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ON REPRESENTATIONS OF SOME PERRON INTEGRABLE FUNCTIONS

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0. In what follows, K always denotes a compact interval on the real line $\mathscr{R}$; $\langle 0, 1\rangle$ will be specified by I. All functions considered are finite, to avoid noninteresting discussions of infinities. We say that f is Newton-integrable in the generalized sense over K, if there exists a function F continuous on K such that $F'(x) = f(x)$ on K, with possible exception of a countable subset $A \subset K$; for $A = \emptyset$ we say that f is Newton-integrable over K. Some function families will now be introduced; by definition:

$f \in \mathscr{S}(K) \Leftrightarrow f$ is a Lebesgue measurable function on K,
$v \in \mathscr{BV}(K) \Leftrightarrow$ the variation $\operatorname{Var}(v; K)$ of v on K is finite,
$n \in \mathscr{N}(K) \Leftrightarrow n$ is Newton-integrable over K,
$n^* \in \mathscr{N}^*(K) \Leftrightarrow n^*$ is Newton-integrable in the generalized sense over K,
$f \in \mathscr{P}(K) \Leftrightarrow f$ is Perron-integrable over K, $-\infty < (P)\int_K f < \infty$,
$l \in \mathscr{L}(K) \Leftrightarrow l$ is Lebesgue-integrable over K, $(L)\int_K |f| < \infty$.

In what follows, we write e.g. $\mathscr{L}(a, b)$ instead of $\mathscr{L}(\langle a, b\rangle)$; also, we put $\int_a^b f = \int_{\langle a,b\rangle} f$. For each $f \in \mathscr{S}(K)$, $\sigma(f)$ denotes the set of $L-$ singular points of f (see [2], p. 255). Given a mapping f of a set A and a nonvoid set B, then $f \mid B$ denotes the mapping of $A \cap B$ (if $\neq \emptyset$) coinciding there with f.

1. We begin with a problem posed in [1], concerning the possibility of multiplication within the class of $\mathscr{F}$ – integrable functions.

An integration $(\mathscr{F}, \iota)$ on $\mathscr{R}$, in the sense of [1], is a correspondence assigning to each K a linear subset $\mathscr{F}(K)$ of $\mathscr{S}(K)$ and a finite functional $f \to (\iota)\int_K f$, $f \in \mathscr{F}(K)$, so that the following is satisfied:

(I) $(\iota)\int_K$ is linear on $\mathscr{F}(K)$,

(II) $f \in \mathscr{L}(K) \Rightarrow f \in \mathscr{F}(K)$ and $(\iota)\int_K f = (L)\int_K f$,

(III) $f \in \mathscr{F}(K)$, $\langle c, d\rangle \subset K \Rightarrow f \mid \langle c, d\rangle \in \mathscr{F}(c, d)$,

(IV) $a < b < c$, $f \mid \langle a, b\rangle \in \mathscr{F}(a, b)$, $f \mid \langle b, c\rangle \in \mathscr{F}(b, c) \Rightarrow f \mid \langle a, c\rangle \in \mathscr{F}(a, c)$ and $(\iota) \int_a^b f + (\iota) \int_b^c f = (\iota) \int_a^c f$,

(V) $f \in \mathscr{F}(a, b)$, $f \geqq 0 \Rightarrow f \in \mathscr{L}(a, b)$,

(VI) $f \in \mathscr{F}(a, b) \Rightarrow (\iota) \int_a^x f$ is continuous on $\langle a, b\rangle$.

Now the above mentioned problem reads as follows (Problem B of [1]): Do there exist an integration $(\mathscr{F}, \iota)$, $f \in \mathscr{F}(I)$ and φ which is absolutely continuous on I such that $f\varphi \notin \mathscr{F}(I)$?

In the next section we answer to this positively.

2. Let $n \in \mathscr{N}(I) - \mathscr{L}(I)$ be such that $\sigma(n) = \{0, 1\}$; put further $n(t) = 0$ for $t \in \mathscr{R} - I$. Define $\mathscr{F}(K) = \{l + \lambda n \mid K;\ l \in \mathscr{L}(K),\ \lambda \in \mathscr{R}\}$, $(\iota) \int_K f = (P) \int_K f$ for $f \in \mathscr{F}(K)$. We show that $\mathscr{F}$ is the desired integration. For, given a suitable φ, absolutely continuous on I, it is not possible to write $n\varphi = l + \lambda n$, as $\sigma(n\varphi)$ may be e.g. equal to $\{0\}$, while for $\lambda \neq 0$ we have $\sigma(l + \lambda n) = \{0, 1\}$.

3. On the other hand, the smallest $\mathscr{F}$ containing $\mathscr{N}$ and fulfilling

(VII) $$f \in \mathscr{F}(K)\,, \quad v \in \mathscr{BV}(K) \Rightarrow fv \in \mathscr{F}(K)$$

is evidently $\mathscr{T}$ defined as follows:

(3.1) $$f \in \mathscr{T}(K) \Leftrightarrow f = l + \sum_{i=1}^{r} n_i v_i\,, \quad l \in \mathscr{L}(K)\,,$$
$$n_i \in \mathscr{N}(K)\,, \quad v_i \in \mathscr{BV}(K)\,, \quad i = 1, \ldots, r\,.$$

As it is well known, $\mathscr{T} \subset \mathscr{P}$, and the question arises whether the inclusion is proper.

4. We prove a stronger result. Write

(4.1) $$f \in \mathscr{T}^*(K) \Leftrightarrow f = l + \sum_{i=1}^{r} n_i^* v_i\,, \quad l \in \mathscr{L}(K)\,,$$
$$n_i^* \in \mathscr{N}^*(K)\,, \quad v_i \in \mathscr{BV}(K)\,, \quad i = 1, \ldots, r\,.$$

Then

(4.2) $$\mathscr{T} \subset \mathscr{T}^* \subset \mathscr{P}$$

and first we show that $\mathscr{T}^*$ lies properly in $\mathscr{P}$.

Remark. Now, all will be related to I; so we write simply $\mathscr{L}$ instead of $\mathscr{L}(I)$, etc.

5. Lemma. *Let $f \in \mathscr{P}$, $v \in \mathscr{BV}$. Put $F(x) = \int_0^x f$, $H(x) = \int_0^x fv$. Suppose that $F(x) = O(x)$, $x \to 0+$. Put $S(x) = H(x) - v(0+)\, F(x)$. Then $S'^{+}(0) = 0$.*

Proof. 1° Suppose first that v is nondecreasing, $v(0) = v(0+) = 0$. Let $c \in \mathscr{R}$ be such that $|x^{-1} F(x)| \leqq c$. We have $H(x) = v(x)(F(x) - F(\xi))$, $0 \leqq \xi \leqq x$; hence $|x^{-1} H(x)| \leqq 2c\, v(x)$, $S'^{+}(0) = 0$.

2° In the general case there are nondecreasing v_1, v_2 such that $v_j(0) = v_j(0+) = 0$, $j = 1, 2$, and $v = v(0+) + v_1 - v_2$ on $(0, 1\rangle$. Put $S_j(x) = \int_0^x f v_j$. Then $H = v(0+) F + S_1 - S_2$, etc.

6. Corollary. 1° $n^* \in \mathscr{N}^*$, $v \in \mathscr{BV} \Rightarrow n^* v \in \mathscr{N}^*$.

2° *Let $f \in \mathscr{N}$, $v \in \mathscr{BV}$, $H(x) = \int_0^x fv$. Then $H'^{+}(0) = v(0+) f(0)$.*

From 1° we infer that it is sufficient to prove the following theorem.

7. Theorem. *There exists $f \in \mathscr{P}$ not expressible in the form $f = l + n^*$, $l \in \mathscr{L}$, $n^* \in \mathscr{N}^*$.*

Proof. Let D denote the Cantor discontinuum. To each interval $J = (a, b)$ contiguous to D there exists a natural number r such that $r(b - a) > 1$. To each such J and r there exist numbers α_j and a continuously differentiable function φ_J on $\bar{J}$ such that $\varphi(a) = \varphi(b) = 0$,

$$|\varphi| \leqq 2(b - a) \quad \text{on} \quad J\,, \tag{7.1}$$

$a < \alpha_0 < \alpha_1 < \ldots < \alpha_r < b$ and

$$\varphi(\alpha_j)(-1)^j > b - a\,, \quad j = 0, 1, \ldots, r\,. \tag{7.2}$$

Now put $f(x) = F(x) = 0$, $x \in D$, and $F(x) = \varphi_J(x)$, $f(x) = \varphi_J'(x)$ on each J. Using (7.1), we get from Lemma (3.4) of [2], p. 249, that $F(x) = (P) \int_0^x f$. Suppose now that there exist $l \in \mathscr{L}$, $n^* \in \mathscr{N}^*$ such that $f = l + n^*$ on I; hence also $F = L + N^*$, where $L(x) = \int_0^x l$, $N^*(x) = \int_0^x n^*$. As N^* is differentiable on I with possible exception of a denumerable set, there exists $\beta \in D$ such that $N^{*\prime}(\beta)$ exists and

(7.3) there are infinitely many intervals (a, b) contiguous to D such that $2a - b < \beta < a < b$.

We may assume that $N^*(\beta) = N^{*\prime}(\beta) = 0$. Let $\gamma > \beta$ be such that

$$x \in (\beta, \gamma) \Rightarrow |N^*(x)| < 2^{-2}(x - \beta)\,. \tag{7.4}$$

Then, according to (7.2), (7.3) and (7.4), $\operatorname{Var}(F - N^*; \langle a, b\rangle) \geqq \sum_{j=1}^{r} |(F - N^*) \,.\, (\alpha_j) - (F - N^*)(\alpha_{j-1})| \geqq \sum_{j=1}^{r} |F(\alpha_j) - F(\alpha_{j-1})| - \sum_{j=1}^{r} |N^*(\alpha_j)| - \sum_{j=1}^{r} |N^*(\alpha_{j-1})| \geqq \sum_{j=1}^{r} 2(b - a) - 2\sum_{j=1}^{r} 2^{-2} \,.\, 2(b - a) = r(b - a) > 1$, for each contiguous interval

(a, b) such that $2a - b < \beta < b < \gamma$. Hence $\operatorname{Var}(L; I) = \operatorname{Var}(F - N^*; I) \geqq \geqq \operatorname{Var}(F - N^*; \langle \beta, 1 \rangle) = \infty$; a contradiction.

8. We are now going to show that also the first inclusion in (4.2) is proper. First, we prove a lemma.

9. Lemma. *Let* $1 > x_1 > y_1 > x_2 > y_2 > \ldots, x_r \to 0, \sum_{r=1}^{\infty} x_r = \infty$. *Let* F, H *be functions on* I. *Let* $F(x_r) \geqq x_r, F(y_r) \leqq -y_r, r = 1, 2, \ldots$; *let* $H'^+(0)$ *be finite. Then* $\operatorname{Var}(F + H; I) = \infty$.

Proof. Put $H_1(x) = H(x) - H(0) - xH'^+(0)$. Then $H_1(0) = H_1'^+(0) = 0$. There exists an index m such that $x \in (0, x_m) \Rightarrow |H_1(x)| < \frac{1}{2}x$. Put $R = F + H_1$. Then $p > m \Rightarrow |R(y_p) - R(x_{p+1})| + |R(x_p) - R(y_p)| + \ldots + |R(y_m) - R(x_{m+1})| + |R(x_m) - R(y_m)| > 2(x_{p+1} + \ldots + x_{m+1})$; hence $\operatorname{Var}(R; I) = \infty$, and also $\operatorname{Var}(F + H; I) = \infty$.

10. Theorem. *Let* $F(x) = x \sin x^{-1}, f(x) = F'(x), x > 0$. *Then* $f \in \mathscr{N}^* - \mathscr{T}$.

Proof. Let on the contrary $f = l + \sum_{i=1}^{r} n_i v_i$, $l \in \mathscr{L}$, $n_i \in \mathscr{N}$, $v_i \in \mathscr{BV}$, $i = 1, \ldots, r$. Put $H(x) = \sum_{i=1}^{r} \int_0^x n_i v_i$; then, according to 2° in corollary 6, a finite $H'^+(0)$ exists. Put $F(0) = 0$. From Lemma 9 we infer that $\operatorname{Var}(F - H; I) = \infty$; hence contradiction.

11. Comparing theorems 6 and 10, a natural problem arises: Let $n_i \in \mathscr{N}$, $v_i \in \mathscr{BV}$ $i = 1, 2$. Do there exist $n \in \mathscr{N}$, $v \in \mathscr{BV}$ such that $nv = n_1 v_1 + n_2 v_2$?

12. We close this paper with a theorem asserting that the representation of a Perron integrable function f in the form $f = l + n^*$ is possible, supposing $\sigma(f)$ is countable.

13. Lemma. *Let* $\varepsilon > 0$, *let* $J \subset \mathscr{R}$ *be an open interval and let* f *be a function on* J. *Then there exists a function* g *on* J *such that* 1° g *is continuous on* $J - \sigma(f)$ 2° $\int_J |f - g| < \varepsilon$.

Proof. Let $\mathfrak{A}$ denote the system of components of $J - \sigma(f)$. Let $\varepsilon_A > 0$ correspond to $A \in \mathfrak{A}$ so that $\sum_{A \in \mathfrak{A}} \varepsilon_A < \varepsilon$. Let $A \in \mathfrak{A}$, $a = \inf A$, $b = \sup A$. For each $r = 0, \pm 1, \pm 2, \ldots$ let $c_r \in \mathscr{R}$ be such that $\ldots < c_{r-1} < c_r < \ldots$, $a = \inf c_r$, $b = \sup c_r$. Further, let g_r be continuous on J, with compact support in (c_{r-1}, c_r), and such that $\sum_r \int_A |f - g_r| < \varepsilon_A$. Put $g_A = \sum_r g_r$. Then evidently $\int_A |f - g_A| < \varepsilon_A$, and g_A is continuous on A. Let further χ denote the characteristic function of the set $\sigma(f)$. Now it is sufficient to put $g = \chi f + \sum_{A \in \mathfrak{A}} g_A$.

14. Theorem. *Let $f \in \mathscr{P}$ and let $\varepsilon > 0$. Let $\sigma(f)$ be countable. Then there exist $l \in \mathscr{L}$ and $n^* \in \mathscr{N}$ such that $f = l + n^*$, $\int_I |l| < \varepsilon$.*

Proof. Let $n^* = g$ of Lemma 13 and put $l = f - n^*$, $G(x) = \int_0^x n^*$. Then G is continuous on I, $G'(x) = n^*(x)$ on $I - \sigma(f)$; hence the theorem.

References

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