# On resistance matrices of weighted balanced digraphs 

R. Balaji, R.B. Bapat and Shivani Goel

November 4, 2021


#### Abstract

Let $G$ be a connected graph with $V(G)=\{1, \ldots, n\}$. Then the resistance distance between any two vertices $i$ and $j$ is given by $r_{i j}:=l_{i i}^{\dagger}+l_{j j}^{\dagger}-2 l_{i j}^{\dagger}$, where $l_{i j}^{\dagger}$ is the $(i, j)^{\text {th }}$ entry of the Moore-Penrose inverse of the Laplacian matrix of $G$. For the resistance matrix $R:=\left[r_{i j}\right]$, there is an elegant formula to compute the inverse of $R$. This says that $$
R^{-1}=-\frac{1}{2} L+\frac{1}{\tau^{\prime} R \tau} \tau \tau^{\prime},
$$ where $$
\tau:=\left(\tau_{1}, \ldots, \tau_{n}\right)^{\prime} \text { and } \tau_{i}:=2-\sum_{\{j \in V(G):(i, j) \in E(G)\}} r_{i j} \quad i=1, \ldots, n .
$$

A far reaching generalization of this result that gives an inverse formula for a generalized resistance matrix of a strongly connected and matrix weighted balanced directed graph is obtained in this paper. When the weights are scalars, it is shown that the generalized resistance is a non-negative real number. We also obtain a perturbation result involving resistance matrices of connected graphs and Laplacians of digraphs.


Keywords. Balanced digraphs, Laplacian matrices, resistance matrices, row diagonally dominant matrices, Jacobi identity.
AMS CLASSIFICATION. 05C50

## 1 Introduction

Let $G$ be a simple connected graph. Suppose $x$ and $y$ are any two vertices of $G$. The length of the shortest path connecting $x$ and $y$ in $G$ is the natural way to define the distance between $x$ and $y$. This classical distance has certain limitations. For instance, consider two graphs $G_{1}$ and $G_{2}$ such that
(i) $V\left(G_{1}\right)=V\left(G_{2}\right)=\{1, \ldots, n\}$.
(ii) $i$ and $j$ are adjacent in both $G_{1}$ and $G_{2}$.
(iii) There is only path between $i$ and $j$ in $G_{1}$ and there are multiple paths connecting $i$ and $j$ in $G_{2}$.

Then the shortest distance between $i$ and $j$ in both $G_{1}$ and $G_{2}$ is one. However, since there are multiple paths connecting $i$ and $j$ in $G_{2}$, the communication between $i$ and $j$ in $G_{2}$ is better than in $G_{1}$. This significance is not reflected in the shortest distance. Several applications require to overcome this limitation. Instead of the classical distance, the so-called resistance distance is used widely in many situations like in electrical networks, chemistry and random walks: see for example [1] and [2]. If there are multiple paths between two vertices, then the resistance distance is less than the shortest distance. The resistance matrix is now the matrix with $(i, j)^{\text {th }}$ entry equal to the resistance distance between $i$ and $j$. Resistance matrices are non-singular and the inverse is given by an elegant formula that can be computed directly from the graph. The main purpose of this paper is to deduce a formula for the inverse of a generalized resistance matrix of a simple digraph with some special properties. This new formula generalizes the following known results.

### 1.1 Inverse of the resistance matrix of a connected graph

Let $G$ be a connected graph with $V(G)=\{1, \ldots, n\}$. Let $\delta_{i}$ denote the degree of the vertex $i$ and $A$ be the adjacency matrix of $G$. Then the Laplacian matrix of $G$ is $L=\operatorname{Diag}\left(\delta_{1}, \ldots, \delta_{n}\right)-A$. Now the resistance between any two vertices $i$ and $j$ in $G$ is

$$
\begin{equation*}
r_{i j}:=l_{i i}^{\dagger}+l_{j j}^{\dagger}-2 l_{i j}^{\dagger} \tag{1}
\end{equation*}
$$

where $l_{i j}^{\dagger}$ is the $(i, j)^{\text {th }}$ entry of the Moore Penrose inverse of $L$. Define $R:=\left[r_{i j}\right]$. Then the inverse of $R$ is given by

$$
\begin{equation*}
R^{-1}=-\frac{1}{2} L+\frac{1}{\tau^{\prime} R \tau} \tau \tau^{\prime} \tag{2}
\end{equation*}
$$

where

$$
\tau:=\left(\tau_{1}, \ldots, \tau_{n}\right)^{\prime} \text { and } \tau_{i}:=2-\sum_{\{j \in V(G):(i, j) \in E\}} r_{i j} \quad i=1, \ldots, n .
$$

The proof of (2) is given in Theorem 9.1.2 in [1].

### 1.2 Inverse of the distance matrix of a tree

Let $T$ be a tree with $V(T)=\{1, \ldots, n\}$ and $r_{i j}$ (defined in (1)) be the resistance distance between any two vertices $i$ and $j$. If $d_{i j}$ is the length of the shortest path connecting $i$ and $j$ in $T$, then by an induction argument, it can be shown that $d_{i j}=r_{i j}$. Define $D:=\left[d_{i j}\right]$. Specializing formula (2) to $T$ gives

$$
\begin{equation*}
D^{-1}=-\frac{1}{2} L+\frac{\left(2-\delta_{1}, \ldots, 2-\delta_{n}\right)^{\prime}\left(2-\delta_{1}, \ldots, 2-\delta_{n}\right)}{2(n-1)} \tag{3}
\end{equation*}
$$

where $\delta_{i}$ is the degree of the vertex $i$ and $L$ is the Laplacian matrix of $T$. This formula is obtained by Graham and Lovász in [3].

### 1.3 Inverse of the distance matrix of a weighted tree

Formula (3) can be generalized to weighted trees. We first need to define the Laplacian matrix of a weighted tree. Consider a tree $G=(V, \Omega)$ with $V=\{1, \ldots, n\}$. To an edge $(i, j) \in \Omega$, we assign a positive real number $w_{i j}$. Define

$$
l_{i j}:=\left\{\begin{array}{rl}
-\frac{1}{w_{i j}} & (i, j) \in \Omega \\
0 & i \neq j \text { and }(i, j) \notin \Omega \\
\sum_{\{k:(i, k) \in \Omega\}} \frac{1}{w_{i k}} & i=j .
\end{array}\right.
$$

Then the Laplacian matrix of $G$ is $L:=\left[l_{i j}\right]$. The distance matrix of $G$ is the symmetric matrix $D$ with $(i, j)^{\text {th }}$ entry equal to sum of all the weights that lie in the path connecting $i$ and $j$. In this case, by an induction argument, it can be shown that $L D L+2 L=0$ and from this identity it is easy to show that $d_{i j}=l_{i i}^{\dagger}+l_{j j}^{\dagger}-2 l_{i j}^{\dagger}$, where $l_{i j}^{\dagger}$ is the $(i, j)^{\mathrm{th}}$ entry of the Moore Penrose inverse of $L$. Let $\delta_{i}$ be the degree of the vertex $i$. In this setting, the following inverse formula is obtained in [4]:

$$
\begin{equation*}
D^{-1}=-\frac{1}{2}\left(L+\frac{\tau \tau^{\prime}}{\sum_{i, j} w_{i j}}\right) \tag{4}
\end{equation*}
$$

where $\tau$ is the vector $\left(2-\delta_{1}, \ldots, 2-\delta_{n}\right)^{\prime}$.

### 1.4 Inverse of the distance matrix of a tree with matrix weights

Formula (4) can be generalized. Consider a tree on $n$ vertices with vertex set $V(T)=$ $\{1, \ldots, n\}$ and edge set $E(T)$. To an edge $(i, j)$ in $T$, assign a positive definite matrix $W_{i j}$ of some fixed order $s$. Define

$$
L_{i j}:=\left\{\begin{array}{rl}
-W_{i j}^{-1} & (i, j) \in E(T) \\
O_{s} & i \neq j \text { and }(i, j) \notin E(T) \\
\sum_{\{k:(i, k) \in E(T)\}} W_{i k}^{-1} & i=j .
\end{array}\right.
$$

(Here $O_{s}$ is the $s \times s$ matrix with all entries equal to zero.) The Laplacian matrix $L$ of $T$ is then the $n s \times n s$ matrix with $(i, j)^{\text {th }}$ block equal to $L_{i j}$. The distance between any two vertices $i$ and $j$ in $T$ is the sum of all positive definite matrices that lie in the path connecting $i$ and $j$. Let the $(i, j)^{\text {th }}$ block of the Moore-Penrose inverse of $L$ be given by $M_{i j}$. Then, by induction it can be shown that

$$
D_{i j}=M_{i i}+M_{j j}-2 M_{i j} .
$$

The inverse of $D$ is

$$
\begin{equation*}
D^{-1}=-\frac{1}{2}\left(L+F S^{-1} F^{\prime}\right), \tag{5}
\end{equation*}
$$

where

$$
S=\sum_{i} \sum_{j} W_{i j} \text { and } F=\left(2-\delta_{1}, \ldots, 2-\delta_{n}\right)^{\prime} \otimes I_{s} .
$$

Formula (5) is obtained in [5].

### 1.5 Inverse of the resistance matrix of a directed graph

Let $G=(V, \mathcal{E})$ be a digraph with vertex set $V=\{1, \ldots, n\}$. A directed edge from a vertex $i$ to a vertex $j$ in $G$ will be denoted by $(i, j)$. Recall that $G$ is said to be strongly connected if there is a directed path between any two vertices $i$ and $j$. A vertex $i$ is said to be balanced if the outdegree of $i$ and the indegree of $i$ are equal. If all the vertices are balanced, then we say that $G$ is balanced. We now assume that $G$ is a strongly connected and balanced digraph. If $i$ and $j$ are any two vertices, define

$$
a_{i j}:= \begin{cases}1 & (i, j) \in \mathcal{E} \\ 0 & \text { otherwise }\end{cases}
$$

The Laplacian matrix of $G$ is the matrix $L=\left[l_{i j}\right]$ such that

$$
l_{i j}:=\left\{\begin{array}{rl}
-a_{i j} & i \neq j \\
\sum_{\{k: k \neq i\}} a_{i k} & i=j .
\end{array}\right.
$$

Now, let $L^{\dagger}:=\left[l_{i j}^{\dagger}\right]$ be the Moore-Penrose inverse of $L$. Then the resistance between any two vertices $i$ and $j$ in $G$ is given by

$$
r_{i j}:=l_{i i}^{\dagger}+l_{j j}^{\dagger}-2 l_{i j}^{\dagger} .
$$

For the resistance matrix $R=\left[r_{i j}\right]$ of $G$, we have the following inverse formula from [6]:

$$
\begin{equation*}
R^{-1}=-\frac{1}{2} L+\frac{1}{\tau^{\prime} R \tau}\left(\tau\left(\tau^{\prime}+\mathbf{1}^{\prime} \operatorname{diag}\left(L^{\dagger}\right) M\right)\right) \tag{6}
\end{equation*}
$$

where

$$
M:=L-L^{\prime}, \quad \tau_{i}:=2-\sum_{\{j:(i, j) \in \mathcal{E}\}} r_{j i},
$$

and $\mathbf{1}$ is the vector of all ones in $\mathbb{R}^{n}$.
We now ask if there is a formula that unifies (2), (3), (4), (5) and (6).

### 1.6 Results obtained

We consider a simple digraph $\mathcal{G}=(V, \mathcal{E})$ with $V=\{1, \ldots, n\}$. An element in $\mathcal{E}$ will be denoted by $(i, j)$. Precisely, $(i, j) \in \mathcal{E}$ means that there is a directed edge from a vertex $i$ to a vertex $j$ in $\mathcal{G}$. All edges are assigned a positive definite matrix of some fixed order $s$. These positive definite matrices will be called weights. Let $W_{i j}$ be the weight of the edge $(i, j)$. In this set up, we define the following.
(i) Laplacian of $\mathcal{G}$ : If $i$ and $j$ are any two distinct vertices in $\mathcal{G}$, define

$$
L_{i j}:=\left\{\begin{aligned}
-W_{i j}^{-1} & (i, j) \in \mathcal{E} \\
O_{s} & \text { otherwise }
\end{aligned}\right.
$$

(Here, $O_{s}$ is the $s \times s$ matrix with all entries equal to zero.) The Laplacian of $\mathcal{G}$ is then the $n s \times n s$ matrix

$$
L(\mathcal{G}):=\left[\begin{array}{rrrrr}
-\sum_{\{j: j \neq 1\}} L_{1 j} & & L_{12} & \ldots & \\
L_{21} & -\sum_{\{j: j \neq 2\}} L_{2 j} & \cdots & & L_{1 n} \\
\vdots & \vdots & \vdots & & L_{2 n} \\
L_{n 1} & & L_{n 2} & \ldots & -\sum_{\{j: j \neq n\}} L_{n j}
\end{array}\right]
$$

We shall say that $L_{i j}$ is the $(i, j)^{\text {th }}$ block of $L(\mathcal{G})$. We note that since $\mathcal{G}$ is a digraph, $L(\mathcal{G})$ is not symmetric in general.
(ii) Resistance matrix of $\mathcal{G}$ : Let $i$ and $j$ be any two vertices in $\mathcal{G}$. Fix $a, b>0$. Then, a generalized resistance distance between $i$ and $j$ is the $s \times s$ matrix

$$
R_{i j}=a^{2} K_{i i}+b^{2} K_{j j}-2 a b K_{i j},
$$

where $K_{i j}$ is the $(i, j)^{\text {th }}$ block of the Moore Penrose inverse of $L$. The resistance matrix corresponding to $a$ and $b$ is then the $n s \times n s$ matrix with $(i, j)^{\text {th }}$ block equal to $R_{i j}$.
(iii) Balanced vertices: We say that a vertex $j \in V$ is balanced if

$$
\sum_{\{i \in V:(i, j) \in \mathcal{E}\}} W_{i j}^{-1}=\sum_{\{i \in V:\{(j, i) \in \mathcal{E}\}} W_{j i}^{-1} .
$$

(iv) Balanced digraphs: If every vertex of $\mathcal{G}$ is balanced, then we shall say that $\mathcal{G}$ is balanced.

We obtain the following result in this paper.
Theorem 1. Let $\mathcal{G}=(V, \mathcal{E})$ be a weighted, balanced and strongly connected digraph, where $V=\{1, \ldots, n\}$. Let $W_{i j}$ be the weight of the edge $(i, j)$. Fix $a, b>0$. If $R$ is the resistance matrix of $\mathcal{G}, L$ is the Laplacian of $\mathcal{G}$, and $L^{\dagger}$ is the Moore-Penrose inverse of $\mathcal{G}$, then

$$
R^{-1}=-\frac{1}{2 a b} L+\tau\left(\tau^{\prime} R \tau\right)^{-1}\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right)
$$

where $\tau=\left(\tau_{1}, \ldots, \tau_{n}\right)^{\prime}$ is given by

$$
\tau_{i}:=2 a b I_{s}+L_{i i} R_{i i}-\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} R_{j i},
$$

$U=[\underbrace{I_{s}, \ldots, I_{s}}_{n}]$ and $\Delta\left(L^{\dagger}\right)=\operatorname{Diag}\left(K_{11}, \ldots, K_{n n}\right)$, with $K_{i i}$ being the $i^{\text {th }}$ diagonal block of $L^{\dagger}$.

To illustrate, we give an example.
Example 1. Consider the following graph $\mathcal{G}$ on four vertices. Let


Figure 1: Directed Graph $\mathcal{G}$.

$$
W_{14}=W_{21}=\frac{1}{5}\left[\begin{array}{rr}
3 & -4 \\
-4 & 7
\end{array}\right], \quad W_{43}=W_{32}=\left[\begin{array}{rr}
1 & -1 \\
-1 & 2
\end{array}\right],
$$

and

$$
W_{42}=\left[\begin{array}{rr}
2 & -3 \\
-3 & 5
\end{array}\right] .
$$

The graph $\mathcal{G}$ is balanced with the above matrix weights. The Laplacian of $\mathcal{G}$ is

$$
L=\left[L_{i j}\right]=\left[\begin{array}{rr|rr|rr|rr}
7 & 4 & 0 & 0 & 0 & 0 & -7 & -4 \\
4 & 3 & 0 & 0 & 0 & 0 & -4 & -3 \\
\hline-7 & -4 & 7 & 4 & 0 & 0 & 0 & 0 \\
-4 & -3 & 4 & 3 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & -2 & -1 & 2 & 1 & 0 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
\hline 0 & 0 & -5 & -3 & -2 & -1 & 7 & 4 \\
0 & 0 & -3 & -2 & -1 & -1 & 4 & 3
\end{array}\right] .
$$

The Moore-Penrose inverse of $L$ is the matrix

$$
L^{\dagger}=\left[K_{i j}\right]=\frac{1}{80}\left[\begin{array}{rr|rr|rr|rr}
20 & -25 & -16 & 23 & -12 & 11 & 8 & -9 \\
-25 & 45 & 23 & -39 & 11 & -23 & -9 & 17 \\
\hline 8 & -9 & 20 & -25 & -24 & 27 & -4 & 7 \\
-9 & 17 & -25 & 45 & 27 & -51 & 7 & -11 \\
\hline-12 & 11 & 0 & -5 & 36 & -33 & -24 & 27 \\
11 & -23 & -5 & 5 & -33 & 69 & 27 & -51 \\
\hline-16 & 23 & -4 & 7 & 0 & -5 & 20 & -25 \\
23 & -39 & 7 & -11 & -5 & 5 & -25 & 45
\end{array}\right] .
$$

Suppose $a=2$ and $b=3$. The resistance matrix of $\mathcal{G}$ is

$$
\begin{aligned}
R & =\left[4 K_{i i}+9 K_{j j}-12 K_{i j}\right] \\
& =\frac{1}{80}\left[\begin{array}{rrr|rr|rr}
20 & -25 & 452 & -601 & 548 & -529 & 164 \\
\hline-25 & 45 & -601 & 1053 & -529 & 1077 & -217 \\
-2881 \\
\hline 164 & -217 & 20 & -25 & 692 & -721 & 308 \\
\hline-409 \\
-217 & 381 & -25 & 45 & -721 & 1413 & -409 \\
\hline 468 & -489 & 324 & -297 & 36 & -33 & 612 \\
\hline-689 & 957 & -297 & 621 & -33 & 69 & -681 \\
\hline 452 & -601 & 308 & -409 & 404 & -337 & 20 \\
\hline-601 & 1053 & -409 & 717 & -337 & 741 & -25 \\
\hline 45
\end{array}\right] .
\end{aligned}
$$

Next, we compute $\tau=\left(\tau_{1}, \tau_{2}, \tau_{3}, \tau_{4}\right)^{\prime}$. Recall that

$$
\tau_{i}:=2 a b I_{s}+L_{i i} R_{i i}-\sum_{\{j:(i, j) \in E\}} W_{i j}^{-1} R_{j i} .
$$

Thus,

$$
\begin{gathered}
\tau=\frac{1}{5}\left[\begin{array}{rr|rr|rr|rr}
15 & 0 & 15 & 0 & 21 & 2 & 9 & -2 \\
0 & 15 & 0 & 15 & 2 & 19 & -2 & 11
\end{array}\right]^{\prime}, \\
\tau^{\prime} R \tau=\frac{1}{5}\left[\begin{array}{rr}
2916 & -3159 \\
-3159 & 6075
\end{array}\right],
\end{gathered}
$$

and

$$
\tau^{\prime}-4 U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+9 U^{\prime} \Delta\left(L^{\dagger}\right) L=\frac{1}{10}\left[\begin{array}{rr|rr|rr|rr}
30 & 0 & 3 & -9 & 57 & 9 & 30 & 0 \\
0 & 30 & -9 & 12 & 9 & 48 & 0 & 30
\end{array}\right]
$$

Hence, we have

$$
\begin{aligned}
& -\frac{1}{2 a b} L+\tau\left(\tau^{\prime} R \tau\right)^{-1}\left(\tau^{\prime}-4 U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+9 U^{\prime} \Delta\left(L^{\dagger}\right) L\right) \\
& =\frac{1}{42444}\left[\begin{array}{rr|rr|rr|rr}
-23259 & -13368 & -84 & -138 & 3084 & 1698 & 26259 & 14928 \\
-13368 & -9891 & -138 & 54 & 1698 & 1386 & 14928 & 11331 \\
\hline 26259 & 14928 & -24843 & -14286 & 3084 & 1698 & 1500 & 780 \\
14928 & 11331 & -14286 & -10557 & 1698 & 1386 & 780 & 720 \\
\hline 2204 & 1188 & 6938 & 3351 & -2530 & -975 & 2204 & 1188 \\
1188 & 1016 & 3351 & 3587 & -975 & -1555 & 1188 & 1016 \\
\hline 796 & 372 & 17653 & 10521 & 8698 & 4371 & -23963 & -13776 \\
372 & 424 & 10521 & 7132 & 4371 & 4327 & -13776 & -10187
\end{array}\right]
\end{aligned}
$$

which is equal to $R^{-1}$.

### 1.7 Other results

We obtain the following two results after proving Theorem 1.

- By numerical computations, we observe that for any $a, b>0$,

$$
R_{i j}=a^{2} K_{i i}+b^{2} K_{j j}-2 a b K_{i j}
$$

is positive semidefinite. We do not know how to prove this result in general. However, when the weights in $\mathcal{G}$ of Theorem 1 are positive scalars, we show that $R_{i j}$ is always a non-negative real number.

- Let $T$ be a tree on $n$ vertices. Suppose $D$ and $L$ are the distance and Laplacian matrices of $T$. Then, from (3) it can be deduced that

$$
\left(D^{-1}-L\right)^{-1}=\frac{1}{3} D+\frac{1}{3}\left(\sum_{i, j} w_{i j}\right) \mathbf{1 1}^{\prime} .
$$

In particular, this equation says that every entry in $\left(D^{-1}-L\right)^{-1}$ is non-negative. Suppose $M$ is the Laplacian matrix of an arbitrary tree on $n$ vertices. It can be shown that $D^{-1}-M$ is non-singular. We now say that $\left(D^{-1}-M\right)^{-1}$ is a perturbation of the distance matrix $D$. In [4], it is shown that all perturabations of $D$ are non-negative matrices. We now assume that $R$ is the resistance matrix of a connected graph (defined in Section 1.1) on $n$ vertices. Now consider the Laplacian matrix $L$ of $\mathcal{G}$ in Theorem 1. Suppose all the weights in $\mathcal{G}$ are positive scalars. It can be shown that $R^{-1}-L$ is always non-singular. We now say that $\left(R^{-1}-L\right)^{-1}$ is a perturabation of $R$. By performing certain numerical experiments, we observed that similar to the result in [4], all perturbations of $R$
are non-negative matrices. Since $\mathcal{G}$ is a digraph, $L$ is not symmetric in general and hence all perturbations of $R$ are not symmetric matrices. Despite this difficulty, by using a argument different from [4], we show that all perturbations of $R$ are non-negative. This result is proved in the final part of this paper.

## 2 Preliminaries

We mention the notation and some basic results that will be used in the paper.
(i) We reserve $\mathcal{G}$ to denote a simple, strongly connected, weighted, and balanced digraph with vertex set $V=\{1, \ldots, n\}$. A directed edge from $i$ to $j$ in $\mathcal{G}$ will be denoted by $(i, j)$. We use $\mathcal{E}$ to denote the edge set of $\mathcal{G}$. The weight of an edge $(i, j)$ will be denoted by $W_{i j}$. All weights will be symmetric positive definite matrices and have fixed order $s$.
(ii) Let $B^{n s}$ be the set of all real $n s \times n s$ matrices. A matrix $A$ in $B^{n s}$ will be denoted by $\left[A_{i j}\right]$, where $A_{i j}$ is an $s \times s$ matrix. We shall say that $A_{i j}$ is the $(i, j)^{\text {th }}$ block of $A$. There are $n$ blocks in $A$. The null-space of a matrix $A$ is denoted by $\operatorname{null}(A)$ and the column space by $\operatorname{col}(A)$.
(iii) The vector of all ones in $\mathbb{R}^{n}$ will be denoted by 1 . The matrix $\mathbf{1}^{\prime} \otimes I_{s}$ will be denoted by $U^{\prime}$, i.e.

$$
U:=\left[I_{s}, \ldots, I_{s}\right]^{\prime}
$$

where $I_{s}$ appears $n$ times. We use $J$ to denote the matrix in $B^{n s}$ with all blocks equal to $I_{s}$. Note that $J=U U^{\prime}$.
(iv) The Laplacian matrix of $\mathcal{G}$ will be denoted by $L$ and its Moore-Penrose by $L^{\dagger}$. We note that $L$ and $L^{\dagger}$ belong to $B^{n s}$. The $(i, j)^{\text {th }}$ block of $L^{\dagger}$ will be denoted by $K_{i j}$. We use $\Delta\left(L^{\dagger}\right)$ to denote the block diagonal matrix

$$
\operatorname{Diag}\left(K_{11}, \ldots, K_{n n}\right)
$$

Let $\mathcal{G}^{\prime}$ be the digraph such that $V\left(\mathcal{G}^{\prime}\right):=\{1, \ldots, n\}$ and

$$
E\left(\mathcal{G}^{\prime}\right):=\{(j, i):(i, j) \in E(\mathcal{G})\}
$$

To an edge $(i, j)$ of $\mathcal{G}^{\prime}$, we assign the weight $W_{j i}$. Again $\mathcal{G}^{\prime}$ will be strongly connected, and balanced. The Laplacian of $\mathcal{G}^{\prime}$ is clearly $L^{\prime}$.
(v) Fix $a, b>0$. The generalized resistance matrix of $\mathcal{G}$ corresponding to $a$ and $b$ will be denoted by $R_{a, b}$. Thus, $R_{a, b}$ is an element in $B^{n s}$ with $(i, j)^{\text {th }}$ block equal to

$$
a^{2} K_{i i}+b^{2} K_{j j}-2 a b K_{i j} .
$$

(vi) Let $A$ be an $m \times m$ matrix.
(a) We say that $A$ is positive semidefinite if $x^{\prime} A x \geq 0$ for all $x \in \mathbb{R}^{m}$ and positive definite if $x^{\prime} A x>0$ for all $0 \neq x \in \mathbb{R}^{m}$. (To define positive semidefiniteness, we do not assume that $A$ is symmetric.)
(b) Following [7], we say that $A$ is almost positive definite if for each $x \in \mathbb{R}^{m}$, either $x^{\prime} A x>0$ or $A x=0$. Suppose $A$ is almost positive definite. Then the Moore-Penrose inverse of $A$ is also almost positive definite: see Corollary 2 in [7].
(vii) Let $B=\left[A_{1}, \ldots, A_{n}\right]$, where each $A_{i}$ is an $s \times s$ matrix. As before, we say that $A_{j}$ is the $j^{\text {th }}$ block of $B$. Let $\operatorname{Diag}(B)$ be the $n s \times n s$ block matrix

$$
\operatorname{Diag}\left(A_{1}, \ldots, A_{n}\right)
$$

(vii) We use $[n]$ for $\{1, \ldots, n\}$. The zero matrix of order $s \times s(s \geq 2)$ will be denoted by $O_{s}$. The identity matrix of order $k$ will be denoted by $I_{k}$.
(viii) Let $A=\left[a_{i j}\right]$ be an $m \times m$ matrix. We say that $A$ is row diagonally dominant if

$$
\left|a_{i i}\right| \geq \sum_{\{j: j \neq i\}}\left|a_{i j}\right|
$$

We shall use the following well known result on diagonally dominant matrices.
Theorem 2. Let $A$ be row diagonally dominant. Suppose $A$ is non-singular. Let $B:=A^{-1}$ and $B=\left[b_{i j}\right]$. Then,

$$
\left|b_{i i}\right| \geq\left|b_{j i}\right| \quad \forall j .
$$

## 3 Results

To prove our main result, we need to show that the Laplacian of $\mathcal{G}$ has certain properties.

### 3.1 Properties of the Laplacian

From the digraph $\mathcal{G}=(V, \mathcal{E})$, we define a simple undirected graph $\widetilde{G}$ as follows.
Definition 1. Let $V(\widetilde{G}):=\{1, \ldots, n\}$. We say that any two vertices $i, j \in V(\widetilde{G})$ are adjacent in $\widetilde{G}$ if and only if either $(i, j) \in \mathcal{E}$ or $(j, i) \in \mathcal{E}$.

Definition 2. To an edge $(i, j)$ of $\widetilde{G}$, define

$$
\widetilde{W}_{i j}:=\left\{\begin{array}{cl}
\left(W_{i j}^{-1}+W_{j i}^{-1}\right)^{-1} & (i, j) \in \mathcal{E} \text { and }(j, i) \in \mathcal{E}  \tag{7}\\
W_{i j} & (i, j) \in \mathcal{E} \text { and }(j, i) \notin \mathcal{E} \\
W_{j i} & (i, j) \notin \mathcal{E} \text { and }(j, i) \in \mathcal{E}
\end{array}\right.
$$

We now have the weighted graph $\widetilde{G}$. Let $E$ be the set of all edges of $\widetilde{G}$.
Definition 3. The Laplacian of $\widetilde{G}$ is the matrix $L(\widetilde{G})=\left[M_{i j}\right] \in B^{n s}$, where the $(i, j)^{\text {th }}$ block is defined as follows:

$$
M_{i j}:=\left\{\begin{array}{cl}
-\widetilde{W}_{i j}^{-1} & (i, j) \in E  \tag{8}\\
O_{s} & i \neq j \text { and }(i, j) \notin E \\
\sum_{\{k: k \neq i\}} \widetilde{W}_{i k}^{-1} & i=j
\end{array}\right.
$$

We now have the following proposition. Recall that $L$ is the Laplacian of $\mathcal{G}$.
Proposition 1. $L(\widetilde{G})=L+L^{\prime}$.
Proof. We shall write

$$
A=L+L^{\prime} \quad \text { and } \quad M=L(\widetilde{G})
$$

Let the $(i, j)^{\text {th }}$ block of $A$ be $A_{i j}$ and $L$ be $L_{i j}$. We need to show that

$$
A_{i j}=M_{i j} \text { for all } i, j \in V
$$

where $M_{i j}$ is defined in (8). Note that $A_{i j}=L_{i j}+L_{j i}$. Partition the set of all edges $E$ of $\widetilde{G}$ as follows:

$$
\begin{aligned}
S_{1} & :=\{(i, j) \in E:(i, j) \in \mathcal{E} \text { and }(j, i) \notin \mathcal{E}\} \\
S_{2} & :=\{(i, j) \in E:(i, j) \notin \mathcal{E} \text { and }(j, i) \in \mathcal{E}\} \\
S_{3} & :=\{(i, j) \in E:(i, j) \in \mathcal{E} \text { and }(j, i) \in \mathcal{E}\}
\end{aligned}
$$

Fix $i$ and $j$ in $\{1, \ldots, n\}$. We consider the following cases.
Case 1: Suppose $i$ and $j$ are not adjacent in $\widetilde{G}$. This means $(i, j) \notin \mathcal{E}$ and $(j, i) \notin \mathcal{E}$. So, $L_{i j}=O_{s}$ and $L_{j i}=O_{s}$. By (8), $M_{i j}=O_{s}$. Therefore,

$$
A_{i j}=L_{i j}+L_{j i}=O_{s}=M_{i j} .
$$

Case 2: Suppose $i$ and $j$ are adjacent in $\widetilde{G}$. We consider three possible sub-cases. Case (i): Let $(i, j) \in S_{1}$. Then, by (7), $\widetilde{W}_{i j}=W_{i j}$. So, $M_{i j}=-W_{i j}^{-1}$. Because the weight of the edge $(i, j)$ in $\mathcal{G}$ is $W_{i j}, L_{i j}=-W_{i j}^{-1}$. Since $(j, i) \notin \mathcal{E}, L_{j i}=O_{s}$ and therefore,

$$
\begin{equation*}
A_{i j}=L_{i j}+L_{j i}=-W_{i j}^{-1}=M_{i j} \tag{9}
\end{equation*}
$$

Case (ii): Let $(i, j) \in S_{2}$. Then, by (7), $\widetilde{W}_{i j}=W_{j i}$. So, $M_{i j}=-W_{j i}^{-1}$. As $(i, j) \notin \mathcal{E}, L_{i j}=O_{s}$. The weight of the edge $(j, i)$ in $\mathcal{G}$ is $W_{j i}^{-1}$. So, $L_{j i}=-W_{j i}^{-1}$; and hence

$$
\begin{equation*}
A_{i j}=L_{i j}+L_{j i}=-W_{j i}^{-1}=M_{i j} . \tag{10}
\end{equation*}
$$

Case 3: Let $(i, j) \in S_{3}$. Then,

$$
\widetilde{W}_{i j}=\left(W_{i j}^{-1}+W_{j i}^{-1}\right)^{-1}
$$

So,

$$
M_{i j}=-\widetilde{W}_{i j}^{-1}=-\left(W_{i j}^{-1}+W_{j i}^{-1}\right)
$$

The weights of the edges $(i, j)$ and $(j, i)$ in $\mathcal{G}$ are respectively, $W_{i j}$ and $W_{j i}$. So,

$$
L_{i j}=-W_{i j}^{-1} \quad \text { and } \quad L_{j i}=-W_{j i}^{-1} .
$$

Thus,

$$
\begin{equation*}
A_{i j}=L_{i j}+L_{j i}=-\left(W_{i j}^{-1}+W_{j i}^{-1}\right)=M_{i j} \tag{11}
\end{equation*}
$$

Since

$$
\left(L+L^{\prime}\right) U=M U=O_{s}
$$

it follows that $A_{i i}=M_{i i}$ for each $i=1, \ldots, n$. The proof is complete.
We now deduce some properties of the Laplacian matrix $L$ of $\mathcal{G}$.
Proposition 2. The following are true.
(i) $L$ is positive semidefinite.
(ii) $\operatorname{null}(L)=\operatorname{null}\left(L^{\prime}\right)=\operatorname{col}(J)$.
(iii) $L^{\dagger}$ is almost positive definite.
(iv) $L L^{\dagger}=L^{\dagger} L=I_{n s}-\frac{J}{n}$.

Proof. Consider the undirected graph $\widetilde{G}=(V, E)$ in Definition 1. Put $\widetilde{L}=L(\widetilde{G})$ and $S_{i j}=\widetilde{W}_{i j}^{-1}$, where $\widetilde{W}_{i j}$ are defined in (7). Corresponding to an edge $(p, q)$ in $\widetilde{G}$, we now define $\widetilde{S}(p, q) \in B^{n s}$ with $(i, j)^{\text {th }}$ block given by

$$
\widetilde{S}(p, q)_{i j}:=\left\{\begin{array}{rl}
-S_{p q} & (i, j)=(p, q) \quad \text { or }(i, j)=(q, p) \\
S_{p q} & i=j=p \text { or } i=j=q \\
O_{s} & \text { else. }
\end{array}\right.
$$

Now,

$$
\widetilde{L}=\sum_{(p, q) \in E} \widetilde{S}(p, q)
$$

Let $x \in \mathbb{R}^{n s}$. Write

$$
x=\left(x^{1}, \ldots, x^{j}, \ldots, x^{n}\right)^{\prime}, \quad \text { where each } x^{i} \in \mathbb{R}^{s} .
$$

By an easy verification, we find that, if $(p, q) \in E$, then

$$
x^{\prime} \widetilde{S}(p, q) x=\left\langle S_{p q}\left(x^{p}-x^{q}\right), x^{p}-x^{q}\right\rangle .
$$

Thus,

$$
\begin{equation*}
x^{\prime} \widetilde{L} x=\sum_{(i, j) \in E}\left\langle S_{i j}\left(x^{i}-x^{j}\right), x^{i}-x^{j}\right\rangle . \tag{12}
\end{equation*}
$$

Each $S_{i j}$ is a positive definite matrix. So, $x^{\prime} \widetilde{L} x \geq 0$. By Proposition $1, \widetilde{L}=L+L^{\prime}$. Therefore, $x^{\prime} \widetilde{L} x=2 x^{\prime} L x$. So, $x^{\prime} L x \geq 0$. This proves (i).

Let $x \in \operatorname{null}(L)$. As, $\widetilde{L}=L+L^{\prime}$, we see that $x^{\prime} \widetilde{L} x=0$. By (12), if $(i, j) \in E$, then $x^{i}=x^{j}$. Because $\mathcal{G}$ is strongly connected, $\widetilde{G}$ is connected. So, $x^{i}=x^{j}$ for all $i, j$. Thus,

$$
x \in \operatorname{span}\left\{(w, \ldots, w)^{\prime} \in \mathbb{R}^{n s}: w \in \mathbb{R}^{s}\right\} .
$$

Since, $\operatorname{col}(J)=\operatorname{span}\left\{(w, \ldots, w): w \in \mathbb{R}^{s}\right\}$, we see that $x=J p$ for some $w \in \mathbb{R}^{s}$. So, $\operatorname{null}(L)=\operatorname{col}(J)$. Now, $\mathcal{G}^{\prime}$ is a strongly connected, and balanced digraph. Because $L^{\prime}$ is the Laplacian of $\mathcal{G}^{\prime}$, we see that $\operatorname{null}\left(L^{\prime}\right)=\operatorname{col}(J)$. This proves (ii).

We now prove (iii). Let $y \in \mathbb{R}^{n s}$. Since $L$ is positive semidefinite, $y^{\prime} L y \geq 0$. Suppose $y^{\prime} L y=0$. Then, $y^{\prime} \widetilde{L} y=0$. By equation (12), it follows that $y \in \operatorname{col}(J)$. Since $\operatorname{col}(J)=\operatorname{null}(L)$, we have $L y=0$. Thus, either $y^{\prime} L y>0$ or $L y=0$. So, $L$ is almost positive definite. By item (vi) in Section $2, L^{\dagger}$ is almost positive definite as well. This proves (iii).

To prove (iv), we show that

$$
L L^{\dagger} v=v \text { for all } v \in \operatorname{null}(J)
$$

Let $v \in \operatorname{null}(J)$. Suppose $L L^{\dagger} v=w$. Then,

$$
J w=0 \text { and } L^{\dagger} L L^{\dagger} v=L^{\dagger} w
$$

Since $L^{\dagger} L L^{\dagger}=L^{\dagger}$, we get $L^{\dagger} v=L^{\dagger} w$ and hence $v-w \in \operatorname{null}\left(L^{\dagger}\right)$. As null $\left(L^{\dagger}\right)=$ $\operatorname{col}(J)$, we get

$$
v-w \in \operatorname{col}(J) .
$$

Since $J L L^{\dagger} v=J w$, and $J L=O_{n s}, J w=0$. As $v \in \operatorname{null}(J), J v=0$. So, $J(v-w)=0$ and hence

$$
v-w \in \operatorname{null}(J)
$$

We now have $v-w \in \operatorname{null}(J) \cap \operatorname{col}(J)$. So, $v=w$. The proof is complete.
Proposition 3. $\Delta\left(L^{\dagger}\right)$ is a positive definite matrix.

Proof. We recall that $\Delta\left(L^{\dagger}\right)=\operatorname{Diag}\left(K_{11}, \ldots, K_{n n}\right)$. Fix $i \in\{1, \ldots, n\}$. We show that $K_{i i}$ is positive definite. Let $y \in \mathbb{R}^{s}$. Define $q:=\left(q^{1}, \ldots, q^{n}\right)^{\prime} \in \mathbb{R}^{n s}$ by

$$
q^{j}:= \begin{cases}y & j=i \\ 0 & \text { else }\end{cases}
$$

In view of previous lemma, $L^{\dagger}$ is almost positive definite. So, $q^{\prime} L^{\dagger} q>0$ or $L^{\dagger} q=0$. We note that $q^{\prime} L^{\dagger} q=y^{\prime} K_{i i} y$. Hence, if $q^{\prime} L^{\dagger} q>0$, then $y^{\prime} K_{i i} y>0$. Suppose $L^{\dagger} q=0$. Since $\operatorname{null}\left(L^{\dagger}\right)=\operatorname{null}\left(L^{\prime}\right)$ and $\operatorname{null}\left(L^{\prime}\right)=\operatorname{col}(J), q=0$. This means $y$ is zero. So, $K_{i i}$ is positive definite. The proof is complete.

### 3.2 Inverse formula

Recall that the generalized resistance matrix of $\mathcal{G}$ corresponding to two positive real numbers $a$ and $b$ is

$$
R_{a, b}:=\left[R_{i j}\right]=\left[a^{2} L_{i i}^{\dagger}+b^{2} L_{j j}^{\dagger}-2 a b L_{i j}^{\dagger}\right] .
$$

Define

$$
\begin{equation*}
\tau_{i}:=2 a b I_{s}+L_{i i} R_{i i}-\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} R_{j i} \text { for all } i=1, \ldots, n . \tag{13}
\end{equation*}
$$

Now, let $\tau$ be the $n s \times s$ matrix $\left(\tau_{1}, \ldots, \tau_{n}\right)^{\prime}$. The inverse formula will be proved by using the following lemma.

Lemma 1. The following are true.
(i) $\tau=a^{2} L \Delta\left(L^{\dagger}\right) U+\frac{2 a b}{n} U$.
(ii) $\tau^{\prime}+a^{2} U^{\prime} \Delta\left(L^{\dagger}\right)\left(L-L^{\prime}\right)=a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U^{\prime}$.
(iii) $L R_{a, b}+2 a b I_{n s}=\tau U^{\prime}$.
(iv) $R_{a, b} L+2 a b I_{n s}=U \tau^{\prime}-a^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L$.
(v) $U^{\prime} \tau=2 a b I_{s}$.
(vi) $\tau^{\prime} R_{a, b} \tau=2 a^{3} b^{3} \tilde{X}^{\prime} L \tilde{X}+\frac{1}{n} 4 a^{2} b^{2}\left(a^{2}+b^{2}\right) \sum_{i=1}^{n} K_{i i}$, where $\tilde{X}:=\Delta\left(L^{\dagger}\right) U$.
(vii) $\tau^{\prime} R_{a, b} \tau$ is a positive definite matrix.

Proof. Fix $i \in[n]$. For simplicity, we shall use $R$ for $R_{a, b}$. Since $L=\left[L_{i j}\right]$ and $L^{\dagger}=\left[K_{i j}\right]$, the $(i, j)^{\text {th }}$ block of $L L^{\dagger}$ is the $s \times s$ matrix

$$
L_{i i} K_{i i}-\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j i} .
$$

The $(i, j)^{\text {th }}$ block of $I_{n s}-\frac{J}{n}$ is $\left(1-\frac{1}{n}\right) I_{s}$. Since $L L^{\dagger}=I_{n s}-\frac{1}{n} J$, we see that

$$
L_{i i} K_{i i}-\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j i}=\left(1-\frac{1}{n}\right) I_{s}
$$

Rewriting the above equation, we have

$$
\begin{equation*}
\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j i}=L_{i i} K_{i i}-\left(1-\frac{1}{n}\right) I_{s} \tag{14}
\end{equation*}
$$

By definition,

$$
\tau_{i}=2 a b I_{s}+L_{i i} R_{i i}-\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} R_{j i} .
$$

Because

$$
R_{j i}=a^{2} K_{j j}+b^{2} K_{i i}-2 a b K_{j i} \text { and } R_{i i}=(a-b)^{2} K_{i i},
$$

we have

$$
\begin{equation*}
\tau_{i}=2 a b I_{s}+(a-b)^{2} L_{i i} K_{i i}-\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1}\left(a^{2} K_{j j}+b^{2} K_{i i}-2 a b K_{j i}\right) \tag{15}
\end{equation*}
$$

We recall that

$$
L_{i i}=\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1}
$$

So,

$$
\begin{equation*}
\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{i i}=L_{i i} K_{i i} . \tag{16}
\end{equation*}
$$

Substituting (16) in (15),

$$
\tau_{i}=2 a b I_{s}+(a-b)^{2} L_{i i} K_{i i}-b^{2} L_{i i} K_{i i}-a^{2} \sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j j}+2 a b \sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j i} .
$$

Using (14) in the above equation, we get
$\tau_{i}=2 a b I_{s}+(a-b)^{2} L_{i i} K_{i i}-b^{2} L_{i i} K_{i i}-a^{2} \sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j j}+2 a b L_{i i} K_{i i}-2 a b\left(1-\frac{1}{n}\right) I_{s}$.
After simplification,

$$
\begin{equation*}
\tau_{i}=a^{2} L_{i i} K_{i i}-a^{2} \sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j j}+\frac{2 a b}{n} I_{s} . \tag{17}
\end{equation*}
$$

Let $A:=\operatorname{Diag}(L)-L$. Write $A=\left[A_{i j}\right]$. Then,

$$
\begin{align*}
\sum_{\{j:(i, j) \in \mathcal{E}\}} W_{i j}^{-1} K_{j j} & =\sum_{j=1}^{n} A_{i j} K_{j j}  \tag{18}\\
& =\left(A \Delta\left(L^{\dagger}\right) U\right)_{i} .
\end{align*}
$$

We now compute $A U$. Because

$$
A=\left[\begin{array}{ccccc}
O_{s} & -L_{12} & -L_{13} & \ldots & -L_{1 n} \\
-L_{21} & O_{s} & -L_{23} & \ldots & -L_{2 n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-L_{n 1} & -L_{n 2} & -L_{n 3} & \ldots & O_{s}
\end{array}\right]
$$

it follows that

$$
(A U)_{i}=-\sum_{j \in[n] \backslash\{i\}} L_{i j} .
$$

Put

$$
P:=\Delta\left(L^{\dagger}\right) U
$$

Then,

$$
P_{i}=K_{i i} .
$$

Thus,

$$
(\operatorname{Diag}(A U) P)_{i}=-\sum_{j \in[n] \backslash\{i\}} L_{i j} K_{i i} .
$$

As $\sum_{j=1}^{n} L_{i j}=O_{s}$, we get

$$
\begin{equation*}
(\operatorname{Diag}(A U) P)_{i}=-\sum_{j \in[n] \backslash\{i\}} L_{i j} K_{i i}=L_{i i} K_{i i} . \tag{19}
\end{equation*}
$$

By (17), (18) and (19),

$$
\begin{equation*}
\tau_{i}=a^{2}(\operatorname{Diag}(A U) P-A P)_{i}+\frac{2 a b}{n} I_{s} \tag{20}
\end{equation*}
$$

Put

$$
\widetilde{A}:=\operatorname{Diag}(A U)-A
$$

In view of (20),

$$
\begin{equation*}
\tau_{i}=a^{2}(\widetilde{A} P)_{i}+\frac{2 a b}{n} I_{s} \tag{21}
\end{equation*}
$$

But a direct verification tells that

$$
\widetilde{A}=L
$$

Therefore by (21),

$$
\tau=a^{2} L \Delta\left(L^{\dagger}\right) U+\frac{2 a b}{n} U
$$

This proves (i).
We now prove (ii). Put $M:=L-L^{\prime}$. Then by (i),

$$
\begin{align*}
a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) M+\tau^{\prime} & =a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+\frac{2 a b}{n} U^{\prime}  \tag{22}\\
& =a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U^{\prime}
\end{align*}
$$

The proof of (ii) is complete.
We now prove (iii). Since

$$
R_{i j}=a^{2} L_{i i}^{\dagger}+b^{2} L_{j j}^{\dagger}-2 a b L_{i j}^{\dagger}
$$

and $R=\left[R_{i j}\right]$, it is easy to see that

$$
R=a^{2} \Delta\left(L^{\dagger}\right) U U^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right)-2 a b L^{\dagger} .
$$

As $L L^{\dagger}=I_{n s}-\frac{1}{n} U U^{\prime}$ and $L U=O_{s}$, we get

$$
\begin{align*}
L R & =a^{2} L \Delta\left(L^{\dagger}\right) U U^{\prime}-2 a b L L^{\dagger} \\
& =a^{2} L \Delta\left(L^{\dagger}\right) U U^{\prime}+\frac{2 a b}{n} U U^{\prime}-2 a b I_{n s}  \tag{23}\\
& =\left(a^{2} L \Delta\left(L^{\dagger}\right) U+\frac{2 a b}{n}\right) U^{\prime}-2 a b I_{n s} .
\end{align*}
$$

By (i),

$$
\tau=a^{2} L \Delta\left(L^{\dagger}\right) U+\frac{2 a b}{n} U
$$

Hence,

$$
L R=\tau U^{\prime}-2 a b I_{n s} .
$$

This completes the proof of (iii).
To prove (iv), first we observe that

$$
R L=b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L-2 a b L^{\dagger} L
$$

Since $L^{\dagger} L=I_{n s}-\frac{1}{n} U U^{\prime}$, we get

$$
\begin{equation*}
R L+2 a b I_{n s}=b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U U^{\prime} \tag{24}
\end{equation*}
$$

By (i),

$$
\begin{equation*}
U \tau^{\prime}=a^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+\frac{2 a b}{n} U U^{\prime} \tag{25}
\end{equation*}
$$

From (24) and (25),

$$
R L+2 a b I_{n s}=U \tau^{\prime}-a^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L
$$

The proof of (iv) is complete.
By item (i),

$$
U^{\prime} \tau=a^{2} U^{\prime} L \Delta\left(L^{\dagger}\right) U+\frac{2 a b}{n} U^{\prime} U .
$$

As $U^{\prime} U=I_{n s}$ and $U^{\prime} L=O_{s}$, it follows that

$$
U^{\prime} \tau=2 a b I_{n s}
$$

This proves (v).
Put $M=L-L^{\prime}$. By (ii),

$$
\begin{equation*}
\tau^{\prime} R \tau=\left(a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) M\right) R \tau \tag{26}
\end{equation*}
$$

Because $M=L-L^{\prime}$,

$$
a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) M=a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+\frac{2 a b}{n} U^{\prime}
$$

Substituting for $\tau$ from (i) in (26), we get

$$
\tau^{\prime} R \tau=\left(a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+\frac{2 a b}{n} U^{\prime}\right) R\left(a^{2} L \Delta\left(L^{\dagger}\right) U+\frac{2 a b}{n} U\right)
$$

Therefore,

$$
\begin{align*}
\tau^{\prime} R \tau & =a^{4} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R L \Delta\left(L^{\dagger}\right) U+\frac{2 a^{3} b}{n} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R U+\frac{2 a^{3} b}{n} U^{\prime} R L \Delta\left(L^{\dagger}\right) U  \tag{27}\\
& +\frac{4 a^{2} b^{2}}{n^{2}} U^{\prime} R U
\end{align*}
$$

As

$$
L U=L^{\prime} U=O_{s} \text { and } R=a^{2} \Delta\left(L^{\dagger}\right) U U^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right)-2 a b L^{\dagger}
$$

we have

$$
U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R L \Delta\left(L^{\dagger}\right) U=-2 a b U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} L^{\dagger} L \Delta\left(L^{\dagger}\right) U
$$

Since $L^{\dagger} L=I_{n s}-\frac{1}{n} U U^{\prime}$ and $L^{\prime} U=O_{s}$,

$$
\begin{align*}
U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R L \Delta\left(L^{\dagger}\right) U & =-2 a b U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}\left(I_{n s}-\frac{1}{n} U U^{\prime}\right) \Delta\left(L^{\dagger}\right) U  \tag{28}\\
& =-2 a b U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} \Delta\left(L^{\dagger}\right) U
\end{align*}
$$

Define

$$
\widetilde{X}:=\Delta\left(L^{\dagger}\right) U
$$

Then,

$$
\begin{equation*}
\tilde{X}^{\prime} L \widetilde{X}=U^{\prime} \Delta\left(L^{\dagger}\right) L \Delta\left(L^{\dagger}\right) U \tag{29}
\end{equation*}
$$

By (28),

$$
U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R L \Delta\left(L^{\dagger}\right) U=-2 a b \widetilde{X}^{\prime} L \widetilde{X}
$$

In view of (iv) and (i), we have

$$
\begin{gathered}
R L+2 a b I_{n s}=U \tau^{\prime}-a^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L \\
U \tau^{\prime}=a^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+\frac{2 a b}{n} U U^{\prime}
\end{gathered}
$$

These two equations imply

$$
R L=b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U U^{\prime}-2 a b I_{n s}
$$

Hence,

$$
\begin{aligned}
U^{\prime} R L \Delta\left(L^{\dagger}\right) U & =U^{\prime}\left(b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right) L+\frac{2 a b}{n} U U^{\prime}-2 a b I_{n s}\right) \Delta\left(L^{\dagger}\right) U \\
& =\left(b^{2} n\right) U^{\prime} \Delta\left(L^{\dagger}\right) L \Delta\left(L^{\dagger}\right) U
\end{aligned}
$$

By (29),

$$
\begin{equation*}
U^{\prime} R L \Delta\left(L^{\dagger}\right) U=b^{2} n \widetilde{X}^{\prime} L \widetilde{X} \tag{30}
\end{equation*}
$$

We also note that

$$
\begin{align*}
U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R U & =U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}\left(a^{2} \Delta\left(L^{\dagger}\right) U U^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right)-2 a b L^{\dagger}\right) U \\
& =a^{2} n U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} \Delta\left(L^{\dagger}\right) U  \tag{31}\\
& =a^{2} n \widetilde{X}^{\prime} L \widetilde{X}
\end{align*}
$$

where the second equality follows from $L^{\prime} U=L^{\dagger} U=O_{s}$ and the last one from (29). Since

$$
R=a^{2} \Delta\left(L^{\dagger}\right) U U^{\prime}+b^{2} U U^{\prime} \Delta\left(L^{\dagger}\right)-2 a b L^{\dagger}
$$

we see that

$$
\begin{equation*}
U^{\prime} R U=n\left(a^{2}+b^{2}\right) \sum_{i=1}^{n} K_{i i} . \tag{32}
\end{equation*}
$$

Substituting (28), (30), (31) and (32) in (27), we get

$$
\tau^{\prime} R \tau=2 a^{3} b^{3} \tilde{X}^{\prime} L \tilde{X}+\frac{4 a^{2} b^{2}\left(a^{2}+b^{2}\right)}{n} \sum_{i=1}^{n} K_{i i} .
$$

The proof of (vi) is complete.
Since $L$ is positive semidefinite, $\widetilde{X}^{\prime} L \widetilde{X}$ is positive semidefinite. By Proposition (3), each $K_{i i}$ is positive definite. So, $\tau^{\prime} R \tau$ is positive definite. This proves (vii). The proof is complete.

We prove the inverse formula in Theorem 1.

## Theorem 3.

$$
R_{a, b}^{-1}=-\frac{1}{2 a b} L+\tau\left(\tau^{\prime} R \tau\right)^{-1}\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right) .
$$

Proof. Again, as in the proof of above lemma, we shall use $R_{a, b}$ for $R$. By item (iii) of Lemma 1,

$$
L R+2 a b I_{n s}=\tau U^{\prime} .
$$

In view of item (v) of the previous Lemma, $U^{\prime} \tau=2 a b I_{s}$. So,

$$
L R \tau+2 a b \tau=\tau U^{\prime} \tau=2 a b \tau
$$

This implies

$$
L R \tau=O_{s}
$$

We know that

$$
\operatorname{null}(L)=\operatorname{span}\left\{(p, \ldots, p)^{\prime}: p \in \mathbb{R}^{s}\right\}
$$

So,

$$
R \tau=U C
$$

where $C$ is a $s \times s$ matrix. Since $\tau^{\prime} R \tau$ is a positive definite matrix, $R \tau$ cannot be zero. Hence, $C \neq O_{s}$. As $\tau^{\prime} U=2 a b I_{s}$, we get

$$
C=\frac{1}{2 a b} \tau^{\prime} R \tau
$$

Therefore,

$$
\begin{equation*}
R \tau=\frac{1}{2 a b} U\left(\tau^{\prime} R \tau\right) \tag{33}
\end{equation*}
$$

Since $L^{\prime} U=O_{s}$, from item (iv) of Lemma 1, we deduce that

$$
\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right)\left(R L+2 a b I_{n s}\right)=2 a b\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right)
$$

Simplifying the above equation, we get

$$
\begin{equation*}
\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right) R L=O_{s} \tag{34}
\end{equation*}
$$

We now claim that

$$
\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right) R \neq O_{s} .
$$

If not, then

$$
\begin{equation*}
\tau^{\prime} R \tau-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime} R \tau+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L R \tau=O_{s} \tag{35}
\end{equation*}
$$

By (33),

$$
R \tau=\frac{1}{2 a b} U \tau^{\prime} R \tau
$$

So,

$$
L R \tau=O_{s} \quad \text { and } \quad L^{\prime} R \tau=O_{s}
$$

Hence (35) leads to $\tau^{\prime} R \tau=O_{s}$. This contradicts that $\tau^{\prime} R \tau$ is positive definite. Hence,

$$
\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right) R \neq O_{s}
$$

Since nullity of $L^{\prime}$ is $s$ and $L^{\prime} U=O_{s}$, by (34), there exists an $s \times s$ matrix $\widetilde{C}$ such that

$$
\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right)\left(L^{\dagger}\right) L\right) R=\tilde{C} U^{\prime}
$$

We know that $U^{\prime} \tau=2 a b I_{s}$. So, from the previous equation,

$$
\tilde{C}=\frac{1}{2 a b} \tau^{\prime} R \tau
$$

Thus,

$$
\begin{equation*}
\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right) R=\frac{\tau^{\prime} R \tau}{2 a b} U^{\prime} \tag{36}
\end{equation*}
$$

We now have

$$
\begin{aligned}
Q:= & \left(-\frac{1}{2 a b} L+\tau\left(\tau^{\prime} R \tau\right)^{-1}\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right)\right) R \\
& =-\frac{1}{2 a b} L R+\tau\left(\tau^{\prime} R \tau\right)^{-1}\left(\tau^{\prime}-a^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+b^{2} U^{\prime} \Delta\left(L^{\dagger}\right) L\right) R
\end{aligned}
$$

By (36), we have

$$
\begin{equation*}
Q=-\frac{1}{2 a b}\left(L R-\tau U^{\prime}\right) \tag{37}
\end{equation*}
$$

Item (iii) of Lemma 1 says that

$$
L R+2 a b I_{n s}=\tau U^{\prime}
$$

Substituting back in (37), we get $Q=I_{n s}$. The proof is complete.

### 3.3 Special cases

(i) Suppose all the weights in $\mathcal{G}$ are equal to 1. Choose $a=b=1$. We shall denote $R_{i j}$ by $r_{i j}$ and define $R:=\left[r_{i j}\right]$. Now by (13),

$$
\tau_{i}=2-\sum_{\{j:(i, j) \in \mathcal{E}\}} r_{j i} .
$$

We note that $r_{i i}=0$ and $U=\mathbf{1}$. Hence, by our formula in Theorem 1,

$$
\begin{aligned}
R^{-1} & =-\frac{1}{2} L+\tau\left(\tau^{\prime} R \tau\right)^{-1}\left(\tau^{\prime}-\mathbf{1}^{\prime} \Delta\left(L^{\dagger}\right) L^{\prime}+\mathbf{1}^{\prime} \Delta\left(L^{\dagger}\right) L\right) \\
& =-\frac{1}{2} L+\frac{\tau}{\tau^{\prime} R \tau}\left(\tau^{\prime}-\mathbf{1}^{\prime} \Delta\left(L^{\dagger}\right)\left(L-L^{\prime}\right)\right)
\end{aligned}
$$

Thus we get (6).
(ii) Suppose $T$ is a tree with $V(T)=\{1, \ldots, n\}$. To denote an edge in $T$, we shall use the notation $i j$. Let the weight of an edge $i j$ be $W_{i j}$. Assume that all weights are positive definite matrices of order $s$. Now, define a directed graph $\widetilde{T}$ as follows. Let $V(\widetilde{T})=\{1, \ldots, n\}$. We use the notation $(i, j)$ to denoted a directed edge from $i$ to $j$. Now, we define $E(\widetilde{T}):=\{(i, j),(j, i): i j \in E(T)\}$. Now we assign the weight $W_{p q}$ to an edge $(p, q)$ in $\widetilde{T}$. It is clear that $\widetilde{T}$ is strongly connected, weighted and balanced digraph. Now define the Laplacian matrix of $T$, say, $L(T)$ as given in 1.4 and the Laplacian of $\widetilde{T}$, say, $L(\widetilde{T})$ as given in item (i) of 1.6. We note that $L(T)=L(\widetilde{T})$. Fix $a=b=1$. Let $R_{i j}$ be the resistance between $i$ and $j$. Then,

$$
R_{i j}=M_{i i}+M_{j j}-2 M_{i j}
$$

where $M_{i j}$ is the $(i, j)^{\text {th }}$ block of the Moore Penrose inverse of $L(\widetilde{T})$. If $D_{i j}$ is the shortest distance between $i$ and $j$ in $T$, then by the argument mentioned in 1.4, $D_{i j}=R_{i j}$. Define $D:=\left[D_{i j}\right]$. Because $D_{i i}=O_{s}$, by Theorem 1, we have

$$
\begin{aligned}
\tau_{i}: & =2 I_{s}-\sum_{\{j:(i, j) \in E\}} W_{i j}^{-1} R_{j i} \\
& =2 I_{s}-\sum_{\{j:(i, j) \in E\}} W_{i j}^{-1} W_{i j} \\
& =\left(2-\delta_{i}\right) I_{s},
\end{aligned}
$$

where $\delta_{i}$ is the out-degree of the vertex $i$. We now define

$$
\tau:=\left(2-\delta_{1}, \ldots, 2-\delta_{n}\right)^{\prime} \otimes I_{s}
$$

By an induction argument it follows that if $S$ is the sum of all the weights in $T$, then

$$
D \tau=[S, \ldots, S]^{\prime} .
$$

(See Lemma 1 in [8]). Because $T$ is a tree, $\sum_{i=1}^{n} \delta_{i}=2(n-1)$. Thus, $\sum_{i=1}^{n} \tau_{i}=$ $2 I_{s}$ and hence $\tau^{\prime} D \tau=2 S$. By our formula in Theorem 1,

$$
\begin{align*}
D^{-1} & =-\frac{1}{2} L(T)+\tau\left(\tau^{\prime} D \tau\right)^{-1} \tau^{\prime}  \tag{38}\\
& =-\frac{1}{2} L(T)+\frac{1}{2} \tau S \tau^{\prime}
\end{align*}
$$

This is formula (5).
In a similar manner, we get (2), (3), and (4).

### 3.4 Non-negativity of the resistance

From numerical computations, we observe that $R_{i j}=a^{2} l_{i i}^{\dagger}+b^{2} l_{j j}^{\dagger}-2 a b l_{i j}^{\dagger}$ is always positive semidefinite. But at this stage, we do not know how to prove this. However, when all the weights in $\mathcal{G}$ are positive scalars, we now show that the resistance is always non-negative. We need the following lemma.
Lemma 2. Let $A=\left[a_{i j}\right]$ be an $n \times n$ matrix with the following properties.
(i) All the off-diagonal entries are non-positive.
(ii) $A \mathbf{1}=A^{\prime} \mathbf{1}=0$.
(iii) $\operatorname{rank}\left(A+A^{\prime}\right)=n-1$.
(iv) $A$ is positive semidefinite.

Let $A^{\dagger}:=\left[p_{i j}\right]$ be the Moore-Penrose inverse of $A$. Then,

$$
p_{i i} \geq p_{i j} \quad \text { and } \quad p_{i i} \geq p_{j i} \quad \forall j .
$$

Proof. By a permutation similarity argument, without loss of generality, we may assume that $i=1$ and $j=n$. We now show that $p_{11} \geq p_{1 n}$ and $p_{11} \geq p_{n 1}$. By symmetry of our assumptions, it suffices to show that $p_{11} \geq p_{1 n}$.

Let $\mathbf{1}_{n-1}$ be the vector of all ones in $\mathbb{R}^{n-1}$. By (ii) we can partition $A$ as follows:

$$
A=\left[\begin{array}{cc}
B & -B \mathbf{1}_{n-1} \\
-\mathbf{1}_{n-1}^{\prime} B & \mathbf{1}_{n-1}^{\prime} B \mathbf{1}_{n-1}
\end{array}\right]
$$

All the row sums of $A+A^{\prime}$ are equal to zero. So, all the cofactors of $A+A^{\prime}$ are equal. As $\operatorname{rank}\left(A+A^{\prime}\right)=n-1$, we now deduce that the common cofactor of $A+A^{\prime}$ is non-zero. In particular, $\operatorname{det}\left(B+B^{\prime}\right) \neq 0$. Since $A$ is positive semidefinite, $B+B^{\prime}$ is positive semidefinite. Because $B+B^{\prime}$ is non-singular, $B+B^{\prime}$ is positive definite. So, $B$ is positive definite. All the off-diagonal entries of $B$ are non-positive. By a well-known theorem on $\mathbf{Z}$-matrices, $B$ is non-singular and all entries of $B^{-1}$ are non-negative. By a direct verification,

$$
A^{\dagger}=\left[\begin{array}{cc}
B^{-1}-\frac{1}{n} \mathbf{1}_{n-1} \mathbf{1}_{n-1}^{\prime} B^{-1}-\frac{1}{n} B^{-1} \mathbf{1}_{n-1} \mathbf{1}_{n-1}^{\prime} & -\frac{1}{n} B^{-1} \mathbf{1}_{n-1}  \tag{39}\\
-\frac{1}{n} \mathbf{1}_{n-1}^{\prime} B^{-1}
\end{array}\right]+\frac{\mathbf{1}_{n-1}^{\prime} B^{-1} \mathbf{1}_{n-1}}{n^{2}}\left(\mathbf{1 1}^{\prime}\right)
$$

Put

$$
C=\left[c_{i j}\right]:=B^{-1} \text { and } \delta:=\frac{1}{n^{2}} \mathbf{1}_{n-1}^{\prime} B^{-1} \mathbf{1}_{n-1}
$$

Then, $c_{i j} \geq 0 \quad \forall i, j$ and

$$
\begin{gathered}
p_{11}=c_{11}-\frac{1}{n} \sum_{j=1}^{n-1} c_{j 1}-\frac{1}{n} \sum_{j=1}^{n-1} c_{1 j}+\delta \\
p_{1 n}=-\frac{1}{n} \sum_{j=1}^{n-1} c_{1 j}+\delta
\end{gathered}
$$

Now,

$$
p_{11}-p_{1 n}=c_{11}-\frac{1}{n} \sum_{i=1}^{n-1} c_{i 1} .
$$

By Theorem 2,

$$
c_{11} \geq c_{j 1} \quad \forall j=1, \ldots, n-1
$$

So,

$$
-\frac{1}{n} \sum_{i=1}^{n-1} c_{i 1} \geq-\left(\frac{n-1}{n}\right) c_{11} .
$$

Hence,

$$
c_{11}-\frac{1}{n} \sum_{i=1}^{n-1} c_{i 1} \geq c_{11}-\frac{n-1}{n} c_{11}=\frac{1}{n} c_{11} .
$$

Since $c_{11} \geq 0$, we conclude that

$$
p_{11}-p_{1 n} \geq 0
$$

The proof is complete.
Now it can be easily shown that any generalized resistance is non-negative.
Theorem 4. Suppose all the weights in $\mathcal{G}$ are positive scalars. Let $a, b>0$. Let $L^{\dagger}=\left[k_{i j}\right]$ be the Moore-Penrose inverse of the Laplacian of $\mathcal{G}$. Then,

$$
r_{i j}:=a^{2} k_{i i}+b^{2} k_{j j}-2 a b k_{i j} \geq 0
$$

Proof. We note that the Laplacian matrix $L$ of $\mathcal{G}$ satisfies all the conditions of the previous lemma. Moreover, by Proposition $3, k_{i i}$ and $k_{j j}$ are positive. As a consequence of Lemma 2, we deduce that

$$
\min \left(k_{i i}, k_{j j}\right) \geq \max \left(k_{i j}, k_{j i}\right)
$$

Now

$$
a^{2} k_{i i}+b^{2} k_{j j}-2 a b k_{i j} \geq 0,
$$

follows from the arithmetic mean and geometric mean inequality.

### 3.5 A perturbation result

We now show that if $R$ is the resistance matrix of a connected graph with $n$ vertices, and if $L$ is the Laplacian matrix of $\mathcal{G}$ with positive scalar weights, then $\left(R^{-1}-L\right)^{-1}$ has all entries non-negative.

Theorem 5. Let $H$ be a simple (undirected) connected graph with $n$ vertices and $R$ be the resistance matrix of $H$. Assume that all the weights in $\mathcal{G}$ are positive scalars. Then, $R^{-1}-L$ is non-singular and every entry in $\left(R^{-1}-L\right)^{-1}$ is non-negative.

Proof. Let $M=\left[m_{i j}\right]$ be the Moore-Penrose inverse of the Laplacian matrix of $H$. Then the $(i, j)^{\text {th }}$ entry $r_{i j}$ of $R$ is given by

$$
r_{i j}=m_{i i}+m_{j j}-2 m_{i j} .
$$

Fix $\alpha \geq 0$. Define $S:=\alpha L$. We complete the proof by using the following claims.
Claim 1: $R^{-1}-S$ is non-singular.
To prove this claim, we can assume that $S=\alpha L$, where $\alpha>0$. By Proposition 2, $\operatorname{rank}(S)=n-1, S+S^{\prime}$ is positive semidefinite and $S^{\prime} \mathbf{1}=S \mathbf{1}=0$. Let $x \in \mathbb{R}^{n}$ be such that

$$
\begin{equation*}
\left(R^{-1}-S\right)(x)=0 \tag{40}
\end{equation*}
$$

Put $u:=R^{-1} x$. Assuming $u \neq 0$, we now get a contradiction. As $1^{\prime} S=0$, it follows that $\mathbf{1}^{\prime} R^{-1} x=0$ and hence $u \in \mathbf{1}^{\perp}$. Writing

$$
R=\operatorname{Diag}(M) \mathbf{1 1}^{\prime}+\mathbf{1 1}^{\prime} \operatorname{Diag}(M)-2 M,
$$

we get

$$
\begin{aligned}
u^{\prime} R u & =u^{\prime}\left(\operatorname{Diag}(M) \mathbf{1 1}^{\prime}+\mathbf{1 1}^{\prime} \operatorname{Diag}(M)-2 M\right) u \\
& =-2 u^{\prime} M u
\end{aligned}
$$

Since $\operatorname{null}(M)=\operatorname{span}\{\mathbf{1}\}, M$ is positive definite on $\mathbf{1}^{\perp}$. So, $u^{\prime} M u>0$. Hence,

$$
\begin{equation*}
u^{\prime} R u<0 \tag{41}
\end{equation*}
$$

It is easy to see that

$$
\begin{equation*}
u^{\prime} R u=x^{\prime} R^{-1} x \tag{42}
\end{equation*}
$$

By (40),

$$
x^{\prime} R^{-1} x=x^{\prime} S x .
$$

Since $S+S^{\prime}$ is positive semidefinite, $x^{\prime} S x \geq 0$. So, $x^{\prime} R^{-1} x \geq 0$. Hence by (42),

$$
\begin{equation*}
u^{\prime} R u \geq 0 \tag{43}
\end{equation*}
$$

Thus, we get a contradiction from (41) and (43). Therefore, $u=R^{-1} x=0$. This implies $x=0$. So, $R^{-1}-S$ is non-singular. The claim is proved.

Claim 2: If $C$ is a $k \times k$ proper principal submatrix of $S-R^{-1}$, then

$$
q^{\prime} C q>0 \text { for all } 0 \neq q \in \mathbb{R}^{k}
$$

If $A$ is an $n \times n$ matrix, we shall use the notation $A[i]$ to denote the principal submatrix of $A$ obtained by deleting the $i^{\text {th }}$ row and $i^{\text {th }}$ column of $A$. Fix $1 \leq i \leq n$ and define

$$
B:=S[i]-R^{-1}[i]
$$

Since $R$ is negative definite on $\mathbf{1}^{\perp}$ and the diagonal entries are zero, $R$ has exactly one positive eigenvalue. By an application of interlacing theorem, we see that $-R^{-1}[i]$ is positive semidefinite. Hence

$$
\begin{equation*}
-p^{\prime} R^{-1}[i] p \geq 0 \text { for all } w \in \mathbb{R}^{n-1} . \tag{44}
\end{equation*}
$$

As the row sums and the column sums of $S$ are equal to zero, it follows that all the cofactors of $S$ are equal. Because $S+S^{\prime}$ is positive semidefinite and has rank $n-1$, it follows that every proper principal submatrix of $S+S^{\prime}$ is positive definite. So, we have

$$
\begin{equation*}
p^{\prime} S[i] p>0 \text { for all } 0 \neq p \in \mathbb{R}^{n-1} . \tag{45}
\end{equation*}
$$

By (44) and (45),

$$
p^{\prime} B p>0 \text { for all } 0 \neq p \in \mathbb{R}^{n-1}
$$

The claim is proved.
In particular, we note that all principal minors of $S-R^{-1}$ with order less than $n$ are positive.

Claim 3: $\operatorname{det}\left(S-R^{-1}\right)<0$.
Because $\gamma L-R^{-1}$ is non-singular for every $\gamma \geq 0$,

$$
\operatorname{sgn} \operatorname{det}\left(\gamma L-R^{-1}\right)=\operatorname{sgn} \operatorname{det}\left(-R^{-1}\right)
$$

Since $-R$ has exactly one negative eigenvalue, $\operatorname{det}\left(-R^{-1}\right)<0$. So,

$$
\operatorname{det}\left(\gamma L-R^{-1}\right)<0 \quad \forall \gamma \geq 0
$$

This proves the claim.
Claim 4: All principal minors of $\left(S-R^{-1}\right)^{-1}$ are negative.
Put

$$
G:=\left(S-R^{-1}\right)^{-1} \text { and } H:=S-R^{-1} .
$$

Let $s<n$, and let $\widehat{G}$ be a $s \times s$ principal submatrix of $G$. Suppose $\widehat{H}$ is the complementary submatrix of $\widehat{G}$ in $H$. By Jacobi identity,

$$
\operatorname{det}(\widehat{G})=\frac{\operatorname{det}(\widehat{H})}{\operatorname{det}(H)}
$$

By claim 2 and 3,

$$
\operatorname{sgn}(\operatorname{det} \widehat{H})>0 \text { and } \operatorname{sgn}(\operatorname{det}(H))<0 .
$$

So, $\operatorname{det}(\widehat{G})<0$. The claim is proved.
We now complete the proof of the theorem. Given $\beta \geq 0$, let

$$
\left[\begin{array}{ll}
y_{i i}(\beta) & y_{i j}(\beta) \\
y_{j i}(\beta) & y_{j j}(\beta)
\end{array}\right]
$$

denote a $2 \times 2$ principal submatrix of $\left(\beta L-R^{-1}\right)^{-1}$. By claim 4 ,

$$
\begin{equation*}
y_{i i}(\beta)<0 \text { and } y_{j j}(\beta)<0 \text { for all } \beta \geq 0 \tag{46}
\end{equation*}
$$

We now show that $y_{i j}(\beta)<0$ for all $\beta \geq 0$. Since $y_{i j}(0)=-r_{i j}, y_{i j}(0)<0$. If $y_{i j}(\alpha)>0$ for some $\alpha>0$, then by continuity, $y_{i j}(\delta)=0$ for some $\delta>0$. Hence by (46),

$$
\operatorname{det}\left(\left[\begin{array}{ll}
y_{i i}(\delta) & y_{i j}(\delta) \\
y_{j i}(\delta) & y_{j j}(\delta)
\end{array}\right]\right)=y_{i i}(\delta) y_{j j}(\delta)>0
$$

However by claim 4,

$$
\operatorname{det}\left(\left[\begin{array}{ll}
y_{i i}(\delta) & y_{i j}(\delta) \\
y_{j i}(\delta) & y_{j j}(\delta)
\end{array}\right]\right)<0
$$

Thus, we have a contradiction. So, $y_{i j}(\alpha) \leq 0$ for all $\alpha \geq 0$. We now conclude that every entry in $\left(L-R^{-1}\right)^{-1}$ is negative. The proof is complete.

We illustrate the above result with an example.
Example 2. Consider the graphs $H$ and $\mathcal{G}$ given in Figure 2. Let the positive scalar

(a)

(b)

Figure 2: (a) Graph $H$, and (b) Graph $\mathcal{G}$.
weights $w_{i j}$ assigned to each edge $(i, j)$ of $\mathcal{G}$ be

$$
w_{14}=w_{21}=\frac{10}{7}, w_{32}=w_{43}=5 \text { and } w_{42}=2 .
$$

The Laplacian matrix of $\mathcal{G}$ is

$$
L=\frac{1}{10}\left[\begin{array}{rrrr}
7 & 0 & 0 & -7 \\
-7 & 7 & 0 & 0 \\
0 & -2 & 2 & 0 \\
0 & -5 & -2 & 7
\end{array}\right]
$$

The resistance matrix $R$ of $H$ is

$$
R=\frac{1}{4}\left[\begin{array}{llll}
0 & 3 & 4 & 3 \\
3 & 0 & 3 & 4 \\
4 & 3 & 0 & 3 \\
3 & 4 & 3 & 0
\end{array}\right]
$$

Now,

$$
\left(R^{-1}-L\right)^{-1}=\frac{1}{8612}\left[\begin{array}{cccc}
2335 & 6555 & 7515 & 5125 \\
5125 & 2585 & 6905 & 6915 \\
7515 & 5975 & 1135 & 6905 \\
6555 & 6415 & 5975 & 2585
\end{array}\right]
$$

which is a non-negative matrix.

## Funding

The second author acknowledges the support of the Indian National Science Academy under the INSA Senior Scientist scheme.

## References

[1] Bapat RB. Graphs and matrices. New Delhi: Hindustan Book Agency; 2018.
[2] Klein DJ, Rándic M. Resistance distance. J Math Chem. 1993;12:81-95.
[3] Graham RL, Lovasz L. Distance matrix polynomials of trees. Adv in Math. 1978;29(1):60-88.
[4] Bapat R, Kirkland SJ, Neumann M. On distance matrices and Laplacians. Linear Algebra Appl. 2005;401:193-209.
[5] Balaji R, Bapat RB. Block distance matrices. Electron J Linear Algebra. 2007;16:435-443.
[6] Balaji R, Bapat RB and and Goel S. Resistance matrices of balanced directed graphs. Linear Multilinear Algebra. 2020;1-22. DOI:10.1080/03081087.2020.1748850
[7] Lewis TO, Newman TG. Pseudoinverses of positive semidefinite matrices. SIAM J Appl Math. 1968;16:701-703.
[8] Bapat RB. Determinant of the distance matrix of a tree with matrix weights. Linear Algebra Appl. 2006;416:2-7.

