## On resistance matrices of weighted balanced digraphs

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#### Abstract

Let G be a connected graph with  $V(G) = \{1, \ldots, n\}$ . Then the resistance distance between any two vertices i and j is given by  $r_{ij} := l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}$ , where  $l_{ij}^{\dagger}$  is the  $(i, j)^{\text{th}}$  entry of the Moore-Penrose inverse of the Laplacian matrix of G. For the resistance matrix  $R := [r_{ij}]$ , there is an elegant formula to compute the inverse of R. This says that

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R \tau} \tau \tau',$$

where

$$\tau := (\tau_1, \dots, \tau_n)'$$
 and  $\tau_i := 2 - \sum_{\{j \in V(G): (i,j) \in E(G)\}} r_{ij}$   $i = 1, \dots, n.$ 

A far reaching generalization of this result that gives an inverse formula for a generalized resistance matrix of a strongly connected and matrix weighted balanced directed graph is obtained in this paper. When the weights are scalars, it is shown that the generalized resistance is a non-negative real number. We also obtain a perturbation result involving resistance matrices of connected graphs and Laplacians of digraphs.

Keywords. Balanced digraphs, Laplacian matrices, resistance matrices, row diagonally dominant matrices, Jacobi identity. AMS CLASSIFICATION. 05C50

## 1 Introduction

Let G be a simple connected graph. Suppose x and y are any two vertices of G. The length of the shortest path connecting x and y in G is the natural way to define the distance between x and y. This classical distance has certain limitations. For instance, consider two graphs  $G_1$  and  $G_2$  such that

- (i)  $V(G_1) = V(G_2) = \{1, \dots, n\}.$
- (ii) i and j are adjacent in both  $G_1$  and  $G_2$ .
- (iii) There is only path between i and j in  $G_1$  and there are multiple paths connecting i and j in  $G_2$ .

Then the shortest distance between i and j in both  $G_1$  and  $G_2$  is one. However, since there are multiple paths connecting i and j in  $G_2$ , the communication between i and j in  $G_2$  is better than in  $G_1$ . This significance is not reflected in the shortest distance. Several applications require to overcome this limitation. Instead of the classical distance, the so-called resistance distance is used widely in many situations like in electrical networks, chemistry and random walks: see for example [1] and [2]. If there are multiple paths between two vertices, then the resistance distance is less than the shortest distance. The resistance matrix is now the matrix with (i, j)<sup>th</sup> entry equal to the resistance distance between i and j. Resistance matrices are non-singular and the inverse is given by an elegant formula that can be computed directly from the graph. The main purpose of this paper is to deduce a formula for the inverse of a generalized resistance matrix of a simple digraph with some special properties. This new formula generalizes the following known results.

## **1.1** Inverse of the resistance matrix of a connected graph

Let G be a connected graph with  $V(G) = \{1, \ldots, n\}$ . Let  $\delta_i$  denote the degree of the vertex i and A be the adjacency matrix of G. Then the Laplacian matrix of G is  $L = \text{Diag}(\delta_1, \ldots, \delta_n) - A$ . Now the resistance between any two vertices i and j in G is

$$r_{ij} := l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}, \tag{1}$$

where  $l_{ij}^{\dagger}$  is the  $(i, j)^{\text{th}}$  entry of the Moore Penrose inverse of L. Define  $R := [r_{ij}]$ . Then the inverse of R is given by

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R \tau} \tau \tau',$$
(2)

where

$$\tau := (\tau_1, \dots, \tau_n)'$$
 and  $\tau_i := 2 - \sum_{\{j \in V(G): (i,j) \in E\}} r_{ij}$   $i = 1, \dots, n.$ 

The proof of (2) is given in Theorem 9.1.2 in [1].

### **1.2** Inverse of the distance matrix of a tree

Let T be a tree with  $V(T) = \{1, \ldots, n\}$  and  $r_{ij}$  (defined in (1)) be the resistance distance between any two vertices i and j. If  $d_{ij}$  is the length of the shortest path connecting i and j in T, then by an induction argument, it can be shown that  $d_{ij} = r_{ij}$ . Define  $D := [d_{ij}]$ . Specializing formula (2) to T gives

$$D^{-1} = -\frac{1}{2}L + \frac{(2-\delta_1,\dots,2-\delta_n)'(2-\delta_1,\dots,2-\delta_n)}{2(n-1)},$$
(3)

where  $\delta_i$  is the degree of the vertex *i* and *L* is the Laplacian matrix of *T*. This formula is obtained by Graham and Lovász in [3].

#### **1.3** Inverse of the distance matrix of a weighted tree

Formula (3) can be generalized to weighted trees. We first need to define the Laplacian matrix of a weighted tree. Consider a tree  $G = (V, \Omega)$  with  $V = \{1, \ldots, n\}$ . To an edge  $(i, j) \in \Omega$ , we assign a positive real number  $w_{ij}$ . Define

$$l_{ij} := \begin{cases} -\frac{1}{w_{ij}} & (i,j) \in \Omega\\ 0 & i \neq j \text{ and } (i,j) \notin \Omega\\ \sum_{\{k:(i,k)\in\Omega\}} \frac{1}{w_{ik}} & i = j. \end{cases}$$

Then the Laplacian matrix of G is  $L := [l_{ij}]$ . The distance matrix of G is the symmetric matrix D with  $(i, j)^{\text{th}}$  entry equal to sum of all the weights that lie in the path connecting i and j. In this case, by an induction argument, it can be shown that LDL + 2L = 0 and from this identity it is easy to show that  $d_{ij} = l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}$ , where  $l_{ij}^{\dagger}$  is the  $(i, j)^{\text{th}}$  entry of the Moore Penrose inverse of L. Let  $\delta_i$  be the degree of the vertex i. In this setting, the following inverse formula is obtained in [4]:

$$D^{-1} = -\frac{1}{2} \left( L + \frac{\tau \tau'}{\sum_{i,j} w_{ij}} \right), \tag{4}$$

where  $\tau$  is the vector  $(2 - \delta_1, \ldots, 2 - \delta_n)'$ .

# 1.4 Inverse of the distance matrix of a tree with matrix weights

Formula (4) can be generalized. Consider a tree on n vertices with vertex set  $V(T) = \{1, \ldots, n\}$  and edge set E(T). To an edge (i, j) in T, assign a positive definite matrix  $W_{ij}$  of some fixed order s. Define

$$L_{ij} := \begin{cases} -W_{ij}^{-1} & (i,j) \in E(T) \\ O_s & i \neq j \text{ and } (i,j) \notin E(T) \\ \sum_{\{k:(i,k) \in E(T)\}} W_{ik}^{-1} & i = j. \end{cases}$$

(Here  $O_s$  is the  $s \times s$  matrix with all entries equal to zero.) The Laplacian matrix L of T is then the  $ns \times ns$  matrix with  $(i, j)^{\text{th}}$  block equal to  $L_{ij}$ . The distance between any two vertices i and j in T is the sum of all positive definite matrices that lie in the path connecting i and j. Let the  $(i, j)^{\text{th}}$  block of the Moore-Penrose inverse of L be given by  $M_{ij}$ . Then, by induction it can be shown that

 $D_{ij} = M_{ii} + M_{jj} - 2M_{ij}$ 

The inverse of D is

$$D^{-1} = -\frac{1}{2}(L + FS^{-1}F'), \tag{5}$$

where

$$S = \sum_{i} \sum_{j} W_{ij}$$
 and  $F = (2 - \delta_1, \dots, 2 - \delta_n)' \otimes I_s$ 

Formula (5) is obtained in [5].

### **1.5** Inverse of the resistance matrix of a directed graph

Let  $G = (V, \mathcal{E})$  be a digraph with vertex set  $V = \{1, \ldots, n\}$ . A directed edge from a vertex *i* to a vertex *j* in *G* will be denoted by (i, j). Recall that *G* is said to be strongly connected if there is a directed path between any two vertices *i* and *j*. A vertex *i* is said to be balanced if the outdegree of *i* and the indegree of *i* are equal. If all the vertices are balanced, then we say that *G* is balanced. We now assume that *G* is a strongly connected and balanced digraph. If *i* and *j* are any two vertices, define

$$a_{ij} := \begin{cases} 1 & (i,j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

The Laplacian matrix of G is the matrix  $L = [l_{ij}]$  such that

$$l_{ij} := \begin{cases} -a_{ij} & i \neq j \\ \sum_{\{k:k\neq i\}} a_{ik} & i = j. \end{cases}$$

Now, let  $L^{\dagger} := [l_{ij}^{\dagger}]$  be the Moore-Penrose inverse of L. Then the resistance between any two vertices i and j in G is given by

$$r_{ij} := l_{ii}^{\dagger} + l_{jj}^{\dagger} - 2l_{ij}^{\dagger}.$$

For the resistance matrix  $R = [r_{ij}]$  of G, we have the following inverse formula from [6]:

$$R^{-1} = -\frac{1}{2}L + \frac{1}{\tau' R \tau} (\tau (\tau' + \mathbf{1}' \operatorname{diag}(L^{\dagger})M)),$$
(6)

where

$$M := L - L', \quad \tau_i := 2 - \sum_{\{j:(i,j)\in\mathcal{E}\}} r_{ji},$$

and **1** is the vector of all ones in  $\mathbb{R}^n$ .

We now ask if there is a formula that unifies (2), (3), (4), (5) and (6).

### **1.6** Results obtained

We consider a simple digraph  $\mathcal{G} = (V, \mathcal{E})$  with  $V = \{1, \ldots, n\}$ . An element in  $\mathcal{E}$  will be denoted by (i, j). Precisely,  $(i, j) \in \mathcal{E}$  means that there is a directed edge from a vertex *i* to a vertex *j* in  $\mathcal{G}$ . All edges are assigned a positive definite matrix of some fixed order *s*. These positive definite matrices will be called *weights*. Let  $W_{ij}$  be the weight of the edge (i, j). In this set up, we define the following. (i) Laplacian of  $\mathcal{G}$ : If *i* and *j* are any two distinct vertices in  $\mathcal{G}$ , define

$$L_{ij} := \begin{cases} -W_{ij}^{-1} & (i,j) \in \mathcal{E} \\ O_s & \text{otherwise.} \end{cases}$$

(Here,  $O_s$  is the  $s \times s$  matrix with all entries equal to zero.) The Laplacian of  $\mathcal{G}$  is then the  $ns \times ns$  matrix

$$L(\mathcal{G}) := \begin{bmatrix} -\sum_{\{j:j\neq 1\}} L_{1j} & L_{12} & \dots & L_{1n} \\ L_{21} & -\sum_{\{j:j\neq 2\}} L_{2j} & \dots & L_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} & \dots & -\sum_{\{j:j\neq n\}} L_{nj} \end{bmatrix}$$

We shall say that  $L_{ij}$  is the  $(i, j)^{\text{th}}$  block of  $L(\mathcal{G})$ . We note that since  $\mathcal{G}$  is a digraph,  $L(\mathcal{G})$  is not symmetric in general.

(ii) **Resistance matrix of**  $\mathcal{G}$ : Let *i* and *j* be any two vertices in  $\mathcal{G}$ . Fix a, b > 0. Then, a generalized resistance distance between *i* and *j* is the  $s \times s$  matrix

$$R_{ij} = a^2 K_{ii} + b^2 K_{jj} - 2ab K_{ij},$$

where  $K_{ij}$  is the  $(i, j)^{\text{th}}$  block of the Moore Penrose inverse of L. The resistance matrix corresponding to a and b is then the  $ns \times ns$  matrix with  $(i, j)^{\text{th}}$  block equal to  $R_{ij}$ .

(iii) **Balanced vertices:** We say that a vertex  $j \in V$  is balanced if

$$\sum_{\{i \in V: (i,j) \in \mathcal{E}\}} W_{ij}^{-1} = \sum_{\{i \in V: (j,i) \in \mathcal{E}\}} W_{ji}^{-1}.$$

(iv) **Balanced digraphs:** If every vertex of  $\mathcal{G}$  is balanced, then we shall say that  $\mathcal{G}$  is balanced.

We obtain the following result in this paper.

**Theorem 1.** Let  $\mathcal{G} = (V, \mathcal{E})$  be a weighted, balanced and strongly connected digraph, where  $V = \{1, \ldots, n\}$ . Let  $W_{ij}$  be the weight of the edge (i, j). Fix a, b > 0. If Ris the resistance matrix of  $\mathcal{G}$ , L is the Laplacian of  $\mathcal{G}$ , and  $L^{\dagger}$  is the Moore-Penrose inverse of  $\mathcal{G}$ , then

$$R^{-1} = -\frac{1}{2ab}L + \tau(\tau'R\tau)^{-1}(\tau' - a^2U'\Delta(L^{\dagger})L' + b^2U'\Delta(L^{\dagger})L),$$

where  $\tau = (\tau_1, \ldots, \tau_n)'$  is given by

$$\tau_i := 2abI_s + L_{ii}R_{ii} - \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}R_{ji},$$

 $U = [\underbrace{I_s, \ldots, I_s}_{n}]$  and  $\Delta(L^{\dagger}) = \text{Diag}(K_{11}, \ldots, K_{nn})$ , with  $K_{ii}$  being the *i*<sup>th</sup> diagonal block of  $L^{\dagger}$ .

To illustrate, we give an example.

**Example 1.** Consider the following graph  $\mathcal{G}$  on four vertices. Let

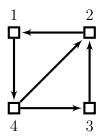


Figure 1: Directed Graph  $\mathcal{G}$ .

$$W_{14} = W_{21} = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ -4 & 7 \end{bmatrix}, \quad W_{43} = W_{32} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix},$$

and

$$W_{42} = \left[ \begin{array}{cc} 2 & -3 \\ -3 & 5 \end{array} \right].$$

The graph  $\mathcal G$  is balanced with the above matrix weights. The Laplacian of  $\mathcal G$  is

$$L = [L_{ij}] = \begin{bmatrix} 7 & 4 & 0 & 0 & 0 & 0 & -7 & -4 \\ 4 & 3 & 0 & 0 & 0 & 0 & -4 & -3 \\ -7 & -4 & 7 & 4 & 0 & 0 & 0 & 0 \\ -4 & -3 & 4 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -5 & -3 & -2 & -1 & 7 & 4 \\ 0 & 0 & -3 & -2 & -1 & -1 & 4 & 3 \end{bmatrix}$$

The Moore-Penrose inverse of L is the matrix

	20	-25	-16	23	-12	11	8	-9	1
$L^{\dagger} = [K_{ij}] = \frac{1}{80}$	-25	45	23	-39	11	-23	-9	17	
	8	-9	20	-25	-24	27	-4	7	
	-9	17	-25	45	27		7		
	-12	11	0	-5	36	-33	-24	27	•
	11	-23	-5	5	-33	69	27	-51	
	-16	23	-4	7	0	-5	20	-25	
	23	-39	7	-11	-5	5	-25	45	

Suppose a = 2 and b = 3. The resistance matrix of  $\mathcal{G}$  is

$R = [4K_{ii} + 9K_{jj} - 12K_{ij}]$									
	20	-25	$452 \\ -601$	-601	548	-529	164	$-217^{-1}$	1
	-25	45			-529	1077	-217	381	
	164	-217	20	-25	692	-721	308	-409	
_ 1	-217	381	-25	45	-721	1413	-409	717	
$-\frac{1}{80}$	468	-489	324	-297	36	-33	612	-681	'
	-489	957	-297	621	-33	69	-681	1293	
	452			-409			20	-25	
	-601	1053	-409	717	-337	741	-25	45	

.

Next, we compute  $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)'$ . Recall that

$$\tau_i := 2abI_s + L_{ii}R_{ii} - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1}R_{ji}.$$

Thus,

$$\tau = \frac{1}{5} \begin{bmatrix} 15 & 0 & | & 15 & 0 & | & 21 & 2 & | & 9 & -2 \\ 0 & 15 & | & 0 & 15 & | & 2 & 19 & | & -2 & 11 \end{bmatrix}',$$
  
$$\tau' R \tau = \frac{1}{5} \begin{bmatrix} 2916 & -3159 \\ -3159 & 6075 \end{bmatrix},$$

and

$$\tau' - 4U'\Delta(L^{\dagger})L' + 9U'\Delta(L^{\dagger})L = \frac{1}{10} \begin{bmatrix} 30 & 0 & 3 & -9 & 57 & 9 & 30 & 0 \\ 0 & 30 & -9 & 12 & 9 & 48 & 0 & 30 \end{bmatrix}.$$

Hence, we have

$=\frac{1}{42444}$	-23259	-13368	-84	-138	3084	1698	26259	14928
	-13368	-9891	-138	54	1698	1386	14928	11331
	26259	14928	-24843	-14286	3084	1698	1500	780
	14928	11331	-14286	-10557	1698	1386	780	720
	2204	1188	6938	3351	-2530	-975	2204	1188
	1188	1016	3351	3587	-975	-1555	1188	1016
	796	372	17653	10521	8698	4371	-23963	-13776
	372	424	10521	7132	4371	4327	-13776	-10187

$$-\frac{1}{2ab}L + \tau(\tau'R\tau)^{-1}(\tau'-4U'\Delta(L^{\dagger})L'+9U'\Delta(L^{\dagger})L)$$

which is equal to  $R^{-1}$ .

## 1.7 Other results

We obtain the following two results after proving Theorem 1.

• By numerical computations, we observe that for any a, b > 0,

$$R_{ij} = a^2 K_{ii} + b^2 K_{jj} - 2abK_{ij}$$

is positive semidefinite. We do not know how to prove this result in general. However, when the weights in  $\mathcal{G}$  of Theorem 1 are positive scalars, we show that  $R_{ij}$  is always a non-negative real number.

• Let T be a tree on n vertices. Suppose D and L are the distance and Laplacian matrices of T. Then, from (3) it can be deduced that

$$(D^{-1} - L)^{-1} = \frac{1}{3}D + \frac{1}{3}(\sum_{i,j} w_{ij})\mathbf{11'}.$$

In particular, this equation says that every entry in  $(D^{-1}-L)^{-1}$  is non-negative. Suppose M is the Laplacian matrix of an arbitrary tree on n vertices. It can be shown that  $D^{-1} - M$  is non-singular. We now say that  $(D^{-1} - M)^{-1}$  is a perturbation of the distance matrix D. In [4], it is shown that all perturbations of D are non-negative matrices. We now assume that R is the resistance matrix of a connected graph (defined in Section 1.1) on n vertices. Now consider the Laplacian matrix L of  $\mathcal{G}$  in Theorem 1. Suppose all the weights in  $\mathcal{G}$  are positive scalars. It can be shown that  $R^{-1} - L$  is always non-singular. We now say that  $(R^{-1} - L)^{-1}$  is a perturbation of R. By performing certain numerical experiments, we observed that similar to the result in [4], all perturbations of R are non-negative matrices. Since  $\mathcal{G}$  is a digraph, L is not symmetric in general and hence all perturbations of R are not symmetric matrices. Despite this difficulty, by using a argument different from [4], we show that all perturbations of R are non-negative. This result is proved in the final part of this paper.

## 2 Preliminaries

We mention the notation and some basic results that will be used in the paper.

- (i) We reserve  $\mathcal{G}$  to denote a simple, strongly connected, weighted, and balanced digraph with vertex set  $V = \{1, \ldots, n\}$ . A directed edge from *i* to *j* in  $\mathcal{G}$  will be denoted by (i, j). We use  $\mathcal{E}$  to denote the edge set of  $\mathcal{G}$ . The weight of an edge (i, j) will be denoted by  $W_{ij}$ . All weights will be symmetric positive definite matrices and have fixed order *s*.
- (ii) Let  $B^{ns}$  be the set of all real  $ns \times ns$  matrices. A matrix A in  $B^{ns}$  will be denoted by  $[A_{ij}]$ , where  $A_{ij}$  is an  $s \times s$  matrix. We shall say that  $A_{ij}$  is the  $(i, j)^{\text{th}}$  block of A. There are n blocks in A. The null-space of a matrix A is denoted by null(A) and the column space by col(A).
- (iii) The vector of all ones in  $\mathbb{R}^n$  will be denoted by **1**. The matrix  $\mathbf{1}' \otimes I_s$  will be denoted by U', i.e.

$$U := [I_s, \ldots, I_s]',$$

where  $I_s$  appears n times. We use J to denote the matrix in  $B^{ns}$  with all blocks equal to  $I_s$ . Note that J = UU'.

(iv) The Laplacian matrix of  $\mathcal{G}$  will be denoted by L and its Moore-Penrose by  $L^{\dagger}$ . We note that L and  $L^{\dagger}$  belong to  $B^{ns}$ . The  $(i, j)^{\text{th}}$  block of  $L^{\dagger}$  will be denoted by  $K_{ij}$ . We use  $\Delta(L^{\dagger})$  to denote the block diagonal matrix

$$\operatorname{Diag}(K_{11},\ldots,K_{nn}).$$

Let  $\mathcal{G}'$  be the digraph such that  $V(\mathcal{G}') := \{1, \ldots, n\}$  and

$$E(\mathcal{G}') := \{(j,i) : (i,j) \in E(\mathcal{G})\}$$

To an edge (i, j) of  $\mathcal{G}'$ , we assign the weight  $W_{ji}$ . Again  $\mathcal{G}'$  will be strongly connected, and balanced. The Laplacian of  $\mathcal{G}'$  is clearly L'.

(v) Fix a, b > 0. The generalized resistance matrix of  $\mathcal{G}$  corresponding to a and b will be denoted by  $R_{a,b}$ . Thus,  $R_{a,b}$  is an element in  $B^{ns}$  with  $(i, j)^{\text{th}}$  block equal to

$$a^2 K_{ii} + b^2 K_{jj} - 2ab K_{ij}.$$

- (vi) Let A be an  $m \times m$  matrix.
  - (a) We say that A is positive semidefinite if  $x'Ax \ge 0$  for all  $x \in \mathbb{R}^m$  and positive definite if x'Ax > 0 for all  $0 \ne x \in \mathbb{R}^m$ . (To define positive semidefiniteness, we do not assume that A is symmetric.)
  - (b) Following [7], we say that A is almost positive definite if for each  $x \in \mathbb{R}^m$ , either x'Ax > 0 or Ax = 0. Suppose A is almost positive definite. Then the Moore-Penrose inverse of A is also almost positive definite: see Corollary 2 in [7].
- (vii) Let  $B = [A_1, \ldots, A_n]$ , where each  $A_i$  is an  $s \times s$  matrix. As before, we say that  $A_j$  is the  $j^{\text{th}}$  block of B. Let Diag(B) be the  $ns \times ns$  block matrix

$$\operatorname{Diag}(A_1,\ldots,A_n).$$

- (vii) We use [n] for  $\{1, \ldots, n\}$ . The zero matrix of order  $s \times s$   $(s \ge 2)$  will be denoted by  $O_s$ . The identity matrix of order k will be denoted by  $I_k$ .
- (viii) Let  $A = [a_{ij}]$  be an  $m \times m$  matrix. We say that A is row diagonally dominant if

$$|a_{ii}| \ge \sum_{\{j: j \neq i\}} |a_{ij}|$$

We shall use the following well known result on diagonally dominant matrices.

**Theorem 2.** Let A be row diagonally dominant. Suppose A is non-singular. Let  $B := A^{-1}$  and  $B = [b_{ij}]$ . Then,

$$|b_{ii}| \ge |b_{ji}| \quad \forall j.$$

## 3 Results

To prove our main result, we need to show that the Laplacian of  $\mathcal{G}$  has certain properties.

## 3.1 Properties of the Laplacian

From the digraph  $\mathcal{G} = (V, \mathcal{E})$ , we define a simple undirected graph  $\widetilde{G}$  as follows.

**Definition 1.** Let  $V(\widetilde{G}) := \{1, \ldots, n\}$ . We say that any two vertices  $i, j \in V(\widetilde{G})$  are adjacent in  $\widetilde{G}$  if and only if either  $(i, j) \in \mathcal{E}$  or  $(j, i) \in \mathcal{E}$ .

**Definition 2.** To an edge (i, j) of  $\widetilde{G}$ , define

$$\widetilde{W}_{ij} := \begin{cases} (W_{ij}^{-1} + W_{ji}^{-1})^{-1} & (i,j) \in \mathcal{E} \text{ and } (j,i) \in \mathcal{E} \\ W_{ij} & (i,j) \in \mathcal{E} \text{ and } (j,i) \notin \mathcal{E} \\ W_{ji} & (i,j) \notin \mathcal{E} \text{ and } (j,i) \in \mathcal{E}. \end{cases}$$
(7)

We now have the weighted graph  $\widetilde{G}$ . Let E be the set of all edges of  $\widetilde{G}$ .

**Definition 3.** The Laplacian of  $\widetilde{G}$  is the matrix  $L(\widetilde{G}) = [M_{ij}] \in B^{ns}$ , where the (i, j)<sup>th</sup> block is defined as follows:

$$M_{ij} := \begin{cases} -\widetilde{W}_{ij}^{-1} & (i,j) \in E\\ O_s & i \neq j \text{ and } (i,j) \notin E\\ \sum_{\{k:k\neq i\}} \widetilde{W}_{ik}^{-1} & i = j. \end{cases}$$

$$\tag{8}$$

We now have the following proposition. Recall that L is the Laplacian of  $\mathcal{G}$ .

## **Proposition 1.** $L(\widetilde{G}) = L + L'$ .

*Proof.* We shall write

$$A = L + L'$$
 and  $M = L(G)$ 

Let the  $(i, j)^{\text{th}}$  block of A be  $A_{ij}$  and L be  $L_{ij}$ . We need to show that

$$A_{ij} = M_{ij}$$
 for all  $i, j \in V$ ,

where  $M_{ij}$  is defined in (8). Note that  $A_{ij} = L_{ij} + L_{ji}$ . Partition the set of all edges E of  $\tilde{G}$  as follows:

 $S_1 := \{(i, j) \in E : (i, j) \in \mathcal{E} \text{ and } (j, i) \notin \mathcal{E}\}$  $S_2 := \{(i, j) \in E : (i, j) \notin \mathcal{E} \text{ and } (j, i) \in \mathcal{E}\}$  $S_3 := \{(i, j) \in E : (i, j) \in \mathcal{E} \text{ and } (j, i) \in \mathcal{E}\}.$ 

Fix i and j in  $\{1, \ldots, n\}$ . We consider the following cases.

**Case 1:** Suppose *i* and *j* are not adjacent in  $\widetilde{G}$ . This means  $(i, j) \notin \mathcal{E}$  and  $(j, i) \notin \mathcal{E}$ . So,  $L_{ij} = O_s$  and  $L_{ji} = O_s$ . By (8),  $M_{ij} = O_s$ . Therefore,

$$A_{ij} = L_{ij} + L_{ji} = O_s = M_{ij}.$$

**Case 2:** Suppose *i* and *j* are adjacent in  $\widetilde{G}$ . We consider three possible sub-cases. **Case** (i): Let  $(i, j) \in S_1$ . Then, by (7),  $\widetilde{W}_{ij} = W_{ij}$ . So,  $M_{ij} = -W_{ij}^{-1}$ . Because the weight of the edge (i, j) in  $\mathcal{G}$  is  $W_{ij}$ ,  $L_{ij} = -W_{ij}^{-1}$ . Since  $(j, i) \notin \mathcal{E}$ ,  $L_{ji} = O_s$  and therefore,

$$A_{ij} = L_{ij} + L_{ji} = -W_{ij}^{-1} = M_{ij}.$$
(9)

**Case** (ii): Let  $(i, j) \in S_2$ . Then, by (7),  $\widetilde{W}_{ij} = W_{ji}$ . So,  $M_{ij} = -W_{ji}^{-1}$ . As  $(i, j) \notin \mathcal{E}$ ,  $L_{ij} = O_s$ . The weight of the edge (j, i) in  $\mathcal{G}$  is  $W_{ji}^{-1}$ . So,  $L_{ji} = -W_{ji}^{-1}$ ; and hence

$$A_{ij} = L_{ij} + L_{ji} = -W_{ji}^{-1} = M_{ij}.$$
 (10)

**Case** 3: Let  $(i, j) \in S_3$ . Then,

$$\widetilde{W}_{ij} = (W_{ij}^{-1} + W_{ji}^{-1})^{-1}$$

So,

$$M_{ij} = -\widetilde{W}_{ij}^{-1} = -(W_{ij}^{-1} + W_{ji}^{-1}).$$

The weights of the edges (i, j) and (j, i) in  $\mathcal{G}$  are respectively,  $W_{ij}$  and  $W_{ji}$ . So,

$$L_{ij} = -W_{ij}^{-1}$$
 and  $L_{ji} = -W_{ji}^{-1}$ 

Thus,

$$A_{ij} = L_{ij} + L_{ji} = -(W_{ij}^{-1} + W_{ji}^{-1}) = M_{ij}.$$
(11)

Since

$$(L+L')U = MU = O_s,$$

it follows that  $A_{ii} = M_{ii}$  for each i = 1, ..., n. The proof is complete.

We now deduce some properties of the Laplacian matrix L of  $\mathcal{G}$ .

**Proposition 2.** The following are true.

- (i) L is positive semidefinite.
- (ii)  $\operatorname{null}(L) = \operatorname{null}(L') = \operatorname{col}(J).$
- (iii)  $L^{\dagger}$  is almost positive definite.
- (iv)  $LL^{\dagger} = L^{\dagger}L = I_{ns} \frac{J}{n}.$

Proof. Consider the undirected graph  $\widetilde{G} = (V, E)$  in Definition 1. Put  $\widetilde{L} = L(\widetilde{G})$  and  $S_{ij} = \widetilde{W}_{ij}^{-1}$ , where  $\widetilde{W}_{ij}$  are defined in (7). Corresponding to an edge (p,q) in  $\widetilde{G}$ , we now define  $\widetilde{S}(p,q) \in B^{ns}$  with  $(i,j)^{\text{th}}$  block given by

$$\widetilde{S}(p,q)_{ij} := \begin{cases} -S_{pq} & (i,j) = (p,q) \text{ or } (i,j) = (q,p) \\ S_{pq} & i = j = p \text{ or } i = j = q \\ O_s & \text{else.} \end{cases}$$

Now,

$$\widetilde{L} = \sum_{(p,q)\in E} \widetilde{S}(p,q).$$

Let  $x \in \mathbb{R}^{ns}$ . Write

$$x = (x^1, \dots, x^j, \dots, x^n)', \text{ where each } x^i \in \mathbb{R}^s.$$

By an easy verification, we find that, if  $(p,q) \in E$ , then

$$x'\widetilde{S}(p,q)x = \langle S_{pq}(x^p - x^q), x^p - x^q \rangle$$

Thus,

$$x'\widetilde{L}x = \sum_{(i,j)\in E} \langle S_{ij}(x^i - x^j), x^i - x^j \rangle.$$
(12)

Each  $S_{ij}$  is a positive definite matrix. So,  $x'\widetilde{L}x \ge 0$ . By Proposition 1,  $\widetilde{L} = L + L'$ . Therefore,  $x'\widetilde{L}x = 2x'Lx$ . So,  $x'Lx \ge 0$ . This proves (i).

Let  $x \in \text{null}(L)$ . As,  $\tilde{L} = L + L'$ , we see that  $x'\tilde{L}x = 0$ . By (12), if  $(i, j) \in E$ , then  $x^i = x^j$ . Because  $\mathcal{G}$  is strongly connected,  $\tilde{G}$  is connected. So,  $x^i = x^j$  for all i, j. Thus,

$$x \in \operatorname{span}\{(w,\ldots,w)' \in \mathbb{R}^{ns} : w \in \mathbb{R}^s\}.$$

Since,  $\operatorname{col}(J) = \operatorname{span}\{(w, \ldots, w) : w \in \mathbb{R}^s\}$ , we see that x = Jp for some  $w \in \mathbb{R}^s$ . So,  $\operatorname{null}(L) = \operatorname{col}(J)$ . Now,  $\mathcal{G}'$  is a strongly connected, and balanced digraph. Because L' is the Laplacian of  $\mathcal{G}'$ , we see that  $\operatorname{null}(L') = \operatorname{col}(J)$ . This proves (ii).

We now prove (iii). Let  $y \in \mathbb{R}^{ns}$ . Since L is positive semidefinite,  $y'Ly \geq 0$ . Suppose y'Ly = 0. Then,  $y'\widetilde{L}y = 0$ . By equation (12), it follows that  $y \in \operatorname{col}(J)$ . Since  $\operatorname{col}(J) = \operatorname{null}(L)$ , we have Ly = 0. Thus, either y'Ly > 0 or Ly = 0. So, L is almost positive definite. By item (vi) in Section 2,  $L^{\dagger}$  is almost positive definite as well. This proves (iii).

To prove (iv), we show that

$$LL^{\dagger}v = v \text{ for all } v \in \operatorname{null}(J).$$

Let  $v \in \operatorname{null}(J)$ . Suppose  $LL^{\dagger}v = w$ . Then,

$$Jw = 0$$
 and  $L^{\dagger}LL^{\dagger}v = L^{\dagger}w$ .

Since  $L^{\dagger}LL^{\dagger} = L^{\dagger}$ , we get  $L^{\dagger}v = L^{\dagger}w$  and hence  $v - w \in \text{null}(L^{\dagger})$ . As  $\text{null}(L^{\dagger}) = \text{col}(J)$ , we get

$$v - w \in \operatorname{col}(J).$$

Since  $JLL^{\dagger}v = Jw$ , and  $JL = O_{ns}$ , Jw = 0. As  $v \in \text{null}(J)$ , Jv = 0. So, J(v-w) = 0and hence

$$v - w \in \operatorname{null}(J).$$

We now have  $v - w \in \text{null}(J) \cap \text{col}(J)$ . So, v = w. The proof is complete.

**Proposition 3.**  $\Delta(L^{\dagger})$  is a positive definite matrix.

Proof. We recall that  $\Delta(L^{\dagger}) = \text{Diag}(K_{11}, \ldots, K_{nn})$ . Fix  $i \in \{1, \ldots, n\}$ . We show that  $K_{ii}$  is positive definite. Let  $y \in \mathbb{R}^s$ . Define  $q := (q^1, \ldots, q^n)' \in \mathbb{R}^{ns}$  by

$$q^j := \begin{cases} y & j = i \\ 0 & \text{else.} \end{cases}$$

In view of previous lemma,  $L^{\dagger}$  is almost positive definite. So,  $q'L^{\dagger}q > 0$  or  $L^{\dagger}q = 0$ . We note that  $q'L^{\dagger}q = y'K_{ii}y$ . Hence, if  $q'L^{\dagger}q > 0$ , then  $y'K_{ii}y > 0$ . Suppose  $L^{\dagger}q = 0$ . Since  $\operatorname{null}(L^{\dagger}) = \operatorname{null}(L')$  and  $\operatorname{null}(L') = \operatorname{col}(J)$ , q = 0. This means y is zero. So,  $K_{ii}$  is positive definite. The proof is complete.

## **3.2** Inverse formula

Recall that the generalized resistance matrix of  $\mathcal{G}$  corresponding to two positive real numbers a and b is

$$R_{a,b} := [R_{ij}] = [a^2 L_{ii}^{\dagger} + b^2 L_{jj}^{\dagger} - 2ab L_{ij}^{\dagger}].$$

Define

$$\tau_i := 2abI_s + L_{ii}R_{ii} - \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}R_{ji} \text{ for all } i = 1,\dots,n.$$
(13)

Now, let  $\tau$  be the  $ns \times s$  matrix  $(\tau_1, \ldots, \tau_n)'$ . The inverse formula will be proved by using the following lemma.

Lemma 1. The following are true.

- (i)  $\tau = a^2 L \Delta(L^{\dagger}) U + \frac{2ab}{n} U.$ (ii)  $\tau' + a^2 U' \Delta(L^{\dagger}) (L - L') = a^2 U' \Delta(L^{\dagger}) L + \frac{2ab}{n} U'.$
- (iii)  $LR_{a,b} + 2abI_{ns} = \tau U'.$

(iv) 
$$R_{a,b}L + 2abI_{ns} = U\tau' - a^2UU'\Delta(L^{\dagger})L' + b^2UU'\Delta(L^{\dagger})L.$$

(v)  $U'\tau = 2abI_s$ .

(vi) 
$$\tau' R_{a,b} \tau = 2a^3 b^3 \tilde{X}' L \tilde{X} + \frac{1}{n} 4a^2 b^2 (a^2 + b^2) \sum_{i=1}^n K_{ii}$$
, where  $\tilde{X} := \Delta(L^{\dagger}) U$ 

(vii)  $\tau' R_{a,b} \tau$  is a positive definite matrix.

*Proof.* Fix  $i \in [n]$ . For simplicity, we shall use R for  $R_{a,b}$ . Since  $L = [L_{ij}]$  and  $L^{\dagger} = [K_{ij}]$ , the  $(i, j)^{\text{th}}$  block of  $LL^{\dagger}$  is the  $s \times s$  matrix

$$L_{ii}K_{ii} - \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}K_{ji}$$

The (i, j)<sup>th</sup> block of  $I_{ns} - \frac{J}{n}$  is  $(1 - \frac{1}{n})I_s$ . Since  $LL^{\dagger} = I_{ns} - \frac{1}{n}J$ , we see that

$$L_{ii}K_{ii} - \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}K_{ji} = (1-\frac{1}{n})I_s.$$

Rewriting the above equation, we have

$$\sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1} K_{ji} = L_{ii} K_{ii} - (1-\frac{1}{n}) I_s.$$
(14)

By definition,

$$\tau_i = 2abI_s + L_{ii}R_{ii} - \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}R_{ji}.$$

Because

$$R_{ji} = a^2 K_{jj} + b^2 K_{ii} - 2ab K_{ji}$$
 and  $R_{ii} = (a-b)^2 K_{ii}$ ,

we have

$$\tau_i = 2abI_s + (a-b)^2 L_{ii} K_{ii} - \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1} (a^2 K_{jj} + b^2 K_{ii} - 2abK_{ji}).$$
(15)

We recall that

$$L_{ii} = \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}$$

So,

$$\sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1} K_{ii} = L_{ii} K_{ii}.$$
(16)

Substituting (16) in (15),

$$\tau_i = 2abI_s + (a-b)^2 L_{ii}K_{ii} - b^2 L_{ii}K_{ii} - a^2 \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}K_{jj} + 2ab \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}K_{ji}.$$

Using (14) in the above equation, we get

$$\tau_i = 2abI_s + (a-b)^2 L_{ii}K_{ii} - b^2 L_{ii}K_{ii} - a^2 \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1}K_{jj} + 2abL_{ii}K_{ii} - 2ab(1-\frac{1}{n})I_s$$

After simplification,

$$\tau_i = a^2 L_{ii} K_{ii} - a^2 \sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1} K_{jj} + \frac{2ab}{n} I_s.$$
(17)

Let A := Diag(L) - L. Write  $A = [A_{ij}]$ . Then,

$$\sum_{\{j:(i,j)\in\mathcal{E}\}} W_{ij}^{-1} K_{jj} = \sum_{j=1}^{n} A_{ij} K_{jj}$$

$$= (A\Delta(L^{\dagger})U)_i.$$
(18)

We now compute AU. Because

$$A = \begin{bmatrix} O_s & -L_{12} & -L_{13} & \dots & -L_{1n} \\ -L_{21} & O_s & -L_{23} & \dots & -L_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -L_{n1} & -L_{n2} & -L_{n3} & \dots & O_s \end{bmatrix},$$

$$(AU)_i = -\sum_{j \in [n] \setminus \{i\}} L_{ij}.$$

Put

$$P := \Delta(L^{\dagger})U.$$

Then,

$$P_i = K_{ii}$$

Thus,

$$(\operatorname{Diag}(AU)P)_i = -\sum_{j \in [n] \setminus \{i\}} L_{ij} K_{ii}.$$

As  $\sum_{j=1}^{n} L_{ij} = O_s$ , we get

$$(\operatorname{Diag}(AU)P)_i = -\sum_{j \in [n] \setminus \{i\}} L_{ij} K_{ii} = L_{ii} K_{ii}.$$
(19)

By (17), (18) and (19),

$$\tau_i = a^2 (\operatorname{Diag}(AU)P - AP)_i + \frac{2ab}{n} I_s.$$
(20)

Put

$$\widetilde{A} := \operatorname{Diag}(AU) - A.$$

In view of (20),

$$\tau_i = a^2 (\widetilde{A}P)_i + \frac{2ab}{n} I_s.$$
(21)

But a direct verification tells that

$$\widetilde{A} = L.$$

Therefore by (21),

$$\tau = a^2 L \Delta(L^{\dagger}) U + \frac{2ab}{n} U.$$

This proves (i).

We now prove (ii). Put M := L - L'. Then by (i),

$$a^{2}U'\Delta(L^{\dagger})M + \tau' = a^{2}U'\Delta(L^{\dagger})L - a^{2}U'\Delta(L^{\dagger})L' + a^{2}U'\Delta(L^{\dagger})L' + \frac{2ab}{n}U'$$
  
$$= a^{2}U'\Delta(L^{\dagger})L + \frac{2ab}{n}U'.$$
 (22)

The proof of (ii) is complete.

We now prove (iii). Since

$$R_{ij} = a^2 L_{ii}^{\dagger} + b^2 L_{jj}^{\dagger} - 2abL_{ij}^{\dagger},$$

and  $R = [R_{ij}]$ , it is easy to see that

$$R = a^2 \Delta(L^{\dagger}) U U' + b^2 U U' \Delta(L^{\dagger}) - 2ab L^{\dagger}.$$

As  $LL^{\dagger} = I_{ns} - \frac{1}{n}UU'$  and  $LU = O_s$ , we get

$$LR = a^{2}L\Delta(L^{\dagger})UU' - 2abLL^{\dagger}$$
  
=  $a^{2}L\Delta(L^{\dagger})UU' + \frac{2ab}{n}UU' - 2abI_{ns}$   
=  $(a^{2}L\Delta(L^{\dagger})U + \frac{2ab}{n})U' - 2abI_{ns}.$  (23)

.

By (i),

$$\tau = a^2 L \Delta(L^{\dagger}) U + \frac{2ab}{n} U.$$

Hence,

$$LR = \tau U' - 2abI_{ns}.$$

This completes the proof of (iii).

To prove (iv), first we observe that

$$RL = b^2 U U' \Delta(L^{\dagger}) L - 2ab L^{\dagger} L.$$

Since  $L^{\dagger}L = I_{ns} - \frac{1}{n}UU'$ , we get

$$RL + 2abI_{ns} = b^2 U U' \Delta(L^{\dagger}) L + \frac{2ab}{n} U U'.$$
(24)

By (i),

$$U\tau' = a^2 U U' \Delta(L^{\dagger}) L' + \frac{2ab}{n} U U'.$$
<sup>(25)</sup>

From (24) and (25),

$$RL + 2abI_{ns} = U\tau' - a^2 UU'\Delta(L^{\dagger})L' + b^2 UU'\Delta(L^{\dagger})L.$$

The proof of (iv) is complete.

By item (i),

$$U'\tau = a^2 U' L \Delta(L^{\dagger}) U + \frac{2ab}{n} U' U.$$

As  $U'U = I_{ns}$  and  $U'L = O_s$ , it follows that

$$U'\tau = 2abI_{ns}.$$

This proves (v).

Put M = L - L'. By (ii),

$$\tau' R\tau = (a^2 U' \Delta(L^{\dagger})L + \frac{2ab}{n}U' - a^2 U' \Delta(L^{\dagger})M)R\tau.$$
(26)

Because M = L - L',

$$a^2 U' \Delta(L^{\dagger})L + \frac{2ab}{n}U' - a^2 U' \Delta(L^{\dagger})M = a^2 U' \Delta(L^{\dagger})L' + \frac{2ab}{n}U'.$$

Substituting for  $\tau$  from (i) in (26), we get

$$\tau' R \tau = (a^2 U' \Delta(L^{\dagger}) L' + \frac{2ab}{n} U') R(a^2 L \Delta(L^{\dagger}) U + \frac{2ab}{n} U).$$

Therefore,

$$\tau' R\tau = a^4 U' \Delta(L^{\dagger}) L' RL \Delta(L^{\dagger}) U + \frac{2a^3 b}{n} U' \Delta(L^{\dagger}) L' RU + \frac{2a^3 b}{n} U' RL \Delta(L^{\dagger}) U + \frac{4a^2 b^2}{n^2} U' RU.$$
(27)

 $\operatorname{As}$ 

$$LU = L'U = O_s$$
 and  $R = a^2 \Delta(L^{\dagger})UU' + b^2 UU' \Delta(L^{\dagger}) - 2abL^{\dagger}$ ,

we have

$$U'\Delta(L^{\dagger})L'RL\Delta(L^{\dagger})U = -2abU'\Delta(L^{\dagger})L'L^{\dagger}L\Delta(L^{\dagger})U.$$
  
Since  $L^{\dagger}L = I_{ns} - \frac{1}{n}UU'$  and  $L'U = O_s$ ,

$$U'\Delta(L^{\dagger})L'RL\Delta(L^{\dagger})U = -2abU'\Delta(L^{\dagger})L'\left(I_{ns} - \frac{1}{n}UU'\right)\Delta(L^{\dagger})U$$
  
=  $-2abU'\Delta(L^{\dagger})L'\Delta(L^{\dagger})U.$  (28)

Define

$$\widetilde{X} := \Delta(L^{\dagger})U.$$

Then,

$$\widetilde{X}' L \widetilde{X} = U' \Delta(L^{\dagger}) L \Delta(L^{\dagger}) U.$$
<sup>(29)</sup>

By (28),

$$U'\Delta(L^{\dagger})L'RL\Delta(L^{\dagger})U = -2ab\widetilde{X}'L\widetilde{X}.$$

In view of (iv) and (i), we have

$$RL + 2abI_{ns} = U\tau' - a^2 UU' \Delta(L^{\dagger})L' + b^2 UU' \Delta(L^{\dagger})L;$$
$$U\tau' = a^2 UU' \Delta(L^{\dagger})L' + \frac{2ab}{n}UU'.$$

These two equations imply

$$RL = b^2 U U' \Delta(L^{\dagger}) L + \frac{2ab}{n} U U' - 2ab I_{ns}$$

Hence,

$$U'RL\Delta(L^{\dagger})U = U'\left(b^{2}UU'\Delta(L^{\dagger})L + \frac{2ab}{n}UU' - 2abI_{ns}\right)\Delta(L^{\dagger})U$$
$$= (b^{2}n)U'\Delta(L^{\dagger})L\Delta(L^{\dagger})U.$$

By (29),

$$U'RL\Delta(L^{\dagger})U = b^2 n \widetilde{X}' L \widetilde{X}.$$
(30)

We also note that

$$U'\Delta(L^{\dagger})L'RU = U'\Delta(L^{\dagger})L'(a^{2}\Delta(L^{\dagger})UU' + b^{2}UU'\Delta(L^{\dagger}) - 2abL^{\dagger})U.$$
  
$$= a^{2}nU'\Delta(L^{\dagger})L'\Delta(L^{\dagger})U$$
  
$$= a^{2}n\widetilde{X}'L\widetilde{X},$$
  
(31)

where the second equality follows from  $L'U = L^{\dagger}U = O_s$  and the last one from (29). Since

$$R = a^2 \Delta(L^{\dagger}) UU' + b^2 UU' \Delta(L^{\dagger}) - 2abL^{\dagger},$$

we see that

$$U'RU = n(a^2 + b^2) \sum_{i=1}^{n} K_{ii}.$$
(32)

Substituting (28), (30), (31) and (32) in (27), we get

$$\tau' R \tau = 2a^3 b^3 \widetilde{X}' L \widetilde{X} + \frac{4a^2 b^2 (a^2 + b^2)}{n} \sum_{i=1}^n K_{ii}.$$

The proof of (vi) is complete.

Since L is positive semidefinite,  $\tilde{X}'L\tilde{X}$  is positive semidefinite. By Proposition (3), each  $K_{ii}$  is positive definite. So,  $\tau'R\tau$  is positive definite. This proves (vii). The proof is complete.

We prove the inverse formula in Theorem 1.

Theorem 3.

$$R_{a,b}^{-1} = -\frac{1}{2ab}L + \tau(\tau'R\tau)^{-1}(\tau' - a^2U'\Delta(L^{\dagger})L' + b^2U'\Delta(L^{\dagger})L)$$

*Proof.* Again, as in the proof of above lemma, we shall use  $R_{a,b}$  for R. By item (iii) of Lemma 1,

$$LR + 2abI_{ns} = \tau U'.$$

In view of item (v) of the previous Lemma,  $U'\tau = 2abI_s$ . So,

$$LR\tau + 2ab\tau = \tau U'\tau = 2ab\tau.$$

This implies

$$LR\tau = O_s.$$

We know that

$$\operatorname{null}(L) = \operatorname{span}\{(p, \dots, p)' : p \in \mathbb{R}^s\}.$$

So,

$$R\tau = UC$$
,

where C is a  $s \times s$  matrix. Since  $\tau' R \tau$  is a positive definite matrix,  $R \tau$  cannot be zero. Hence,  $C \neq O_s$ . As  $\tau' U = 2abI_s$ , we get

$$C = \frac{1}{2ab}\tau' R\tau.$$

Therefore,

$$R\tau = \frac{1}{2ab}U(\tau'R\tau). \tag{33}$$

Since  $L'U = O_s$ , from item (iv) of Lemma 1, we deduce that

$$(\tau' - a^2 U' \Delta(L^{\dagger}) L' + b^2 U' \Delta(L^{\dagger}) L) (RL + 2abI_{ns}) = 2ab(\tau' - a^2 U' \Delta(L^{\dagger}) L' + b^2 U' \Delta(L^{\dagger}) L).$$

Simplifying the above equation, we get

au

$$(\tau' - a^2 U' \Delta(L^{\dagger}) L' + b^2 U' \Delta(L^{\dagger}) L) RL = O_s.$$
(34)

We now claim that

$$(\tau' - a^2 U' \Delta(L^{\dagger}) L' + b^2 U' \Delta(L^{\dagger}) L) R \neq O_s.$$

If not, then

$${}^{\prime}R\tau - a^{2}U^{\prime}\Delta(L^{\dagger})L^{\prime}R\tau + b^{2}U^{\prime}\Delta(L^{\dagger})LR\tau = O_{s}.$$
(35)

By (33),

$$R\tau = \frac{1}{2ab}U\tau'R\tau$$

So,

$$LR\tau = O_s$$
 and  $L'R\tau = O_s$ .

Hence (35) leads to  $\tau' R \tau = O_s$ . This contradicts that  $\tau' R \tau$  is positive definite. Hence,

$$(\tau' - a^2 U' \Delta(L^{\dagger}) L' + b^2 U' \Delta(L^{\dagger}) L) R \neq O_s.$$

Since nullity of L' is s and  $L'U = O_s$ , by (34), there exists an  $s \times s$  matrix  $\widetilde{C}$  such that

$$(\tau' - a^2 U' \Delta(L^{\dagger})L' + b^2 U' \Delta(L^{\dagger})(L^{\dagger})L)R = \tilde{C}U'.$$

We know that  $U'\tau = 2abI_s$ . So, from the previous equation,

$$\tilde{C} = \frac{1}{2ab}\tau' R\tau$$

Thus,

$$(\tau' - a^2 U' \Delta(L^{\dagger})L' + b^2 U' \Delta(L^{\dagger})L)R = \frac{\tau' R \tau}{2ab} U'.$$
(36)

We now have

$$Q := \left( -\frac{1}{2ab}L + \tau(\tau'R\tau)^{-1}(\tau'-a^{2}U'\Delta(L^{\dagger})L'+b^{2}U'\Delta(L^{\dagger})L) \right)R$$
  
=  $-\frac{1}{2ab}LR + \tau(\tau'R\tau)^{-1}(\tau'-a^{2}U'\Delta(L^{\dagger})L'+b^{2}U'\Delta(L^{\dagger})L)R.$ 

By (36), we have

$$Q = -\frac{1}{2ab}(LR - \tau U'). \tag{37}$$

Item (iii) of Lemma 1 says that

$$LR + 2abI_{ns} = \tau U'$$

Substituting back in (37), we get  $Q = I_{ns}$ . The proof is complete.

## 3.3 Special cases

(i) Suppose all the weights in  $\mathcal{G}$  are equal to 1. Choose a = b = 1. We shall denote  $R_{ij}$  by  $r_{ij}$  and define  $R := [r_{ij}]$ . Now by (13),

$$\tau_i = 2 - \sum_{\{j: (i,j) \in \mathcal{E}\}} r_{ji}.$$

We note that  $r_{ii} = 0$  and U = 1. Hence, by our formula in Theorem 1,

$$R^{-1} = -\frac{1}{2}L + \tau(\tau'R\tau)^{-1}(\tau' - \mathbf{1}'\Delta(L^{\dagger})L' + \mathbf{1}'\Delta(L^{\dagger})L)$$
$$= -\frac{1}{2}L + \frac{\tau}{\tau'R\tau}(\tau' - \mathbf{1}'\Delta(L^{\dagger})(L - L')).$$

Thus we get (6).

(ii) Suppose T is a tree with  $V(T) = \{1, \ldots, n\}$ . To denote an edge in T, we shall use the notation ij. Let the weight of an edge ij be  $W_{ij}$ . Assume that all weights are positive definite matrices of order s. Now, define a directed graph  $\widetilde{T}$  as follows. Let  $V(\widetilde{T}) = \{1, \ldots, n\}$ . We use the notation (i, j) to denoted a directed edge from i to j. Now, we define  $E(\widetilde{T}) := \{(i, j), (j, i) : ij \in E(T)\}$ . Now we assign the weight  $W_{pq}$  to an edge (p,q) in  $\widetilde{T}$ . It is clear that  $\widetilde{T}$  is strongly connected, weighted and balanced digraph. Now define the Laplacian matrix of T, say, L(T) as given in 1.4 and the Laplacian of  $\widetilde{T}$ , say,  $L(\widetilde{T})$  as given in item (i) of 1.6. We note that  $L(T) = L(\widetilde{T})$ . Fix a = b = 1. Let  $R_{ij}$  be the resistance between i and j. Then,

$$R_{ij} = M_{ii} + M_{jj} - 2M_{ij},$$

where  $M_{ij}$  is the  $(i, j)^{\text{th}}$  block of the Moore Penrose inverse of  $L(\widetilde{T})$ . If  $D_{ij}$  is the shortest distance between *i* and *j* in *T*, then by the argument mentioned in 1.4,  $D_{ij} = R_{ij}$ . Define  $D := [D_{ij}]$ . Because  $D_{ii} = O_s$ , by Theorem 1, we have

$$\tau_i := 2I_s - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} R_{ji}$$
$$= 2I_s - \sum_{\{j:(i,j)\in E\}} W_{ij}^{-1} W_{ij}$$
$$= (2 - \delta_i) I_s,$$

where  $\delta_i$  is the out-degree of the vertex *i*. We now define

$$\tau := (2 - \delta_1, \dots, 2 - \delta_n)' \otimes I_s.$$

By an induction argument it follows that if S is the sum of all the weights in T, then

$$D\tau = [S, \ldots, S]'.$$

(See Lemma 1 in [8]). Because T is a tree,  $\sum_{i=1}^{n} \delta_i = 2(n-1)$ . Thus,  $\sum_{i=1}^{n} \tau_i = 2I_s$  and hence  $\tau' D \tau = 2S$ . By our formula in Theorem 1,

$$D^{-1} = -\frac{1}{2}L(T) + \tau(\tau' D\tau)^{-1}\tau'$$
  
=  $-\frac{1}{2}L(T) + \frac{1}{2}\tau S\tau'.$  (38)

This is formula (5).

In a similar manner, we get (2), (3), and (4).

## **3.4** Non-negativity of the resistance

From numerical computations, we observe that  $R_{ij} = a^2 l_{ii}^{\dagger} + b^2 l_{jj}^{\dagger} - 2ab l_{ij}^{\dagger}$  is always positive semidefinite. But at this stage, we do not know how to prove this. However, when all the weights in  $\mathcal{G}$  are positive scalars, we now show that the resistance is always non-negative. We need the following lemma.

**Lemma 2.** Let  $A = [a_{ij}]$  be an  $n \times n$  matrix with the following properties.

- (i) All the off-diagonal entries are non-positive.
- (ii)  $A\mathbf{1} = A'\mathbf{1} = 0.$
- (iii) rank(A + A') = n 1.
- (iv) A is positive semidefinite.

Let  $A^{\dagger} := [p_{ij}]$  be the Moore-Penrose inverse of A. Then,

$$p_{ii} \ge p_{ij}$$
 and  $p_{ii} \ge p_{ji}$   $\forall j$ .

*Proof.* By a permutation similarity argument, without loss of generality, we may assume that i = 1 and j = n. We now show that  $p_{11} \ge p_{1n}$  and  $p_{11} \ge p_{n1}$ . By symmetry of our assumptions, it suffices to show that  $p_{11} \ge p_{1n}$ .

Let  $\mathbf{1}_{n-1}$  be the vector of all ones in  $\mathbb{R}^{n-1}$ . By (ii) we can partition A as follows:

$$A = \begin{bmatrix} B & -B\mathbf{1}_{n-1} \\ -\mathbf{1}'_{n-1}B & \mathbf{1}'_{n-1}B\mathbf{1}_{n-1} \end{bmatrix}.$$

All the row sums of A + A' are equal to zero. So, all the cofactors of A + A' are equal. As rank(A + A') = n - 1, we now deduce that the common cofactor of A + A' is non-zero. In particular,  $det(B + B') \neq 0$ . Since A is positive semidefinite, B + B' is positive semidefinite. Because B + B' is non-singular, B + B' is positive definite. So, B is positive definite. All the off-diagonal entries of B are non-positive. By a well-known theorem on **Z**-matrices, B is non-singular and all entries of  $B^{-1}$  are non-negative. By a direct verification,

$$A^{\dagger} = \begin{bmatrix} B^{-1} - \frac{1}{n} \mathbf{1}_{n-1} \mathbf{1}_{n-1}' B^{-1} - \frac{1}{n} B^{-1} \mathbf{1}_{n-1} \mathbf{1}_{n-1}' & -\frac{1}{n} B^{-1} \mathbf{1}_{n-1} \\ -\frac{1}{n} \mathbf{1}_{n-1}' B^{-1} & 0 \end{bmatrix} + \frac{\mathbf{1}_{n-1}' B^{-1} \mathbf{1}_{n-1}}{n^2} (\mathbf{11}').$$
(39)

Put

$$C = [c_{ij}] := B^{-1}$$
 and  $\delta := \frac{1}{n^2} \mathbf{1}'_{n-1} B^{-1} \mathbf{1}_{n-1}$ 

Then,  $c_{ij} \geq 0 \quad \forall i, j \text{ and}$ 

$$p_{11} = c_{11} - \frac{1}{n} \sum_{j=1}^{n-1} c_{j1} - \frac{1}{n} \sum_{j=1}^{n-1} c_{1j} + \delta,$$
$$p_{1n} = -\frac{1}{n} \sum_{j=1}^{n-1} c_{1j} + \delta.$$

Now,

$$p_{11} - p_{1n} = c_{11} - \frac{1}{n} \sum_{i=1}^{n-1} c_{i1}.$$

By Theorem 2,

$$c_{11} \ge c_{j1} \quad \forall j = 1, \dots, n-1.$$

So,

$$-\frac{1}{n}\sum_{i=1}^{n-1}c_{i1} \ge -(\frac{n-1}{n})c_{11}.$$

Hence,

$$c_{11} - \frac{1}{n} \sum_{i=1}^{n-1} c_{i1} \ge c_{11} - \frac{n-1}{n} c_{11} = \frac{1}{n} c_{11}.$$

Since  $c_{11} \ge 0$ , we conclude that

$$p_{11} - p_{1n} \ge 0.$$

The proof is complete.

Now it can be easily shown that any generalized resistance is non-negative.

**Theorem 4.** Suppose all the weights in  $\mathcal{G}$  are positive scalars. Let a, b > 0. Let  $L^{\dagger} = [k_{ij}]$  be the Moore-Penrose inverse of the Laplacian of  $\mathcal{G}$ . Then,

$$r_{ij} := a^2 k_{ii} + b^2 k_{jj} - 2abk_{ij} \ge 0.$$

*Proof.* We note that the Laplacian matrix L of  $\mathcal{G}$  satisfies all the conditions of the previous lemma. Moreover, by Proposition 3,  $k_{ii}$  and  $k_{jj}$  are positive. As a consequence of Lemma 2, we deduce that

$$\min(k_{ii}, k_{jj}) \ge \max(k_{ij}, k_{ji}).$$

Now

$$a^2k_{ii} + b^2k_{jj} - 2abk_{ij} \ge 0,$$

follows from the arithmetic mean and geometric mean inequality.

## 3.5 A perturbation result

We now show that if R is the resistance matrix of a connected graph with n vertices, and if L is the Laplacian matrix of  $\mathcal{G}$  with positive scalar weights, then  $(R^{-1} - L)^{-1}$ has all entries non-negative.

**Theorem 5.** Let H be a simple (undirected) connected graph with n vertices and R be the resistance matrix of H. Assume that all the weights in  $\mathcal{G}$  are positive scalars. Then,  $R^{-1} - L$  is non-singular and every entry in  $(R^{-1} - L)^{-1}$  is non-negative.

*Proof.* Let  $M = [m_{ij}]$  be the Moore-Penrose inverse of the Laplacian matrix of H. Then the  $(i, j)^{\text{th}}$  entry  $r_{ij}$  of R is given by

$$r_{ij} = m_{ii} + m_{jj} - 2m_{ij}.$$

Fix  $\alpha \geq 0$ . Define  $S := \alpha L$ . We complete the proof by using the following claims.

**Claim 1:**  $R^{-1} - S$  is non-singular. To prove this claim, we can assume that  $S = \alpha L$ , where  $\alpha > 0$ . By Proposition 2,  $\operatorname{rank}(S) = n - 1$ , S + S' is positive semidefinite and  $S'\mathbf{1} = S\mathbf{1} = 0$ . Let  $x \in \mathbb{R}^n$  be such that

$$(R^{-1} - S)(x) = 0. (40)$$

Put  $u := R^{-1}x$ . Assuming  $u \neq 0$ , we now get a contradiction. As  $\mathbf{1}'S = 0$ , it follows that  $\mathbf{1}'R^{-1}x = 0$  and hence  $u \in \mathbf{1}^{\perp}$ . Writing

$$R = \operatorname{Diag}(M)\mathbf{11}' + \mathbf{11}'\operatorname{Diag}(M) - 2M,$$

we get

$$u'Ru = u'(\operatorname{Diag}(M)\mathbf{11}' + \mathbf{11}'\operatorname{Diag}(M) - 2M)u$$
$$= -2u'Mu.$$

Since null(M) = span{1}, M is positive definite on  $1^{\perp}$ . So, u'Mu > 0. Hence,

$$u'Ru < 0. \tag{41}$$

It is easy to see that

 $u'Ru = x'R^{-1}x\tag{42}$ 

By (40),

$$x'R^{-1}x = x'Sx$$

Since S + S' is positive semidefinite,  $x'Sx \ge 0$ . So,  $x'R^{-1}x \ge 0$ . Hence by (42),

$$u'Ru \ge 0. \tag{43}$$

Thus, we get a contradiction from (41) and (43). Therefore,  $u = R^{-1}x = 0$ . This implies x = 0. So,  $R^{-1} - S$  is non-singular. The claim is proved.

**Claim 2:** If C is a  $k \times k$  proper principal submatrix of  $S - R^{-1}$ , then

$$q'Cq > 0$$
 for all  $0 \neq q \in \mathbb{R}^k$ .

If A is an  $n \times n$  matrix, we shall use the notation A[i] to denote the principal submatrix of A obtained by deleting the  $i^{\text{th}}$  row and  $i^{\text{th}}$  column of A. Fix  $1 \le i \le n$  and define

$$B := S[i] - R^{-1}[i].$$

Since R is negative definite on  $\mathbf{1}^{\perp}$  and the diagonal entries are zero, R has exactly one positive eigenvalue. By an application of interlacing theorem, we see that  $-R^{-1}[i]$  is positive semidefinite. Hence

$$-p'R^{-1}[i]p \ge 0 \quad \text{for all } w \in \mathbb{R}^{n-1}.$$
(44)

As the row sums and the column sums of S are equal to zero, it follows that all the cofactors of S are equal. Because S + S' is positive semidefinite and has rank n - 1, it follows that every proper principal submatrix of S + S' is positive definite. So, we have

$$p'S[i]p > 0 \text{ for all } 0 \neq p \in \mathbb{R}^{n-1}.$$
 (45)

By (44) and (45),

p'Bp > 0 for all  $0 \neq p \in \mathbb{R}^{n-1}$ .

The claim is proved.

In particular, we note that all principal minors of  $S - R^{-1}$  with order less than n are positive.

Claim 3:  $det(S - R^{-1}) < 0$ .

Because  $\gamma L - R^{-1}$  is non-singular for every  $\gamma \ge 0$ ,

$$\operatorname{sgn} \det(\gamma L - R^{-1}) = \operatorname{sgn} \det(-R^{-1}).$$

Since -R has exactly one negative eigenvalue,  $det(-R^{-1}) < 0$ . So,

$$\det(\gamma L - R^{-1}) < 0 \quad \forall \gamma \ge 0.$$

This proves the claim.

**Claim 4:** All principal minors of  $(S - R^{-1})^{-1}$  are negative. Put

$$G := (S - R^{-1})^{-1}$$
 and  $H := S - R^{-1}$ .

Let s < n, and let  $\widehat{G}$  be a  $s \times s$  principal submatrix of G. Suppose  $\widehat{H}$  is the complementary submatrix of  $\widehat{G}$  in H. By Jacobi identity,

$$\det(\widehat{G}) = \frac{\det(H)}{\det(H)}$$

By claim 2 and 3,

 $\operatorname{sgn}(\operatorname{det}\widehat{H}) > 0$  and  $\operatorname{sgn}(\operatorname{det}(H)) < 0$ .

So,  $\det(\widehat{G}) < 0$ . The claim is proved.

We now complete the proof of the theorem. Given  $\beta \geq 0$ , let

$$\left[\begin{array}{cc} y_{ii}(\beta) & y_{ij}(\beta) \\ y_{ji}(\beta) & y_{jj}(\beta) \end{array}\right]$$

denote a  $2 \times 2$  principal submatrix of  $(\beta L - R^{-1})^{-1}$ . By claim 4,

 $y_{ii}(\beta) < 0 \text{ and } y_{jj}(\beta) < 0 \text{ for all } \beta \ge 0.$  (46)

We now show that  $y_{ij}(\beta) < 0$  for all  $\beta \ge 0$ . Since  $y_{ij}(0) = -r_{ij}$ ,  $y_{ij}(0) < 0$ . If  $y_{ij}(\alpha) > 0$  for some  $\alpha > 0$ , then by continuity,  $y_{ij}(\delta) = 0$  for some  $\delta > 0$ . Hence by (46),

$$\det\left(\left[\begin{array}{cc}y_{ii}(\delta) & y_{ij}(\delta)\\y_{ji}(\delta) & y_{jj}(\delta)\end{array}\right]\right) = y_{ii}(\delta)y_{jj}(\delta) > 0.$$

However by claim 4,

$$\det(\left[\begin{array}{cc} y_{ii}(\delta) & y_{ij}(\delta) \\ y_{ji}(\delta) & y_{jj}(\delta) \end{array}\right]) < 0.$$

Thus, we have a contradiction. So,  $y_{ij}(\alpha) \leq 0$  for all  $\alpha \geq 0$ . We now conclude that every entry in  $(L - R^{-1})^{-1}$  is negative. The proof is complete.

We illustrate the above result with an example.

**Example 2.** Consider the graphs H and  $\mathcal{G}$  given in Figure 2. Let the positive scalar

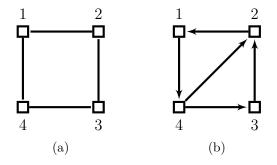


Figure 2: (a) Graph H, and (b) Graph  $\mathcal{G}$ .

weights  $w_{ij}$  assigned to each edge (i, j) of  $\mathcal{G}$  be

$$w_{14} = w_{21} = \frac{10}{7}, \ w_{32} = w_{43} = 5 \text{ and } w_{42} = 2.$$

The Laplacian matrix of  $\mathcal{G}$  is

$$L = \frac{1}{10} \begin{bmatrix} 7 & 0 & 0 & -7 \\ -7 & 7 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -5 & -2 & 7 \end{bmatrix}.$$

The resistance matrix R of H is

$$R = \frac{1}{4} \begin{bmatrix} 0 & 3 & 4 & 3 \\ 3 & 0 & 3 & 4 \\ 4 & 3 & 0 & 3 \\ 3 & 4 & 3 & 0 \end{bmatrix}.$$

Now,

$$(R^{-1} - L)^{-1} = \frac{1}{8612} \begin{bmatrix} 2335 & 6555 & 7515 & 5125\\ 5125 & 2585 & 6905 & 6915\\ 7515 & 5975 & 1135 & 6905\\ 6555 & 6415 & 5975 & 2585 \end{bmatrix}$$

which is a non-negative matrix.

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