

## ON RELIABLE CONTROL SYSTEM DESIGNS\*

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## ABSTRACT

This paper summarizes a research effort which addresses some of the current problems in interfacing systems theory and reliability. Reliability is roughly the probability that a system will perform according to specifications for a given amount of time. The reliability of a system depends on the structure of its components. Systems theory and control theory deal with the response characteristics of a system, which depend on the system dynamics. This report defines the concepts necessary to unify the structural and the dynamic properties of a system. The result is a definition of what constitutes a reliable system, from the viewpoint of systems theory, and a methodology which can be used to determine if a given design allows a reliable control system design.

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## 1. Introduction

A methodology which allows a comparison between alternate designs, based on both the expected system reliability and the expected system performance, is presented. Although the mathematics is restricted to linear systems with quadratic cost criteria and structural dynamics which can be represented by Markov chains, the concept of what constitutes a reliable control system design is applicable to a broad class of systems.

This work demonstrates the classification of a system design as reliable or unreliable, based on the existence of a steady-state switching gain and finite cost for that design. If this gain does not exist with finite cost, then the system design cannot be stabilized; hence, it is unreliable. The only recourse in such a case is to use more reliable components and/or more redundancy. Reliability of a system design can therefore be determined by a test for convergence of a set of coupled Riccati-like equations as the final time goes to infinity.

The reliability of a system is the probability that the system will perform according to specifications for a given amount of time. In a system-theoretic context, the specification which a system must meet is stability; also, since stability is a long-term attribute of a system, the amount of time for which the system must remain stable is taken to be infinite. The following definitions of system reliability are used in this paper:

Definition 1: A system (implying the hardware configuration, or mathematical model of that configuration, and its associated control and estimation structure) has reliability  $r$  where  $r$  is the probability that the system will be stable for all time.

Definition 2: A system is said to be reliable if  $r = 1$ .

Definition 3: A system design, or configuration, is reliable if it is stabilizable with probability one.

These definitions of reliability depend on the definition of stability, and for systems which can have more than one mode of operation, stability is not always easily determined. In this paper, stability will mean cost-stability, which is the property that the accumulated cost of system operation is bounded with probability one. (The definition of cost is deferred.)

A few remarks are now in order. First, the system referenced in the definitions has no specified mathematical form. The definitions are applicable for systems which have jump discontinuities in their dynamics due to failures, or gradual degradations. Second, the definition of stability must account for random structural changes due to failures, reconfigurations, and repairs. A structural model must exist; it is the structural dynamics which determine the probability space over which the expected behavior of the system is determined. Finally, these definitions divide system designs into two classes: A system design is either reliable (stabilizable with respect to an appropriate definition of stability), or it is not. This research is focused on the development of a methodology which can be used to determine if a system design is stabilizable. A system's reliability depends on three elements: The system dynamics, the system structure, and the structural dynamics. The proposed methodology is influenced by all three elements.

## 2. Previous Work

Several authors have studied the optimal control of systems with randomly varying structure. Most notable among these is Wonham [1], where the solution to the continuous time linear regulator problem with randomly jumping parameters is developed. This solution is similar to the discrete time switching gain solution presented in Section 3. Wonham also proves an existence result for the steady-state optimal solution to the control of systems with randomly varying structure; however, the conclusion is only sufficient; it is not necessary. Similar results were obtained in Beard [2] for the existence of a stabilizing gain, where the structures were of a highly specific form; these results were necessary and sufficient algebraic conditions, but cannot be readily generalized to less specific classes of systems. Additional work on the control problem for this class of systems has been done by Sworder [3], Ratner & Luenberger [4], Bar-Shalom & Sivan [5], Willner [6] and Pierce & Sworder [7]. The dual problem of state estimation with a system with random parameter variations over a finite set was studied in Chang & Athans [8].

Some of the preliminary results on which this research was based were presented in unpublished form at the 1977 Joint Automatic Control Conference in San Francisco by Birdwell, and published for the 1977 Conference on Decision and Control Theory in New Orleans by Birdwell & Athans [9]. A survey of the results was presented without proofs in [10]. This paper is based on the results in Birdwell [11].

### 3. Model of System Structure

Models of the structural and the system dynamics will now be presented and used in the sequel to demonstrate the concepts outlined in the introduction. Component failures, repairs, and reconfigurations are modeled by a Markov chain. Only catastrophic changes in the system structure are considered; degradations are not modeled. The hazard rate is assumed to be constant, resulting in an exponential failure distribution. In the discrete-time case, to which the sequel is confined exclusively, the hazard rate becomes the probability of failure (or repair or reconfiguration) between time  $t$  and time  $t + 1$ .

It is now necessary to define precisely the modes of operation and their dynamic transitions. The terms system configuration and system structure will be used.

Definition 4: System Structure: A possible mode of operation for a given system; the components, their interconnections, and the information flow in the system at a given time.

Definition 5: System Configuration: The original design of the system, accounting for all modeled modes of operation, and the Markov chain governing the configuration, or structural, dynamics (transitions among the various structures). In this paper, structures are referenced by the set of non-negative integers

$$I = \{0, 1, 2, \dots, L\} \quad (3.1)$$

Consider the system

$$\underline{x}_{t+1} = \underline{A} \underline{x}_t + \underline{B}_{k(t)} \underline{u}_t \quad (3.2)$$

where

$$\underline{x}_t \in \mathbb{R}^n \quad (3.3)$$

$$\underline{u}_t \in \mathbb{R}^m \quad (3.4)$$

$$\underline{A} \in \mathbb{R}^{n \times m} \quad (3.5)$$

and, for each  $k$ , an element of an indexing set  $I$

$$k \in I = \{0, 1, 2, \dots, L\} \quad (3.6)$$

$$\underline{B}_k \in \mathbb{R}^{n \times m} \quad (3.7)$$

$$\underline{B}_k \in \{\underline{B}_i\}_{i \in I} \quad (3.8)$$

and

$$\underline{B}_i \neq \underline{B}_j \text{ for all } i, j \in I, i \neq j \quad (3.9)$$

The index  $k(t)$  is a random variable taking values in  $I$  which is governed by a Markov chain, and

$$\underline{\pi}_{t+1} = \underline{P} \underline{\pi}_t \quad (3.10)$$

$$\underline{\pi}_t \in \mathbb{R}^{L+1} \quad (3.11)$$

where  $\pi_{i,t}$  is the probability of  $k(t) = i$ , given no on-line information about  $k(t)$ , and  $\pi_0$  is the initial distribution over  $I$ .

It is assumed that the following sequence of events occurs at each time  $t$ :

- 1)  $\underline{x}_t$  is observed exactly

2) then  $B_{k(t-1)}$  switches to  $B_{k(t)}$

3) then  $u_t$  is applied.

Consider the structure set  $\{B_k\}_{k \in I}$  indexed by  $I$ . Define the structural trajectory  $\bar{x}_T$  to be a sequence of elements  $k(t)$  in  $I$  which select a specific structure  $B_{k(t)}$  at time  $t$ ,

$$\bar{x}_T = (k(0), k(1), \dots, k(T-1)) \quad (3.12)$$

The structural trajectory  $\bar{x}_T$  is a random variable with probability of occurrence generated from the Markov equation (3.10).

$$p(\bar{x}_T) = \prod_{t=0}^{T-1} p_{k(t); k(t+1)} \pi_{k(0), 0} \quad (3.13)$$

where the control interval is

$$\{0, 1, 2, \dots, T-1, T\} \quad (3.14)$$

for the finite time problem with terminal time  $T$ . Then for a given state and control trajectory  $(\underline{x}_t, \underline{u}_t)_{t=0}^{T-1}$  generated by (3.2) and  $\bar{x}_T$  from a sequence of controls  $(\underline{u}_t)_{t=0}^{T-1}$  the cost index is to be the standard quadratic cost criterion

$$J_T \left\{ \bar{x}_T \left( \underline{x}_t, \underline{u}_t \right)_{t=0}^{T-1} \right\} = \sum_{t=0}^{T-1} \left( \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t \right) + \underline{x}_T^T \underline{Q} \underline{x}_T \quad (3.15)$$

#### 4. Problem Statement

The objective is to choose a feedback control law, which may depend on any past information about  $\underline{x}_t$  or  $\underline{u}_t$ , mapping  $\underline{x}_t$  into  $\underline{u}_t$

$$\underline{\phi}_t^* : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad (4.1)$$

$$\underline{\phi}_t^* : \underline{x}_t \rightarrow \underline{u}_t \quad (4.2)$$

such that the expected value of the cost function  $J_T$  from equation (3.15)

$$J_T = E \left[ J_T \mid \Pi_0 \right] \quad (4.3)$$

is minimized over all possible mapping  $\underline{\phi}_t$  at  $\underline{\phi}_t^*$ .

Normally, a control law of the form (4.2) must provide both a control and an estimation function in this type of problem; hence the label dual control is used. Here, the structure of the problem allows the exact determination of  $\underline{x}_{t-1}$  from  $\underline{x}_t$ ,  $\underline{x}_{t-1}$  for almost all values of  $\underline{u}_{t-1}$ . This result is stated in the following lemma:

**Lemma 1:** For the set  $\{\underline{B}_k\}_{k \in I}$ , where the  $\underline{B}_k$ 's are distinct, the set  $\{\underline{x}_{k, t+1} = \underline{A} \underline{x}_t + \underline{B}_k \underline{u}_t\}_{k \in I}$  has distinct members for almost all values of  $\underline{u}_t$ .

**Proof:** See Appendix.

Ignoring the set of controls of measure zero for which the members of



$$\left\{ \underline{x}_{k, t+1} \right\}_{k=0}^L \quad (4.3)$$

are not distinct, then for (almost) any control which the optimal algorithm selects, the resulting state  $\underline{x}_{t+1}$  can be compared with the members of the set (4.3) for an exact match (of which there is only one with probability 1), and  $k(t)$  is identified as the generator of that matching member  $\underline{x}_{k, t+1}$ .

This approach is essentially identical to assuming that the structure of the system is perfectly observable. Assuming perfect observability does eliminate any concern about the possibility of encountering a surface of zero measure and causing the control loop to malfunction. However, in a practical application, neither the assumption of perfect state observation nor of perfect structure observation is valid, and in fact the implementer is forced to consider structure identification strategies and the dual effect of control actions on the observation process.

The optimal control law  $\underline{u}_t^* = \underline{\phi}_t^*(\underline{x}_t)$  can be calculated with the assumption that  $k(t-1)$  is known, since this is the case with probability one if no measurement noise is present. Thus, this solution will be labeled the switching gain solution, since, for each time,  $t$ ,  $L + 1$  optimal solutions are calculated apriori, and one solution is chosen on-line based on the past measurements  $\underline{x}_t$ ,  $\underline{x}_{t-1}$  and  $\underline{u}_{t-1}$ , which yield perfect knowledge of  $k(t-1)$ .

## 5. The Optimal Solution

Dynamic programming is used to derive the optimal solution. It is shown in the Appendix that at each time  $t$ , the optimal expected cost-to-go, given the system structure  $k(t-1)$ , is

$$V^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{k,t} \underline{x}_t \quad (5.1)$$

where the  $\underline{S}_{k,t}$  are determined by a set of  $L + 1$  coupled Riccati-like equations (one for each possible configuration):

$$\begin{aligned} \underline{S}_{k,t} = & \underline{A}^T \left\{ \sum_{i=0}^L p_{ik} \underline{S}_{i,t+1} \right. \\ & - \left[ \sum_{i=0}^L p_{ik} \underline{S}_{i,t+1} \underline{B}_i \right] \left[ \underline{R} + \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \\ & \cdot \left. \left[ \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \right] \right\} \underline{A} + \underline{Q} \end{aligned} \quad (5.2)$$

The optimal control, given  $k(t-1) = k$ , is

$$\begin{aligned} \underline{u}_{k,t}^* = & - \left[ \underline{R} + \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \\ & \cdot \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t \end{aligned} \quad (5.3)$$

Writing

$$\underline{u}_{k,t}^* = \underline{G}_{k,t} \underline{x}_t \quad (5.4)$$

then

$$\underline{G}_{k,t} = - \left[ \underline{R} + \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \cdot \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \quad (5.5)$$

Thus,  $\underline{u}_t^* = \underline{\Phi}_t^*(\underline{x}_t)$  is a switching gain linear control law which depends on  $k(t-1)$ .

The variable  $k(t-1)$  is determined by

$$k(t-1) = i \quad \text{iff} \quad \underline{x}_t = \underline{A} \underline{x}_{t-1} + \underline{B}_i \underline{u}_{t-1} \quad (5.6)$$

Note that the  $\underline{S}_{i,t}$ 's and the optimal gains  $\underline{G}_{k,t}$  can be computed off-line and stored. Then at each time  $t$ , the proper gain is selected on-line from  $k(t-1)$ , using equation (5.6), as in Figure 1.

This solution is quite complex relative to the structure of the usual linear quadratic solution. Each of the Riccati-like equations (5.2) involves the same complexity as the Riccati equation for the linear quadratic solution. In addition, there is the on-line complexity arising from the implementation of gain scheduling.

Conditions for the existence of a steady-state solution to equations (5.2) can be developed using the properties of the structural dynamics, as in Chizeck [12]. The steady-state solution would have the advantage that only one set of gains need be stored on-line, instead of requiring a set of gains to be stored for each time  $t$ . The

possibility of limit cycle solutions in the switching gain computations is excluded by the following lemma:

Lemma 2: If the optimal expected cost-to-go at time  $t$  is bounded for all  $t$ , then equation (5.2) converges.

Proof: See Appendix.

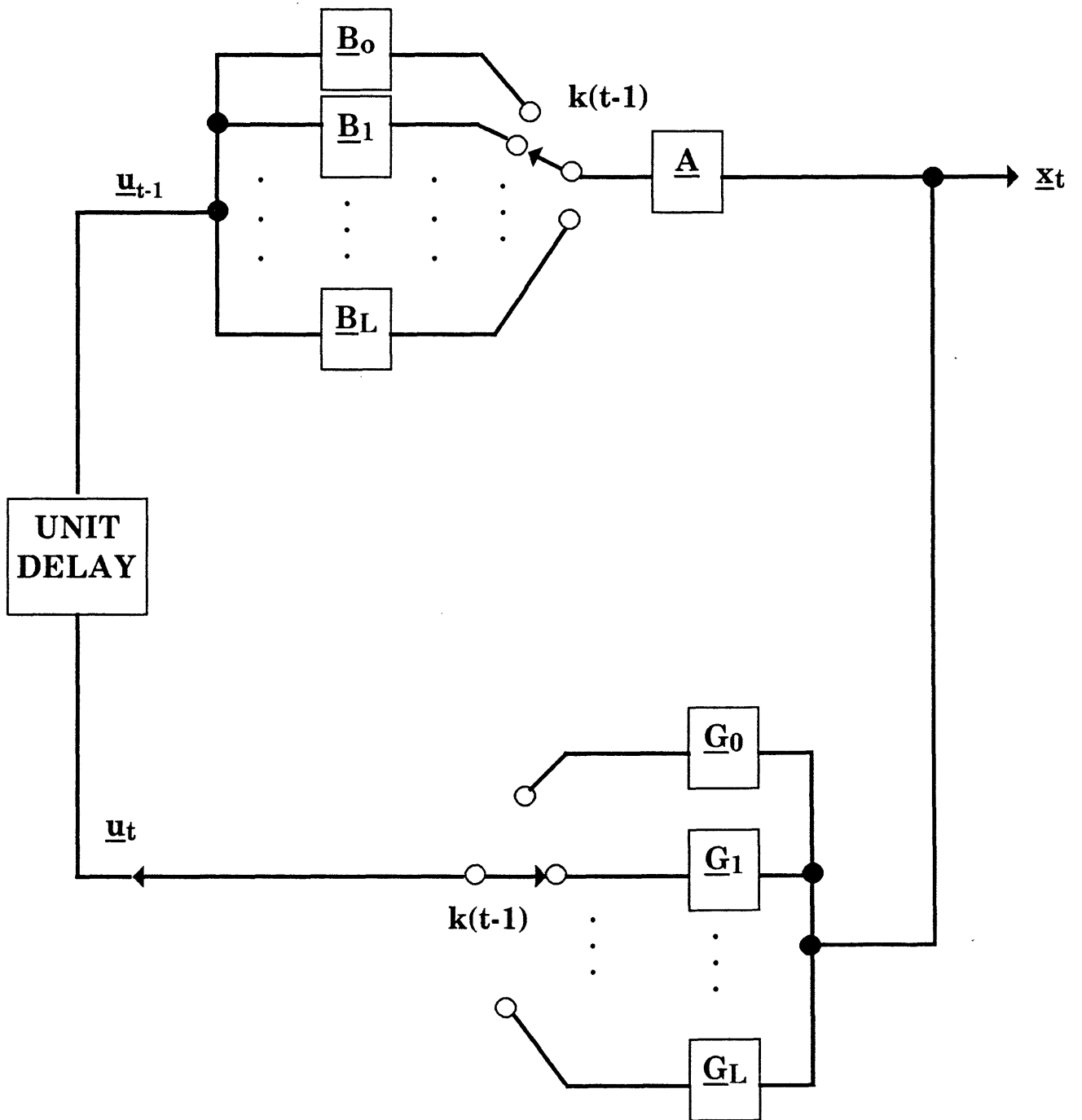


Figure 1: The switching gain control law.

Before we proceed to give necessary and sufficient conditions for the existence of a steady-state solution to equation (5.2), we must define the structure of the Markov chain. The states  $I$  can be divided into closed communicating classes  $C_1, \dots, C_r$  and a group of transient states  $T$ . Let  $n(C_i)$  denote the number of elements in  $C_i$ .

**Lemma 3:** The recursive equations (5.2) converge if and only if there exist feedback control matrices  $\underline{G}_{k(t)}$  and there exist positive definite matrices  $\underline{H}_i, i \in I$ , such that

$$\begin{aligned} \underline{H}_j = & \sum_{t=1}^{\infty} p_{jj}^{t-1} \left( \underline{A} + \underline{B}_j \underline{G}_j \right)^{t-1T} \left( \underline{Q} + \underline{G}_j^T \underline{R} \underline{G}_j \right) \left( \underline{A} + \underline{B}_j \underline{G}_j \right)^{t-1} \\ & + \sum_{\substack{k \in T \\ k \neq j}} \left( \underline{A} + \underline{B}_j \underline{G}_j \right)^{t-1T} \left( \left( \underline{A} + \underline{B}_k \underline{G}_k \right)^T \underline{H}_k \left( \underline{A} + \underline{B}_k \underline{G}_k \right) \right) \left( \underline{A} + \underline{B}_j \underline{G}_j \right)^{t-1} \end{aligned} \quad (5.7)$$

**Proof:** See Appendix.

This lemma is a restatement of the equivalence of Theorem 1 statements i) and iii), but with a different proof explicitly involving the Markov chain's structure. Although necessary and sufficient conditions for the existence of a bounded solution in terms of the system dynamics are unknown, the following lemma supplies sufficient conditions on the matrices  $\underline{A}$ ,  $\underline{B}$ , and  $\underline{P}$ .

**Lemma 4:** The optimal expected cost is bounded for all  $t$  if there exist feedback control laws

$\underline{F}_k, k \in I$ , such that

1. For every state  $k$  in  $C_1, \dots, C_r$ :

If  $p_{kk} < 1$ , then

$$\left(1 - p_{kk}\right) \sum_{t=1}^{\infty} p_{kk}^{t-1} \|\underline{A} - \underline{B}_k \underline{F}_k\|^{t-1} \max_j \|\underline{A} - \underline{B}_j \underline{F}_j\| \leq c < 1 \quad (5.8)$$

where  $\|\underline{A}\|$  is the maximum singular value of the matrix  $\underline{A}$ .

If  $p_{kk} = 1$ , then

$$\sum_{t=1}^{\infty} \|\underline{A} - \underline{B}_k \underline{F}_k\|^{t-1} < \infty \quad (5.9)$$

2. For every transient state  $k$  in  $T$ , let

$$p(k) = 1 - \sum_{j \in T} p_{kj} \quad (5.10)$$

Let  $\bar{p} = \min_{k \in T} p(k)$ . Then

$$\sum_{t=1}^{\infty} \left(1 - \bar{p}\right)^{t-1} \left\{ \max_j \|\underline{A} - \underline{B}_j \underline{F}_j\| \right\}^t < \infty \quad (5.11)$$

Proof: See Appendix.

Note that the sufficiency conditions in Lemma 4 allow the system to have structures for which no stabilizing control gain can be designed. However, the overall system can be considered reliable if the time spent in these structures is sufficiently small, as indicated by the tradeoffs between the singular values of the closed-loop matrices and the self-return probabilities  $p_{kk}$ .



## 6. Implications of the Solution

The existence of a steady-state solution to the switching gain problem establishes a division of system designs into those which are inherently reliable and those which are unreliable. Even though conditions to test for the existence of the steady-state solution are unavailable, software can be used with iteration for the test.

As mentioned earlier, cost stability is the appropriate definition of stability for this problem.

**Definition 6:** (Cost stability). The set of constant gains  $\{\underline{G}_i\}_{i \in I}$  stabilizes the system (3.2) using the control law

$$\underline{u}_t = \underline{G}_k \underline{x}_t \quad (6.1)$$

where  $k$  is determined by Equation (5.6) if and only if the scalar random variable

$$\sum_{t=0}^{\infty} \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t < \infty \quad (6.2)$$

with probability one.

If the infinite time horizon control problem is defined as the minimization of

$$J = \lim_{T \rightarrow \infty} J_T \quad (6.3)$$

then the steady-state values of the gains calculated by Equations (5.2) and (5.5) provide the minimizing control law for Equation (6.3); furthermore, the  $S_{k,t}$  converge if and only if a solution to Equation (6.3) exists.

In addition, the existence of a cost stabilizing set of gains  $\{G_i\}_{i \in I}$  is equivalent to the existence of the infinite time horizon solution. These results are summarized in the following theorem.

Theorem 1: The following statements are equivalent:

- i) Equation (5.2) converges to steady-state values  $\underline{S}_k$  as  $T \rightarrow \infty$  (or  $t \rightarrow -\infty$  for fixed  $T$ ).
- ii) The steady-state set of gains  $\{\underline{G}_k^*\}_{k \in I}$  from Equation (5.5) cost stabilizes the system described by Equations (3.2) and (6.1).
- iii) A set of gains  $\{\underline{G}_k\}_{k \in I}$  exists for which  $J_T$  is bounded.

Proof: See Appendix.

## 7. Example

In this Section, a two-dimensional example is presented with three different switching gain solutions to illustrate the switching gain computational methodology. The computer routines which are used in the calculation of the switching gain solution are documented in [11].

The example is a two-dimensional system with four structural states corresponding to the failure modes of two actuators. In this example, failure of an actuator is modeled as an actuator gain of zero. Thus, the four structures are: I) Both actuators working ( $\underline{B}_0$ ); ii) One actuator failed ( $\underline{B}_1$  and  $\underline{B}_2$ ), and III) Both actuators failed ( $\underline{B}_3$ ). The system is controllable in all structures except for the structure represented by  $\underline{B}_3$ .

Although this example exhibits a very simple structure which models only actuator failure and self-repair, note that the Markov chain formulation does not restrict the configurations of actuators in any structural state. Therefore, this methodology can be used to model and control systems with arbitrary failure, repair, replacement, and reconfiguration structures. Neither is there any restriction that failure and repair/reconfiguration be accomplished within a single structural transition. Therefore, actuator degradation can be modeled as a sequence of discrete failures. The same technique can be applied to repair/reconfiguration modeling.

Actuator failures and repairs are assumed to be independent events with probabilities of failure and repair, per unit time, of  $p_f$  and  $p_r$ , respectively, for both actuators. Note that only exponential failure/repair distributions can be represented.

The matrices  $\underline{Q}$  and  $\underline{R}$  are the quadratic weighting matrices for the state  $\underline{x}_t$  and the control  $u_t$ , respectively. The matrix  $\underline{P}$  is the Markov transition matrix, which is

calculated from knowledge of the system configuration dynamics, represented graphically in Figure 2.

There are three cases in the example. Each case assumes a different failure rate and repair rate for the actuators. Case i) has a high probability of failure and a low probability of repair, relative to Cases ii) and iii). The switching gain solution is not convergent for Case i); the gains themselves converge, but the expected costs do not. Only configuration state 0 is stabilized with its corresponding gain,  $\underline{G}_0$ .

Cases ii) and iii) both assume more reliable actuators than does Case i). Both Cases ii) and iii) have convergent switching gain solutions. Therefore, both Cases ii) and iii) represent reliable configuration designs, while Case i) is unreliable. This difference is due entirely to the different component reliabilities. Equivalently, Cases ii) and iii) are stabilized by the switching gain solution, while Case i) is not. Note that in this Example, stabilizability is not equivalent to stability in each configuration state, or robustness. For this example, no robust gain exists because the system is uncontrollable from configuration state 3.

Case ii) is interesting in that neither the cost nor the gain matrix depends on the structural state. This occurs when all the columns of the Markov transition matrix  $\underline{P}$  are equal. In this case, the on-line implementation is simplified; no switching or detection of structural transitions is required.

## System and Cost Matrices

$$\underline{A} = \begin{bmatrix} 2.71828 & 0.0 \\ 0.0 & .36788 \end{bmatrix}$$

$$\underline{B}_0 = \begin{bmatrix} 1.71828 & 1.71828 \\ -.63212 & .63212 \end{bmatrix}$$

$$\underline{B}_2 = \begin{bmatrix} 1.71828 & 0.0 \\ -.63212 & 0.0 \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} 14. & 8. \\ 8. & 6. \end{bmatrix}$$

$$\underline{B}_1 = \begin{bmatrix} 0.0 & 1.71828 \\ 0.0 & .63212 \end{bmatrix}$$

$$\underline{B}_3 = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

## Markov Transition Matrix

$$\underline{P} = \begin{bmatrix} 1-2p_f+p_f^2 & (1-p_f)p_r & (1-p_f)p_r & p_r^2 \\ p_f(1-p_f)^2 & 1-p_f-p_r+p_f p_r & p_r p_f & p_r(1-p_r) \\ p_f(1-p_f)^2 & p_r p_f & 1-p_f-p_r+p_f p_r & p_r(1-p_r) \\ p_f^2 & (1-p_r)p_f & (1-p_r)p_f & 1-2p_r+p_r^2 \end{bmatrix}$$

The system dynamics are

$$\underline{x}_{t+1} = \underline{A} \underline{x}_t + \underline{B}_{k(t)} u_t ; \quad \underline{x}_t = [x_{1,t} \ x_{2,t}]^T$$

$$k(t) \in \{0,1,2,3\}$$

The cost, which is to be minimized, is

$$J = E \left[ \sum_{t=0}^{\infty} \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t \mid \underline{\pi} \right]$$

Case i)

$$p_f = .3 ; p_r = .7 ; \quad \underline{\pi} = \begin{bmatrix} .49 \\ .21 \\ .21 \\ .09 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

The coupled Riccati equations are non-convergent, but the gains converge:

$$\underline{G}_0 = \begin{bmatrix} -.9636 & 0 \\ -.9134 & 0 \end{bmatrix}$$

$$\underline{G}_1 = \begin{bmatrix} -.9234 & 0 \\ -.8699 & 0 \end{bmatrix}$$

$$\underline{G}_2 = \begin{bmatrix} -.8094 & 0 \\ -1.020 & 0 \end{bmatrix}$$

$$\underline{G}_3 = \begin{bmatrix} -.9636 & 0 \\ -.9134 & 0 \end{bmatrix}$$

Stability tests:

Configuration	Stable
0 ( $\underline{B}_0$ )	yes
1 ( $\underline{B}_1$ )	no
2 ( $\underline{B}_2$ )	no
3 ( $\underline{B}_3$ )	no

Case ii)

$$p_f = .1; \quad p_r = .9$$

$$\underline{\pi} = \begin{bmatrix} .81 \\ .09 \\ .09 \\ .01 \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$

The coupled Riccati equations coverge:

$$\underline{G}_i = \begin{bmatrix} -.8890 & .04222 \\ -.7752 & -.9914 \end{bmatrix}$$

for  $i = 0,1,2,3$

$$\underline{S}_i = \begin{bmatrix} 25.57 & 8.611 \\ 8.611 & 6.398 \end{bmatrix}$$

Stability tests:

Configuration	Stable
0 ( $\underline{B}_0$ )	yes
1 ( $\underline{B}_1$ )	no
2 ( $\underline{B}_2$ )	no
3 ( $\underline{B}_3$ )	no

$$p_f = .1; \quad p_r = .9 ;$$

$$\underline{\pi} =$$

$$\begin{bmatrix} .9799 \\ .009999 \\ .009999 \\ .0001020 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}$$



The coupled Riccati equations coverage:

$$\underline{G}_0 = \begin{bmatrix} -.7558 & .1270 \\ -.8073 & -.1786 \end{bmatrix}$$

$$\underline{S}_0 = \begin{bmatrix} 15.88 & 8.105 \\ 8.105 & 6.137 \end{bmatrix}$$

$$\underline{G}_1 = \begin{bmatrix} -.7060 & .1186 \\ -.8441 & -1.723 \end{bmatrix}$$

$$\underline{S}_1 = \begin{bmatrix} 16.06 & 8.074 \\ 8.074 & 8.143 \end{bmatrix}$$

$$\underline{G}_2 = \begin{bmatrix} -.8375 & .1090 \\ -.7543 & -.1669 \end{bmatrix}$$

$$\underline{S}_2 = \begin{bmatrix} 16.31 & 8.199 \\ 8.199 & 6.158 \end{bmatrix}$$

$$\underline{G}_3 = \begin{bmatrix} -.7863 & .1023 \\ -.7926 & -.1619 \end{bmatrix}$$

$$\underline{S}_3 = \begin{bmatrix} 16.54 & 8.170 \\ 8.170 & 6.162 \end{bmatrix}$$

Stability tests:

Configuration	Stable
0 ( $\underline{B}_0$ )	yes
1 ( $\underline{B}_1$ )	no
2 ( $\underline{B}_2$ )	no
3 ( $\underline{B}_3$ )	no

## 8. Conclusions

The concepts which allow component reliability to influence control system design in a consistent manner have been defined. When specialized to linear systems with quadratic cost functions, an optimal control problem can be defined. The resulting control law depends on the system structure, the structural dynamics, and the system dynamics. The solution to the optimal control problem defines the boundary between reliable (stabilizable) designs and unreliable designs.

In closing, we also note that the restriction that all structural changes occur in the actuator matrix can be easily removed. In this case, a structural state is completely defined by  $\underline{A}_k$  and  $\underline{B}_k$ , rather than by  $\underline{B}_k$  alone. The results in this paper are directly extendible to this case. Many of the details are available in [12].

## 9. Appendix

### A1. Proof of Lemma 1.

Assume  $\underline{x}_{k, t+1} = \underline{x}_{\ell, t+1}$  for  $k \neq \ell$ . Then  $(\underline{B}_k - \underline{B}_\ell)\underline{u}_{t-1} = 0$ , which implies  $\underline{u}_{t-1}$  is in the null space of  $\underline{B}_k - \underline{B}_\ell$ ,  $N(\underline{B}_k - \underline{B}_\ell)$ . Now,  $\text{dimension}(N(\underline{B}_k - \underline{B}_\ell)) < m$  because the  $\underline{B}_k$ 's are distinct. Therefore,

$$\text{dimension} \left( \bigcup_{\substack{k, \ell \\ k \neq \ell}} N(\underline{B}_k - \underline{B}_\ell) \right) < m \quad (\text{A1.1})$$

Therefore the set  $\bigcup_{\substack{k, \ell \\ k \neq \ell}} N(\underline{B}_k - \underline{B}_\ell)$  has measure zero in  $\mathbb{R}^m$ . Q.E.D.

### A.2 Optimal Solution for Deterministic Problem.

For the system described in Section 3, from dynamic programming, the optimal cost-to-go at time  $t$  is given by

$$\begin{aligned} V^*(\underline{x}_t, k(t-1), t) = \min_{\underline{u}_t} & \quad E_{k(t)} \left\{ \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t \right. \\ & \quad \left. \underline{u}_t = \Phi_t(\underline{x}_t) \right. \\ & \quad \left. + V^*(\underline{x}_{t+1}, k(t), t+1) \mid \underline{x}_t \right\} \end{aligned} \quad (\text{A2.1})$$

Assume

$$V^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{k, t} \underline{x}_t \quad (\text{A2.2})$$

This assumption will be verified by induction.

Then

$$\begin{aligned}
\underline{x}_t^T \underline{S}_{k,t} \underline{x}_t = \min_{\underline{u}_t = \Phi_t(\underline{x}_t)} & \left\{ \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t \right. \\
& \left. + \sum_{i=0}^L p_{ik} (\underline{A} \underline{x}_t + \underline{B}_i \underline{u}_t)^T \underline{S}_{i,t+1} (\underline{A} \underline{x}_t + \underline{B}_i \underline{u}_t) \right\}
\end{aligned} \tag{A2.3}$$

and

$$\begin{aligned}
\text{(A2.3)} = \min_{\underline{u}_t = \Phi_t(\underline{x}_t)} & \left\{ \underline{x}_t^T \underline{Q} \underline{x}_t + \underline{u}_t^T \underline{R} \underline{u}_t \right. \\
& \left. + \sum_{i=0}^L p_{ik} \left[ \underline{x}_t^T \underline{A}^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t + \underline{u}_t^T \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \underline{u}_t \right. \right. \\
& \left. \left. + \underline{x}_t^T \underline{A}^T \underline{S}_{i,t+1} \underline{B}_i \underline{u}_t + \underline{u}_t^T \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t \right] \right\}
\end{aligned} \tag{A2.4}$$

Differentiating the r.h.s. of (A2.4) w.r.t.  $\underline{u}_t$  and setting it equal to zero:

$$0 = 2 \underline{R} \underline{u}_t + \sum_{i=0}^L p_{ik} \left[ 2 \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \underline{u}_t + 2 \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t \right] \tag{A2.5}$$

or

$$\begin{aligned}
\underline{u}_{k(t-1),t}^* = - & \left[ \underline{R} + \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{B}_i \right]^{-1} \\
& \cdot \sum_{i=0}^L p_{ik} \underline{B}_i^T \underline{S}_{i,t+1} \underline{A} \underline{x}_t
\end{aligned} \tag{A2.6}$$

is the optimal  $\underline{u}_t^*$ , given  $k(t-1)$ .

Since no noise is present in the system,  $k(t-1)$  is obtained from  $\underline{x}_t$  and  $\underline{x}_{t-1}$ , along with  $\underline{u}_{t-1}$ , as

$$k(t-1) = i \text{ iff } \underline{x}_t = \underline{A} \underline{x}_{t-1} + \underline{B}_1 u_{t-1} \quad (\text{A2.7})$$

Substituting (A2.6) into (A2.4), and eliminating  $\underline{x}_t$  because the equation must be true for all  $\underline{x}_t$ , and the matrix equation is symmetric, on simplification we obtain equation (5.2), which verifies assumption (A2.2) by induction, along with the initial condition

$$\underline{S}_{k, T} = \underline{Q} \quad (\text{A2.8})$$

### A3. Proof of Lemma 2.

Consider the optimization of the cost-to-go given  $k(t-1)$  at time  $t$  with final time  $T$ . This optimal cost-to-go is simply

$$V_T^*(\underline{x}_t, k(t-1), t) \quad (\text{A3.1})$$

where  $T$  denotes the final time. For the process with final time  $T + 1$ , the optimal cost-to-go is

$$\begin{aligned} & V_{T+1}^*(\underline{x}_t, k(t-1), t) \\ &= E \left\{ \sum_{\tau=t}^T \underline{x}_\tau^T \underline{Q} \underline{x}_\tau + \underline{u}_\tau^T \underline{R} \underline{u}_\tau + \underline{x}_{T+1}^T \underline{Q} \underline{x}_{T+1} \mid k(t-1) \right\} \end{aligned} \quad (\text{A3.2})$$

Since this optimal sequence is not necessarily optimal for the problem with final time  $T$ , it must not incur less cost over  $\{t, \dots, T\}$ .

$$\begin{aligned}
& V_{T+1}^*(\underline{x}_t, k(t-1), t) \\
& \geq V_T^*(\underline{x}_t, k(t-1), t) \\
& + E \left\{ \underline{u}_T^T \underline{R} \underline{u}_T + \underline{x}_T^T \underline{Q} \underline{x}_{T+1} \mid k(t-1) \right\}
\end{aligned} \tag{A3.3}$$

Since the expectation term of equation (A3.3) is non-negative,

$$V_{T+1}^*(\underline{x}_t, k(t-1), t) \geq V_T^*(\underline{x}_t, k(t-1), t) \tag{A3.4}$$

Now, note that

$$V_T^*(\underline{x}_t, k(t-1), t) = \underline{x}_t^T \underline{S}_{i, t_T} \underline{x}_t \tag{A3.5}$$

and that equation (5.2) depends only on the number of iterations (T-t) for the calculation of  $\underline{S}_{i, t_T}$ , and therefore,

$$V_T^*(\underline{x}_t, k(t-1), t-1) = V_{T+1}^*(\underline{x}_t, k(t-1), t) \tag{A3.6}$$

Therefore,  $\{\underline{S}_{i, t}\}_{t=T}^{-\infty}$  is an increasing sequence in that

$$\underline{S}_{i, t-1} - \underline{S}_{i, t} \geq 0 \tag{A3.7}$$

Since, by hypothesis,  $V_T^*$  is bounded over t, the  $\underline{S}_{i, t}$  converge.

#### A4. Proof of Lemma 3.

Equation (5.7) implies that the gains  $\underline{G}_k$  result in a finite cost-to-go, expressed as an average of the matrices  $\underline{H}_i$ . Hence, the optimal cost is also finite, and bounded,

so equation (5.2) converges. Similarly, if equation (5.2) converges, selecting  $\underline{H}_k = \underline{S}_k$  and  $\underline{G}_k$  according to equation (5.5) satisfies equation (5.7).

A5. Proof of Lemma 4.

Assume that the control gains  $\underline{F}_k$  are used. Let  $\tau$  be the time of first exit from state  $k$ . Assume  $\tau_0$  is finite with probability 1. Otherwise,  $p_{kk} = 1$  and equation (5.9) applies. Then, equation (5.8) establishes that

$$\|\underline{x}_{-\tau_0}\| \leq c \|\underline{x}_{-\tau}\| \tag{A5.1}$$

Let

$$s = \max_j \|\underline{Q} + \underline{F}_j^T \underline{R} \underline{F}_j\| \tag{A5.2}$$

Then, the cost incurred while in state  $k$  is bounded above by

$$\frac{cs \|\underline{x}_0\|^2}{1 - p_{kk}} \tag{A5.3}$$

Consider now the new state at  $\tau_0$ , and denote the time of first exit  $\tau_1$ . By similar reasoning, we construct the sequence  $\tau_0, \dots, \tau_n, \dots$ .

Let  $C(k)$  be the communicating class of state  $k$ , and

$$q = \max_{j \in C(k)} p_{jj} \tag{A5.4}$$

The overall average cost incurred can be partitioned in terms of the costs incurred between transits  $\tau_i, \tau_{i+1}$ , as



$$\begin{aligned}
& E \left\{ \sum_{t=0}^{\infty} \underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right\} \\
&= E \left\{ \sum_{t=0}^{\tau_1-1} \left( \underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right) + \sum_{t=\tau_1}^{\tau_2-1} \left( \underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right) + \dots \right\}
\end{aligned} \tag{A5.5}$$

$$\leq \frac{cs \|\underline{x}_0\|^2}{1-q} + \frac{cs}{1-q} E \left\{ \|\underline{x}_{\tau_1}\|^2 \right\} + \dots \tag{A5.6}$$

and equation (5.8) implies

$$(A5.6) \leq \frac{cs \|\underline{x}_0\|^2}{1-q} (1 + c + c^2 + \dots) \tag{A5.7}$$

which is finite since  $c < 1$ .

If  $p_{kk} = 1$ , equation (5.9) establishes that, from structure  $k$  at time  $t_0$ ,

$$E \left\{ \sum_{t=t_0}^{\infty} \left( \underline{u}_t^T \underline{R} \underline{u}_t + \underline{x}_t^T \underline{Q} \underline{x}_t \right) \right\} \leq \sum_{t=t_0}^{\infty} \|\underline{A} + \underline{B}_k \underline{F}_k\|^t < \infty. \tag{A5.8}$$

Hence, equations (5.8) and (5.9) establish that, for any initial state  $k(t_0)$ ,  $\underline{x}(t_0)$  in a closed communicating class, the cost-to-go is finite. To show the overall cost is finite, we must establish that from any initial transient state, the cost incurred until a closed communicating class is reached is finite.

Let  $\tau(k)$  denote the time of first exit from  $T$  starting at  $k \in T$ . The expected cost incurred while in  $T$  is

$$\begin{aligned}
& E \left\{ \sum_{t=0}^{\tau} \underline{x}_t^T \left( \underline{F}_{k(t)}^T \underline{R} \underline{F}_{k(t)} + \underline{Q} \right) \underline{x}_t \mid \underline{x}_0, k(0) = k \right\} \\
& \leq s E \left\{ \sum_{t=0}^{\tau} \|\underline{x}_t\|^2 \mid \underline{x}_0, k(0) = k \right\} \\
& \leq s \|\underline{x}_0\|^2 E \left\{ \sum_{t=0}^{\tau} \left\{ \max_{\substack{k \in T \\ j \in T}} \|\underline{A} + \underline{B}_k \underline{F}_j\| \right\}^t \mid k(0) = k \right\}.
\end{aligned} \tag{A5.9}$$

But, from the definition of  $\bar{p}$ ,

$$\text{Prob} \{ \tau > n \} \leq (1 - \bar{p})^n. \tag{A5.10}$$

Hence,

$$\begin{aligned}
& E \left\{ \sum_{t=0}^{\tau} \underline{x}_t^T \left( \underline{F}_{k(t)}^T \underline{R} \underline{F}_{k(t)} + \underline{Q} \right) \underline{x}_t \mid \underline{x}_0, k(0) = k \right\} \\
& \leq s \|\underline{x}_0\|^2 \sum_{t=1}^{\infty} (1 - \bar{p})^t \left( \max_{\substack{k \in T \\ j \in T}} \|\underline{A} + \underline{B}_k \underline{F}_j\| \right)^t < \infty
\end{aligned} \tag{A5.11}$$

by equation (5.11). Hence, the gains  $\underline{F}_j$  result in finite expected cost for all initial states. The optimal expected cost-to-go will be bounded in  $t$  by this cost.

#### A6. Proof of Theorem 1.

i)  $\Rightarrow$  ii):

Suppose  $\{\underline{G}_k^*\}_{k \in I}$  were not cost stabilizing. Then for some set  $M$  of non-zero measure of structural trajectories  $(k(0), k(1), \dots)$ ,  $J_T^*$  on that set is not bounded. But

$$J_T^* \geq \int_M J_T^*(m) dp(m) \rightarrow m \text{ as } T \rightarrow \infty \tag{A6.1}$$

therefore,  $M$  must be of measure zero.

ii)  $\Rightarrow$  iii): The steady-state gains  $\{\underline{G}_k^*\}_{k \in I}$  satisfy iii).

iii)  $\Rightarrow$  i): By assumption, there exists a  $B$  such that

$$J_T \left( \left\{ \underline{G}_k \right\}_{k \in I} \right) < B \text{ for all } T \quad (\text{A6.2})$$

Since

$$J_T^* \leq J_T \left( \left\{ \underline{G}_k \right\}_{k \in I} \right) < B \text{ for all } T, \quad (\text{A6.3})$$

Statement i) is implied by Lemma 2.

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