \frac{\partial f_{il}}{\partial x_l} \Pi_k A_{ik} - 2 \frac{\partial^2 f_{il}}{\partial x_l \partial x_k} A_{ik} - 3f_{ik} f_{kl} A_{li} - \frac{1}{2i} \frac{\partial f_{lk}}{\partial x_k} \Pi_l C + \frac{3}{4} f_{kl} f_{lk} C = 0. \quad (16.1)$$

The tenth condition is found by differentiating (16.1) with respect to the time. One then obtains an expression containing the second time-derivative of C , $A_{\alpha\beta}$, $A_{\alpha 4}$ and A_{44} , but these can be eliminated with the help of equations (15.1), (15.2) and the time-derivatives of (16.3)–(16.6). We shall not give these rather confusing calculations but content ourselves with the knowledge that the 10 subsidiary conditions exist.

The expression for the electric charge-current vector is obtained by forming the derivatives of L with respect to the four-potential. One finds

$$s_k = \frac{e}{hc} [i\{B_{[kil]} A_{il}^* - B_{[ktil]}^* A_{il}\} + A_{ik}^* \Pi_r A_{rl} + A_{lk} \Pi_r^* A_{rl}^* - \frac{1}{2}\{C^* \Pi_r A_{rk} + C \Pi_r^* A_{rk}^* + A_{kr}^* \Pi_r C + A_{kr} \Pi_r^* C^*\} + \frac{3}{8}\{C^* \Pi_k C + C \Pi_k^* C^*\},$$

where $B_{[ikl]} \equiv i(\Pi_k A_{il} - \Pi_l A_{ki})$. For the force-free case, when Π_k becomes $-i\partial/\partial x_k$, this takes on the form given in (A). If one omits from L the term proportional to the field strengths one obtains an expression which is different from the above even in the force-free case, namely,

$$\bar{s}_k = \frac{e}{hc} \left\{ A_{il}^* \frac{\partial A_{il}}{\partial x_k} - A_{il} \frac{\partial A_{il}^*}{\partial x_k} \right\}.$$

This shows that the expression for the current in the limiting case of no forces is not unique, a point which incidentally arises already for the case of spin 1.

II. SPIN 3/2

4. THEORY WITH NO FORCES

In (A) it was shown that a force-free wave field corresponding to particles of spin 3/2 was described by spinors

$$a_{\beta\gamma}^{\dot{\alpha}} = a_{\gamma\beta}^{\dot{\alpha}}, \quad b_{\gamma}^{\dot{\alpha}\beta} = b_{\gamma}^{\beta\dot{\alpha}},$$

which go into one another by reflexion. They satisfy the equations

$$p^{\beta\rho} a_{\rho\gamma}^{\dot{\alpha}} + p^{\dot{\alpha}\rho} a_{\rho\gamma}^{\beta} = 2\kappa b_{\gamma}^{\dot{\alpha}\beta}, \quad p_{\alpha\beta} b_{\beta}^{\dot{\alpha}\gamma} + p_{\beta\beta} b_{\alpha}^{\rho\dot{\gamma}} = 2\kappa a_{\alpha\beta}^{\dot{\gamma}}, \tag{17.1}$$

together with the conditions

$$p_{\dot{\alpha}\beta} a_{\beta\gamma}^{\dot{\alpha}} = 0, \quad p_{\dot{\alpha}\gamma} b_{\gamma}^{\dot{\alpha}\beta} = 0, \tag{17.2}$$

where
$$p_{\dot{\alpha}\beta} = \sigma_{\dot{\alpha}\beta}^k \frac{1}{i} \frac{\partial}{\partial x_k}, \quad \text{with } \sigma_{\dot{\alpha}\beta}^k = (\sigma^x, \sigma^y, \sigma^z, i\mathbf{I})_{\dot{\alpha}\beta}$$

(cf. also the explanations in the Appendix. We only wish to point out here that $\sigma_{\dot{\alpha}\beta}^k$ is the Hermitian conjugate of $-\sigma_{\alpha\beta}^k$). The second order wave equation for $a_{\alpha\beta}^{\dot{\gamma}}$ and $b_{\gamma}^{\dot{\alpha}\beta}$ follows from these equations.

The additional conditions (17.2) mean that no particles of spin 1/2 are to be present. Fields which contain particles of spin 1/2 as well as of spin 3/2 have no longer a definite form for the total charges in the *c*-number theory and so cannot be quantized in accordance with the exclusion principle. On the other hand the latter is physically necessary in order that the energy should be positive in the *q*-number theory (cf. (A)). Equations (17.1) and (17.2) can, as in the previous case, be derived from a variation principle if one introduces auxiliary variables c_{α} and $d^{\dot{\alpha}}$. A suitable choice of the constants in *L* again causes the quantities $c_{\alpha}, d^{\dot{\alpha}}, p_{\dot{\alpha}\beta} a_{\beta\gamma}^{\dot{\alpha}}, p_{\dot{\alpha}\gamma} b_{\gamma}^{\dot{\alpha}\beta}$ (which belong to the spin value 1/2) to vanish as a consequence of the field equations in the force-free case. One has to choose for *L* the following:

$$\begin{aligned} L = & \kappa \{ a_{\dot{\alpha}\beta}^{*\gamma} b_{\gamma}^{\dot{\alpha}\beta} + b_{\gamma}^{*\alpha\beta} a_{\alpha\beta}^{\dot{\gamma}} \} - \{ a_{\dot{\alpha}\beta}^{*\gamma} p^{\beta\rho} a_{\rho\gamma}^{\dot{\alpha}} + b_{\gamma}^{*\alpha\beta} p_{\alpha\beta} b_{\beta}^{\dot{\alpha}\gamma} \} \\ & + \{ a_{\dot{\alpha}\beta}^{*\gamma} p_{\gamma}^{\beta} d^{\dot{\alpha}} + b_{\gamma}^{*\alpha\beta} p^{\dot{\gamma}\beta} c_{\alpha} + \text{conjugate} \} \\ & + 3 \{ d^{*\alpha} p_{\alpha\beta} d^{\beta} + c_{\dot{\alpha}}^{*\beta} p^{\dot{\alpha}\beta} c_{\beta} \} + 6\kappa \{ d^{*\alpha} c_{\alpha} + d^{\dot{\alpha}} c_{\dot{\alpha}}^* \}. \tag{18} \end{aligned}$$

Variation gives the equations

$$\left. \begin{aligned} 2\kappa b_{\gamma}^{\dot{\alpha}\dot{\beta}} - p^{\dot{\beta}\rho} a_{\gamma\rho}^{\dot{\alpha}} - p^{\dot{\alpha}\rho} a_{\gamma\rho}^{\dot{\beta}} + p^{\dot{\beta}}_{\gamma} d^{\dot{\alpha}} + p_{\alpha\dot{\beta}} d^{\dot{\beta}} = 0, \\ 2\kappa a_{\alpha\dot{\beta}}^{\dot{\gamma}} - p_{\alpha\dot{\beta}} b_{\dot{\beta}}^{\dot{\gamma}} - p_{\beta\dot{\beta}} b_{\alpha}^{\dot{\gamma}} + p^{\dot{\gamma}}_{\beta} c_{\alpha} + p^{\dot{\gamma}}_{\alpha} c_{\beta} = 0. \end{aligned} \right\} \quad (19.1)$$

$$\left. \begin{aligned} -p^{\dot{\beta}}_{\gamma} a_{\alpha\dot{\beta}}^{\dot{\gamma}} + 3p_{\alpha\dot{\beta}} d^{\dot{\beta}} + 6\kappa c_{\alpha} = 0, \\ -p^{\dot{\gamma}}_{\beta} b_{\dot{\beta}}^{\dot{\alpha}} + 3p^{\dot{\alpha}\dot{\beta}} c_{\beta} + 6\kappa d^{\dot{\alpha}} = 0. \end{aligned} \right\} \quad (19.2)$$

One can now deduce from these equations that c_{α} and $d^{\dot{\alpha}}$ vanish, as well as that (17.2) is valid. To do this let us apply $p^{\dot{\gamma}}_{\beta}$ to the first of equations (19.1) and $p^{\dot{\alpha}\dot{\beta}}$ to the first of equations (19.2), remembering that

$$p_{\alpha\dot{\beta}} p^{\dot{\beta}\dot{\gamma}} = \delta_{\alpha}^{\dot{\gamma}} \square, \quad p_{\alpha\dot{\beta}} p^{\dot{\beta}}_{\gamma} = \epsilon_{\alpha\dot{\gamma}} \square.$$

$$\text{One finds} \quad \left. \begin{aligned} 2\kappa p^{\dot{\gamma}}_{\beta} b_{\dot{\beta}}^{\dot{\alpha}} - p^{\dot{\alpha}\rho} p^{\dot{\gamma}}_{\beta} a_{\gamma\rho}^{\dot{\beta}} + 3\square d^{\dot{\alpha}} = 0, \\ -p^{\dot{\alpha}\rho} p^{\dot{\gamma}}_{\beta} a_{\gamma\rho}^{\dot{\beta}} + 3\square d^{\dot{\alpha}} + 6\kappa p^{\dot{\alpha}\dot{\beta}} c_{\beta} = 0. \end{aligned} \right\} \quad (20)$$

By subtraction it follows that

$$2\kappa \{ p^{\dot{\gamma}}_{\beta} b_{\dot{\beta}}^{\dot{\alpha}} - 3p^{\dot{\alpha}\dot{\beta}} c_{\beta} \} = 0.$$

Comparing this with the second of equations (19.2) we see that

$$d^{\dot{\alpha}} = 0.$$

Similarly one shows that the reflected quantity c_{α} vanishes. Equations (17.2) then follow from (19.2). Equations (19.1) take the form

$$\left. \begin{aligned} \kappa b_{\gamma}^{\dot{\alpha}\dot{\beta}} = p^{\dot{\beta}\rho} a_{\gamma\rho}^{\dot{\alpha}}, \\ \kappa a_{\alpha\dot{\beta}}^{\dot{\gamma}} = p_{\alpha\dot{\beta}} b_{\dot{\beta}}^{\dot{\gamma}}, \end{aligned} \right\}$$

from which follows the wave equation for $a_{\beta\dot{\gamma}}^{\dot{\alpha}}$ and $b_{\dot{\gamma}}^{\dot{\alpha}\dot{\beta}}$.

5. INTRODUCTION OF FORCES

One can now again introduce electromagnetic forces by replacing $p_{\alpha\dot{\beta}}$ by $\Pi_{\alpha\dot{\beta}}$, where $\Pi_{\alpha\dot{\beta}}$ is the spinor corresponding to the Π_k already defined. One then has the following equations:

$$\left. \begin{aligned} 2\kappa b_{\gamma}^{\dot{\alpha}\dot{\beta}} - \Pi^{\dot{\beta}\rho} a_{\gamma\rho}^{\dot{\alpha}} - \Pi^{\dot{\alpha}\rho} a_{\gamma\rho}^{\dot{\beta}} + \Pi^{\dot{\beta}}_{\gamma} d^{\dot{\alpha}} + \Pi^{\dot{\alpha}}_{\gamma} d^{\dot{\beta}} = 0, \\ 2\kappa a_{\alpha\dot{\beta}}^{\dot{\gamma}} - \Pi_{\alpha\dot{\beta}} b_{\dot{\beta}}^{\dot{\gamma}} - \Pi_{\beta\dot{\beta}} b_{\alpha}^{\dot{\gamma}} + \Pi^{\dot{\gamma}}_{\beta} c_{\alpha} + \Pi^{\dot{\gamma}}_{\alpha} c_{\beta} = 0, \end{aligned} \right\} \quad (21.1)$$

$$\left. \begin{aligned} -\Pi^{\dot{\beta}}_{\gamma} a_{\alpha\dot{\beta}}^{\dot{\gamma}} + 3\Pi_{\alpha\dot{\beta}} d^{\dot{\beta}} + 6\kappa c_{\alpha} = 0, \\ -\Pi^{\dot{\gamma}}_{\beta} b_{\dot{\beta}}^{\dot{\alpha}} + 3\Pi^{\dot{\alpha}\dot{\beta}} c_{\beta} + 6\kappa d^{\dot{\alpha}} = 0. \end{aligned} \right\} \quad (21.2)$$

Eight subsidiary conditions must follow from these equations, which in the force-free case must lead to the vanishing of c_{α} , $d^{\dot{\alpha}}$, and the validity of (17.2).

As the equations (21.1), (21.2) are of first order in the time derivatives the subsidiary conditions must not contain any time derivatives at all. Applying $\Pi\gamma_\beta$ to (21.1), $\Pi^{\alpha\dot{\alpha}}$ to (21.2) we find, analogously to the previous case,

$$2\kappa\Pi\gamma_\beta b^{\beta\dot{\alpha}}_\gamma - \frac{1}{2}(\Pi\gamma_\beta\Pi\rho^{\dot{\beta}} + \Pi\rho_\beta\Pi\gamma^{\dot{\beta}})a^{\dot{\alpha}}_{\gamma\rho} - \Pi\gamma_\beta\Pi^{\dot{\alpha}\rho}a^{\dot{\beta}}_{\gamma\rho} + f^{\dot{\alpha}\beta}d^\beta - 3\Pi^2d^{\dot{\alpha}} = 0,$$

$$-\Pi^{\dot{\alpha}\rho}\Pi\gamma_\beta a^{\dot{\beta}}_{\gamma\rho} + 3f^{\dot{\alpha}\beta}d^\beta - 3\Pi^2d^{\dot{\alpha}} + 6\kappa\Pi\rho^{\dot{\alpha}}c_\rho = 0,$$

where we have substituted

$$\Pi_{\dot{\alpha}\beta}\Pi\dot{\gamma}^\delta - \Pi_{\dot{\gamma}\delta}\Pi^{\dot{\alpha}\beta} = \delta_{\dot{\alpha}}^{\dot{\gamma}}f_{\beta}^{\delta} + \delta_{\beta}^{\delta}f_{\dot{\alpha}}^{\dot{\gamma}},$$

$f^{\beta\delta}$, $f^{\dot{\alpha}\dot{\gamma}}$ being the symmetrical spinors corresponding to the electromagnetic field strengths.

Subtracting the two equations we obtain

$$2\kappa\Pi\gamma_\beta b^{\beta\dot{\alpha}}_\gamma - 2f^{\gamma\rho}a^{\dot{\alpha}}_{\gamma\rho} - 2f_{\rho}^{\dot{\alpha}\beta}d^\beta - 6\kappa\Pi\rho^{\dot{\alpha}}c_\rho = 0. \tag{22}$$

If we now compare this with the second of equations (21.2) we find

$$6\kappa d^{\dot{\alpha}} - \frac{1}{\kappa}f_{\beta}^{\dot{\alpha}\beta}d^\beta = \frac{1}{\kappa}f^{\gamma\rho}a^{\dot{\alpha}}_{\gamma\rho}. \tag{23}$$

Similarly the reflected equation follows:

$$6\kappa c_\alpha - \frac{1}{\kappa}f_{\alpha}^{\beta}c_\beta = \frac{1}{\kappa}f_{\dot{\gamma}\dot{\rho}}b^{\dot{\gamma}\dot{\rho}}_\alpha. \tag{24}$$

This gives four subsidiary conditions. To be able to find four more we must separate time and space derivatives in (21.1), (21.2). For this we consider these equations in a co-ordinate system in which the time co-ordinate is fixed and therefore only require invariance under rotations of space. The spinor $s^{\dot{\alpha}}$ is then equivalent to s_α , and $\Pi^{\dot{\alpha}}_\beta$ is equivalent to $\Pi_{\alpha\beta}$, where

$$\Pi_{\alpha\beta} = \sum_{k=1}^3 \sigma_{\alpha\beta}^k \Pi_k.$$

When written in this form the equations contain Π_4 explicitly as well. The first equations in (21.1), (21.2) become

$$2\kappa b_{\alpha\beta,\gamma} - \Pi_{\beta}^{\rho}a_{\alpha,\gamma\rho} - i\Pi_4 a_{\alpha,\gamma\beta} - \Pi_{\alpha}^{\rho}a_{\beta,\gamma\rho} - i\Pi_4 a_{\beta,\gamma\alpha} + \Pi_{\beta\gamma}d_\alpha + i\epsilon_{\beta\gamma}\Pi_4 d_\alpha + \Pi_{\alpha\gamma}d_\beta + i\epsilon_{\alpha\gamma}\Pi_4 d_\beta = 0, \tag{25}$$

$$-\Pi^{\beta\gamma}a_{\gamma,\alpha\beta} + i\Pi_4 \epsilon^{\beta\gamma}a_{\gamma,\alpha\beta} + 3\Pi_{\alpha}^{\beta}d_\beta + 3i\Pi_4 d_\alpha + 6\kappa c_\alpha = 0. \tag{26}$$

We now multiply (25) by $\epsilon^{\beta\gamma}$ and obtain

$$2\kappa b_{\alpha\beta,\beta} - \Pi_{\beta}^{\rho}a_{\alpha,\rho\beta} - \Pi_{\alpha}^{\rho}a_{\beta,\rho\beta} - i\Pi_4 a_{\beta,\beta\rho} + \Pi_{\beta}^{\beta}d_\alpha + 3i\Pi_4 d_\alpha + \Pi_{\alpha}^{\beta}d_\beta = 0. \tag{27}$$

Downloaded from https://royalsocietypublishing.org/ on 04 August 2022

Forming the difference of (27) and (26) we obtain

$$2\kappa b_{\alpha\beta, \beta} + \Pi^{\beta\rho} a_{\alpha, \beta\rho} + \Pi_{\alpha\rho} a_{\beta, \beta\rho} + \Pi^{\beta\gamma} a_{\gamma, \alpha\beta} + \Pi_{\beta}^{\beta} d_{\alpha} - 2\Pi_{\alpha}^{\beta} d_{\beta} - 6\kappa c_{\alpha} = 0. \tag{28}$$

Similarly for the reflected equation we find

$$2\kappa a_{\alpha\beta, \beta} + \Pi^{\beta\rho} b_{\alpha, \beta\rho} + \Pi_{\alpha\rho} b_{\beta, \beta\rho} + \Pi^{\beta\gamma} b_{\gamma, \alpha\beta} + \Pi_{\beta}^{\beta} c_{\alpha} - 2\Pi_{\alpha}^{\beta} c_{\beta} - 6\kappa d_{\alpha} = 0. \tag{29}$$

Equations (28) and (29) constitute 4 more subsidiary conditions. As a consequence of these 8 conditions the dimensionality of the states is unaltered by a field, and we can develop the theory in powers of the charge.

In conclusion we give the current-charge density vector

$$s_{\alpha\beta} = a_{\mu\beta}^{*\gamma} a_{\gamma\alpha}^{\mu} + b_{\gamma, \alpha}^{*\mu} b_{\beta, \mu}^{\gamma} + \{a_{\mu\beta, \alpha}^{*\mu} d^{\mu} + b_{\beta, \alpha\mu}^{*\mu} c^{\mu} + \text{conjugate}\} - 3\{d_{\alpha}^{*} d_{\beta} + c_{\beta}^{*} c_{\alpha}\}.$$

For the force-free case this reduces to the expression which was denoted by $s_{\alpha\beta}^{(0)}$ in (A).

III. REST-MASS ZERO

6. SPIN 2

One can set κ equal to zero in the formulae derived above for A_{ik} and C and so obtain a theory for zero rest-mass. The equations then run

$$-2\Box A_{ik} + 2\left(\frac{\partial^2 A_{sk}}{\partial x_s \partial x_i} + \frac{\partial^2 A_{si}}{\partial x_s \partial x_k} - \frac{1}{2}\delta_{ik} \frac{\partial^2 A_{rs}}{\partial x_r \partial x_s}\right) - \frac{\partial^2 C}{\partial x_i \partial x_k} + \frac{1}{4}\delta_{ik}\Box C = 0, \tag{30.1}$$

$$\frac{3}{4}\Box C - \frac{\partial^2 A_{rk}}{\partial x_r \partial x_k} = 0. \tag{30.2}$$

It no longer follows from these equations that C and $\partial A_{ik}/\partial x_i$ vanish. Nevertheless, there are four identities which follow from them. For if we differentiate (30.1) with respect to x_i , (30.2) with respect to x_k , we obtain in either case

$$\frac{3}{4}\Box \frac{\partial C}{\partial x_k} - \frac{\partial}{\partial x_k} \frac{\partial^2 A_{rl}}{\partial x_r \partial x_l} = 0, \quad (k = 1, 2, 3, 4). \tag{31}$$

Thus by subtracting the two we get identically zero for each value of k . As a result of these identities it is possible to construct quantities A_{ik}^0, C^0

from an arbitrary vector field f_i , which satisfy the field equations identically. Let us write

$$\left. \begin{aligned} A_{ik}^0 &= \frac{\partial f_i}{\partial x_k} + \frac{\partial f_k}{\partial x_i} - \frac{1}{2} \delta_{ik} \frac{\partial f_l}{\partial x_l} \\ C^0 &= 2 \frac{\partial f_l}{\partial x_l} \end{aligned} \right\} \quad (32)$$

One can easily verify that the Lagrange function to which (11) reduces on setting κ equal to zero is altered by the "gauge transformation"

$$\left. \begin{aligned} A'_{ik} &= A_{ik} + A_{ik}^0 \\ C' &= C + C^0 \end{aligned} \right\} \quad (33)$$

only to the extent of a complete differential.

The formulation in (A) is the same as the one here if one chooses the gauge in such a way that

$$\frac{\partial A_{ik}}{\partial x_i} = 0,$$

which is analogous to the Lorentz condition for the electromagnetic potentials. This condition restricts the gauge transformations to the group discussed in (A). The present scheme is identical with Einstein's "first approximation" of the gravitational equations.

Einstein (1916) considers the equations for the gravitational field in the cases when the deviations from a Euclidean metric are small quantities of the first order. We write

$$g_{ik} = \delta_{ik} + \gamma_{ik}; \quad \gamma_{ii} = \gamma.$$

Now let us write $\gamma_{ik} = A_{ik} + \frac{1}{4} \delta_{ik} C; \quad \gamma = C.$

We obtain the following differential equations for γ_{ik} :

$$\left. \begin{aligned} -\square \gamma_{ik} - \frac{\partial^2 \gamma}{\partial x_i \partial x_k} + \frac{\partial^2 \gamma_{lk}}{\partial x_l \partial x_i} + \frac{\partial^2 \gamma_{li}}{\partial x_l \partial x_k} + \frac{1}{2} \delta_{ik} \left(\square \gamma - \frac{\partial^2 \gamma_{lr}}{\partial x_l \partial x_r} \right) &= 0, \\ \square \gamma - \frac{\partial^2 \gamma_{lr}}{\partial x_l \partial x_r} &= 0. \end{aligned} \right\} \quad (34)$$

These equations are the same as those that Einstein gave for space containing no matter.

The gauge transformation (33) occurs in the gravitational theory as an infinitesimal co-ordinate transformation. When interactions with matter occur and it is no longer sufficient to restrict oneself to the linear terms the

gauge group is altered. This keeps the dimensionality of the possible transformations unchanged; four functions of position always remain arbitrary. It is well known that the existence of an energy-momentum tensor is closely connected with the invariance of the gravitational theory under these transformations. Similarly, the gauge invariance of Maxwell's theory is connected with the conservation of charge.

7. SPIN 3/2

Setting κ equal to zero in the equations for spin 3/2 gives

$$\left. \begin{aligned} -p^{\beta\rho} a_{\gamma\rho}^{\dot{\alpha}} - p^{\dot{\alpha}\rho} a_{\gamma\rho}^{\beta} + p^{\beta}_{\gamma} d^{\dot{\alpha}} + p^{\dot{\alpha}}_{\gamma} d^{\beta} &= 0, \\ -p_{\alpha\beta} b_{\beta}^{\dot{\alpha}\dot{\gamma}} - p_{\beta\dot{\rho}} b_{\alpha}^{\dot{\gamma}\dot{\rho}} + p^{\dot{\gamma}}_{\beta} c_{\alpha} + p^{\dot{\gamma}}_{\alpha} c_{\beta} &= 0; \end{aligned} \right\} \quad (35)$$

$$\left. \begin{aligned} -p^{\beta}_{\dot{\gamma}} a_{\alpha\beta}^{\dot{\gamma}} + 3p_{\alpha\beta} d^{\beta} &= 0, \\ -p^{\gamma}_{\beta} b_{\gamma}^{\dot{\alpha}\dot{\beta}} + 3p^{\dot{\alpha}\dot{\beta}} c_{\beta} &= 0. \end{aligned} \right\} \quad (36)$$

From these equations there follow four, or if we follow the procedure of Majorana (1937) and impose reality conditions on the field quantities, two identities. For, differentiating (35) with $p_{\dot{\alpha}\gamma}$, (36) with $p^{\beta\dot{\alpha}}$ we find in both cases

$$-p^{\beta\rho} p_{\dot{\alpha}\gamma} a_{\gamma\rho}^{\dot{\alpha}} + 3\Box d^{\beta} = 0 \quad (\dot{\beta} = 1, 2),$$

and similarly for the reflected equations. One can therefore find solutions with the help of spinor fields of first rank f_{α} , $g^{\dot{\alpha}}$, which satisfy the equations identically. Let us write

$$\left. \begin{aligned} a_{\alpha\beta}^{0\dot{\gamma}} &= p_{\alpha}^{\dot{\gamma}} f_{\beta} + p_{\beta}^{\dot{\gamma}} f_{\alpha}; & d^{0\dot{\beta}} &= p^{\dot{\beta}\alpha} f_{\alpha}; \\ b_{\gamma}^{0\dot{\alpha}\dot{\beta}} &= p_{\gamma}^{\dot{\alpha}} g^{\dot{\beta}} + p_{\gamma}^{\dot{\beta}} g^{\dot{\alpha}}; & c^0_{\beta} &= p_{\beta\dot{\alpha}} g^{\dot{\alpha}}. \end{aligned} \right\}$$

The effect of the transformation $a'^{\dot{\gamma}}_{\alpha\beta} = a^{0\dot{\gamma}}_{\alpha\beta} + a^{\dot{\gamma}}_{\alpha\beta}$, ... on the Lagrange function is again to add a complete differential.

Whereas the theory for the spin value 2 has an important generalization for force fields, namely the gravitational theory, we here have no such connexion with a known theory. To get a generalization of the theory with interactions, one would first of all have to find a physical interpretation of the gauge group, and of the conservation theorem connected with this group.

8. GENERAL CASE OF ARBITRARY SPIN

To set up a theory with forces for particles of arbitrary spin one again first looks for a variation principle from which the equations of (A) can be derived. The forces can then be introduced by suitable modifications of the

Lagrange function. For instance, the effect of an electromagnetic field can be described by replacing $-i\partial/\partial x_k$ by Π_k . In generalizing the method which we have already used for the spin values 2 and 3/2 we must again introduce auxiliary fields, which, in the force-free case, vanish as a consequence of the field equations. To illustrate the method it will be sufficient to discuss the case of integral spin.

We start from a tensor $A_{ik\dots l}^f$ of rank f , symmetrical in all its indices and whose trace $A_{ii\dots l}^f$ vanishes. Let us further introduce auxiliary fields of rank $f-2, \dots, s, \dots, 1, 0$ which are likewise all symmetrical and of zero trace; there may be several fields with the same rank, which we shall distinguish by an index t . The general field occurring in the Lagrange function we shall thus denote by $A_{ik\dots l}^{st}$, i, k, \dots, l being tensor indices, with respect to which the field is symmetrical, and t distinguishes the different fields of rank s . The index s takes the values $0, 1, \dots, f-2, f$. In general with a field A^s of rank s one can associate $s+1$ kinds of particles, i.e. those of spin $s, s-1, \dots, 1, 0$. The Lagrangian must be so constructed that in the end only those of spin f occur. Or we can say that as a consequence of the field equations all the fields corresponding to particles of spin $f-1, \dots, 1, 0$ vanish. For L we choose the following:

$$L = \sum_s \sum_t \left[\kappa^2 A_{ik\dots}^{st} A_{ik\dots}^{st} + a_1^{st} \frac{\partial A_{ik\dots}^{st}}{\partial x_i} \frac{\partial A_{ik\dots}^{st}}{\partial x_l} + a_2^{st} \frac{\partial A_{ik\dots}^{s+1,t}}{\partial x_i} \frac{\partial A_{ik\dots}^{s+1,t}}{\partial x_l} \right. \\ \left. + \sum_r \left\{ a_3^{s,tr} \frac{\partial A_{ik\dots}^{st}}{\partial x_i} A_{il\dots}^{s+1,r} + a_4^{s,tr} \frac{\partial A_{il\dots}^{st}}{\partial x_k} \frac{\partial A_{miki\dots}^{s+2,r}}{\partial x_m} \right\} \right], \quad (37)$$

where $a_1^{st}, a_2^{st}, a_3^{s,tr}, a_4^{s,tr}$ are constants which must be so chosen that particles of spin s do not occur. In order to get the requisite number of constants we introduce auxiliary fields whose number is given by the following table:

s	Number of fields of rank s	Number of particles of spin s to be removed	Corresponding number of constants at our disposal
$f-1$	0	1	1
$f-2$	1	2	2
$f-3$	1	3	3
$f-4$	1	4	4
$f-5$	2	6	7
$f-6$	3	9	14
...
$f-n$	$n-3$	$\frac{1}{2}(n-2)(n-3)+3$	$2(n-3)^2-n+2$

One sees from the table that in this way there are more constants than are necessary. One can therefore construct the Lagrange function in many ways

such that the same force-free theory follows as in (A). We have not yet been able to find any simple way of avoiding this ambiguity. In particular, it is impossible to manage with only one field of each rank, for one would then obtain n particles of spin $f - n$ and for $n \geq 5$ only 4 constants to remove them.

A completely analogous procedure also works for half-integral spin. The only difference lies in the fact that the Lagrangian contains derivatives of first order only, leading to a slight difference in the number of constants. It is to be remarked that already for $5/2$ one must introduce *two* fields corresponding to spin $1/2$. As the method is otherwise the same as for integral spin there is no need to go into the details.

APPENDIX

(1) *Forms of Lagrangian leading to singular solutions*

As we have already stressed, if one wants to modify the Lagrange function of the field for particles of spin ≥ 1 in a manner corresponding to the interaction with other fields, one must take care that the number of restrictive conditions is not diminished. For then the switching on of the forces would create new particles whose corresponding particular solutions of the equations become singular when the field is switched off again.

As an example of this we shall give an inadmissible form of interaction of particles of spin 1 with a scalar field ψ . Let us write

$$L = \kappa^2 A_i^2 + \left(\frac{\partial A_i}{\partial x_k} \right)^2 - \left(\frac{\partial A_i}{\partial x_i} \right)^2 + \psi \left(\frac{\partial A_i}{\partial x_i} \right)^2. \quad (38)$$

The field equations are

$$\square A_k - \frac{\partial^2 A_i}{\partial x_i \partial x_k} = \kappa^2 A_k - \frac{\partial}{\partial x_k} \left(\psi \frac{\partial A_i}{\partial x_i} \right). \quad (39)$$

From these follows $\frac{\partial A_i}{\partial x_i} = \square \left(\psi \frac{\partial A_i}{\partial x_i} \right)$.

When $\psi \equiv 0$ this is a subsidiary condition for A_i which is, however, removed by the interaction.

To study the character of the new solutions arising from the interaction with ψ let us make the assumption that A_i and ψ depend only on the time. Then we have

$$\frac{\partial^2 A_k}{\partial x_4^2} - \delta_{k4} \frac{\partial^2 A_4}{\partial x_4^2} = \kappa^2 A_k - \delta_{k4} \frac{\partial}{\partial x_4} \left(\psi \frac{\partial A_4}{\partial x_4} \right).$$

The equation for A_4 is of interest since it contains ψ . It runs

$$\kappa^2 A_4 = \frac{\partial}{\partial x_4} \left(\psi \frac{\partial A_4}{\partial x_4} \right). \tag{40}$$

Let us suppose that ψ changes so slowly that $\partial\psi/\partial x_4$ can be neglected. We have then approximately

$$\frac{\kappa^2}{\psi} A_4 = \frac{\partial^2 A_4}{\partial x_4^2}.$$

The waves A_4 therefore correspond to a mass $\kappa/\sqrt{\psi}$. This mass is imaginary for negative ψ ; for $\psi = 0$ it becomes infinite. One therefore obtains strongly singular solutions in the limit $\psi \rightarrow 0$. Equation (40) can be solved explicitly if one takes for ψ the form

$$\psi = \pm e^{-\alpha t}, \quad \alpha > 0.$$

One finds

$$A_4 = \Im \frac{\text{const.}}{\sqrt{\psi}} Z_1(2\kappa/\alpha\sqrt{\psi}),$$

where \Im denotes the imaginary part, and Z_1 is a Bessel function of first order. A_4 diverges exponentially as ψ approaches zero from negative values.

(2) Rules for spinor calculus

In the following we shall collect a few definitions and rules of spinor calculus which have been used in the previous pages.

(1) Spinor indices are raised and lowered according to the following rule:

$$v_1 = -v^2; \quad v_2 = v^1. \tag{1}$$

The scalar product of two spinors is accordingly

$$v_\alpha u^\alpha = -v^2 u_2 = v_1 u_2 - v_2 u_1. \tag{2}$$

This can be expressed in terms of the invariant spinor

$$\epsilon_{12} = -\epsilon_{21} = \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon_{11} = \epsilon_{22} = 0.$$

$$v^\alpha = \epsilon^{\alpha\beta} v_\beta, \quad v_\alpha = v^\beta \epsilon_{\beta\alpha},$$

$$v_\alpha u^\alpha = \epsilon^{\alpha\beta} v_\alpha u_\beta = -\epsilon^{\alpha\beta} u_\alpha v_\beta.$$

Thus to raise a suffix one applies $\epsilon^{\alpha\beta}$ on the left, to lower, on the right.

The transition from spinors to four-dimensional tensors is done by means of the matrices $\sigma_{\alpha\beta}^k$, where k runs from 1 to 4, α and β from 1 to 2. They are defined by

$$\sigma_{\alpha\beta}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{\alpha\beta}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{\alpha\beta}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{\alpha\beta}^4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{3}$$

(This differs from the notation of v. d. Waerden (1932), who uses $\sigma^4 = I$. The above is more convenient here as we are using the imaginary co-ordinate $x_4 = ict$.) From (1) it follows that

$$\begin{aligned} \sigma_{1\dot{2}}^k &= -\sigma^{k, 1\dot{2}}, & \sigma_{2\dot{1}}^k &= -\sigma^{k, 2\dot{1}}, \\ \sigma_{1\dot{1}}^k &= \sigma^{k, 2\dot{2}}, & \sigma_{2\dot{2}}^k &= \sigma^{k, 1\dot{1}}. \end{aligned}$$

As the trace of σ^k vanishes for $k = 1, 2, 3$ we have the following rule:

$$\begin{aligned} \sigma^{k, \dot{\alpha}\beta} &= -\sigma^k_{\dot{\alpha}\beta} \quad \text{for } k = 1, 2, 3, \\ \sigma^{4, \dot{\alpha}\beta} &= \sigma^4_{\dot{\alpha}\beta}. \end{aligned}$$

The σ^k satisfy the commutation rules

$$\left. \begin{aligned} \sigma_{\dot{\alpha}\beta}^k \sigma^{l, \beta\dot{\gamma}} + \sigma_{\dot{\alpha}\beta}^l \sigma^{k, \beta\dot{\gamma}} &= -2\delta_{kl} \delta_{\dot{\alpha}}^{\dot{\beta}}, \\ \sigma_{\dot{\alpha}\beta}^k \sigma^{l, \gamma\dot{\alpha}} + \sigma_{\dot{\alpha}\beta}^l \sigma^{k, \gamma\dot{\alpha}} &= -2\delta_{kl} \delta_{\beta}^{\gamma}. \end{aligned} \right\} \quad (4)$$

One puts a four-vector a_k in correspondence with a spinor $a_{\dot{\alpha}\beta}$ with the help of the $\sigma^k_{\dot{\alpha}\beta}$ as follows:

$$a_{\dot{\alpha}\beta} = a_k \sigma^k_{\dot{\alpha}\beta}. \quad (5)$$

Conversely we have

$$a_{\dot{\alpha}\beta} \sigma^{k, \dot{\alpha}\beta} = -2a_k. \quad (6)$$

The four-vector a_k can also be an operator, for instance $-i\partial/\partial x_k$. If the components of a_k commute with one another,

$$a_i a_k - a_k a_i = 0,$$

then it follows from (4) that

$$\left. \begin{aligned} a_{\dot{\alpha}\beta} a^{\beta\dot{\gamma}} &= -a^2 \delta_{\dot{\alpha}}^{\dot{\gamma}}, \\ a_{\dot{\alpha}\beta} a^{\gamma\dot{\alpha}} &= -a^2 \delta_{\beta}^{\gamma}, \end{aligned} \right\} \quad (7)$$

where

$$a^2 = \sum_{k=1}^4 a_k^2 = -\frac{1}{2} a_{\dot{\alpha}\beta} a^{\beta\dot{\alpha}}.$$

In particular for the spinor

$$p_{\dot{\alpha}\beta} = \frac{1}{i} \frac{\partial}{\partial x_k} \sigma^k_{\dot{\alpha}\beta}, \quad (8)$$

it follows from (7) that

$$p_{\dot{\alpha}\beta} p^{\beta\dot{\gamma}} = \delta_{\dot{\alpha}}^{\dot{\gamma}} \square, \quad (9)$$

where

$$\square \equiv \sum_{k=1}^4 \frac{\partial^2}{\partial x_k^2}.$$

Again, if

$$\Pi_k = \frac{1}{i} \frac{\partial}{\partial x_k} - \frac{e}{hc} \phi_k, \quad (10)$$

then we have

$$\Pi_k \Pi_l - \Pi_l \Pi_k = \frac{ie}{hc} \left(\frac{\partial \phi_l}{\partial x_k} - \frac{\partial \phi_k}{\partial x_l} \right) \equiv f_{kl}. \quad (11)$$

As the components of Π_k do not commute with one another we have, from (4)

$$\begin{aligned} \Pi_{\dot{\alpha}\beta} \Pi^{\beta\dot{\gamma}} &= \sigma_{\dot{\alpha}\beta}^k \sigma^{\dot{\alpha}\beta\dot{\gamma}} \Pi_k \Pi_l \\ &= \frac{1}{2} \sigma_{\dot{\alpha}\beta}^k \sigma^{\dot{\alpha}\beta\dot{\gamma}} \{ \Pi_k \Pi_l + \Pi_l \Pi_k + f_{kl} \} \\ &= -\Pi^2 \delta_{\dot{\alpha}}^{\dot{\gamma}} + f_{\dot{\alpha}}^{\dot{\gamma}}, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \sigma_{\dot{\alpha}\beta}^k \sigma^{\dot{\alpha}\beta\dot{\gamma}\delta} f_{kl} &= \delta_{\dot{\beta}}^{\delta} f_{\dot{\alpha}}^{\dot{\gamma}} + \delta_{\dot{\alpha}}^{\dot{\gamma}} f_{\dot{\beta}}^{\delta} \\ &= \Pi_{\dot{\alpha}\beta} \Pi^{\dot{\gamma}\delta} - \Pi^{\dot{\gamma}\delta} \Pi_{\dot{\alpha}\beta} \end{aligned} \tag{13}$$

is the spinor corresponding to the field strengths (cf. Uhlenbeck and Laporte 1931), with the property that

$$f_{\alpha}^{\alpha} = f_{\dot{\alpha}}^{\dot{\alpha}} = 0.$$

Besides the commutation rules (4) the $\sigma_{\dot{\alpha}\beta}^k$ also satisfy the relations

$$\sum_{k=1}^4 \sigma_{\dot{\alpha}\beta}^k \sigma_{\dot{\gamma}\delta}^k = 2(\delta_{\dot{\alpha}\delta} \delta_{\dot{\gamma}\beta} - \delta_{\dot{\alpha}\beta} \delta_{\dot{\gamma}\delta}) = 2\epsilon_{\dot{\alpha}\dot{\gamma}} \epsilon_{\delta\beta}. \tag{14}$$

Let us now consider the tensor b_{kl} of second rank corresponding to the symmetrical spinor

$$b_{\dot{\alpha}\dot{\beta}, \gamma\delta} = b_{\dot{\beta}\dot{\alpha}, \gamma\delta} = b_{\dot{\alpha}\dot{\beta}, \delta\gamma}.$$

It follows at once from (14) that its trace vanishes,

$$b_{kk} = 0.$$

SUMMARY

The force-free theory of particles with arbitrary spin values already published by one of the authors is generalized to the relativistic wave equations of such particles in an electromagnetic field, with a preliminary restriction to the c -number theory. The spin values $3/2$ and 2 are treated in detail, and for the general case it is merely proved that consistent wave equations exist. The consistency of the system of field equations is attained by deriving them from a Lagrange function containing suitable additional terms which depend on new auxiliary quantities. All the differential equations of the field are derived by variation of the action integral and the vanishing of the auxiliary quantities in the absence of an external field is made to follow as a consequence of them.

In the special case of zero rest-mass there exist identities between the equations, which are now invariant under a group of transformations which is the generalization of the group of gauge transformations in Maxwell's

theory. In the particular case of spin 2, rest-mass zero, the equations agree in the force-free case with Einstein's equations for gravitational waves in general relativity in first approximation; the corresponding group of transformations arises from the infinitesimal co-ordinate transformations.

REFERENCES

- Dirac 1936 *Proc. Roy. Soc. A*, **155**, 447.
 Einstein 1916 *S.B. preuss. Akad. Wiss. (Math. Phys.)*, p. 688.
 — 1918 *S.B. preuss. Akad. Wiss. (Math. Phys.)*, p. 154.
 Fierz 1939 *Helv. Phys. Acta*, **12**, 3 (referred to as (A)).
 Majorana 1937 *Nuovo Cim.* **14**, 171.
 Uhlenbeck and Laporte 1931 *Phys. Rev.* **37**, 1380.
 v. d. Waerden 1932 *Die gruppentheoretische Methode in der Quantenmechanik*, p. 83. Berlin: Springer.

Defect lattices in some ternary alloys

BY H. LIPSON, D.Sc.,

Crystallographic Department, Cavendish Laboratory, Cambridge

AND A. TAYLOR, Ph.D.,

Chemistry Department, King's College, Newcastle-upon-Tyne

(Communicated by A. J. Bradley, F.R.S.—Received 9 June 1939)

In the work on ternary alloys with which the authors have been associated (Bradley, Goldschmidt, Lipson and Taylor 1937), the most outstanding feature has been the large range of composition over which the β (body-centred cubic) structure exists. This was not unexpected in the system Fe-Ni-Al (Bradley and Taylor 1938), where FeAl and NiAl both have this structure and thus may be expected to be isomorphous; but CuAl has quite a different structure, and it was therefore surprising to find the β phase-fields extending towards this composition in the systems Cu-Ni-Al (Bradley and Lipson 1938) and Fe-Cu-Al (Bradley and Goldschmidt 1939).

These extensive phase-fields provide an opportunity for a more detailed examination of the phenomenon of the occurrence of defect lattices which takes place in the Ni-Al system (Bradley and Taylor 1937). In the β