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Ryszard Deszcz, Małgorzata Głogowska, Jan Jełowicki, Miroslava Petrovic-Torgasev ...+1 more authors

Institutions: University of Valenciennes and Hainaut-Cambresis

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ON RIEMANN AND WEYL COMPATIBLE TENSORS

Ryszard Deszcz, Małgorzata Głogowska, Jan Jełowicki, Miroslava Petrović-Torgašev, and Georges Zafindratafa

Dedicated to Professor Witold Roter on his eighty-first birthday

Abstract. We investigate semi-Riemannian manifolds satisfying some curvature conditions. Those conditions are strongly related to pseudosymmetry.

1. Introduction

Let ∇, R, S, $\mathcal{S}$, κ and C be the Levi-Civita connection, the Riemann–Christoffel curvature tensor, the Ricci tensor, the Ricci operator, the scalar curvature and the Weyl conformal curvature tensor of an n-dimensional semi-Riemannian manifold (M,g), respectively. For precise definitions of the symbols used, we refer to Section 2 of this paper and [**27**] and [**29**].

Let A be a symmetric $(0,2)$-tensor and B a generalized curvature tensor on a manifold (M,g), $n \geqslant 3$. According to [**72**, Definition 3.1] (cf. [**73**, Definition 7.1]) the tensor A is called B-compatible if we have on M

$$B(\mathcal{A}X,Y,Z,W) + B(\mathcal{A}Z,Y,W,X) + B(\mathcal{A}W,Y,X,Z) = 0, \tag{1.1}$$

$\mathcal{A}$ is the endomorphism of the Lie algebra $\Xi(M)$ of vector fields on M defined by

$$g(\mathcal{A}X,Y) = A(X,Y), \tag{1.2}$$

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and $X, Y, Z, W \in \Xi(M)$. In particular, a symmetric $(0,2)$-tensor A on M is said to be Riemann compatible (R-compatible) [**73**, Definition 1.1], Weyl compatible (C-compatible) [**74**, Definition 2.1], respectively, if

$$R(\mathcal{A}X, Y, Z, W) + R(\mathcal{A}Z, Y, W, X) + R(\mathcal{A}W, Y, X, Z) = 0, \tag{1.3}$$

$$C(\mathcal{A}X, Y, Z, W) + C(\mathcal{A}Z, Y, W, X) + C(\mathcal{A}W, Y, X, Z) = 0, \tag{1.4}$$

holds on M, respectively. In [**70**, Theorem 3.5] (cf. [**71**, Theorem 4.14]) it was proved that the Ricci tensor S of every Ricci-pseudosymmetric semi-Riemannian manifold ($R \cdot S = L_S Q(g, S)$, see Section 3) is R-compatible, i.e., we have on M

$$R(\mathcal{S}X, Y, Z, W) + R(\mathcal{S}Z, Y, W, X) + R(\mathcal{S}W, Y, X, Z) = 0. \tag{1.5}$$

This result was obtained already in [**2**, Lemma 3.3] and [**28**, Proposition 3.1(iv)] (cf. [**40**, Lemma 2.4]). Unfortunately, [**2**], [**28**] and [**40**] are not cited in [**70**] and [**71**]. We note that (1.5) was also obtained during the study on manifolds satisfying some other curvature conditions of pseudosymmetry type: [**8**, Lemma 3.1, eq. (19)], [**12**, Lemma 3.1, eq. (13); Proposition 3.1, eq. (22)], [**38**, Theorem 4.1, eq. (26)] and [**41**, Proposition 3.9, eq. (43)]. If the Ricci tensor S of a semi-Riemannian manifold (M, g), $n \geqslant 4$, is R-compatible, then also it is C-compatible [**72**, Proposition 3.4]. The converse statement is also true [**74**, Theorem 2.4].

In Section 3 we present definitions of quasi-Einstein, pseudosymmetric and Ricci-pseudosymmetric manifolds. In particular, we present curvature properties of manifolds with parallel Weyl tensor. In Section 4 we show that (1.1), and in particular (1.3) and (1.5), are satisfied on certain semi-Riemannian manifolds (Proposition 4.1, Theorems 4.1–4.4). Finally, in the last section we prove that some warped products manifolds also satisfy (1.5) (Theorem 5.1, Remark 5.1).

2. Preliminaries

Throughout this paper, all manifolds (M, g) are assumed to be connected, paracompact, manifolds of class C^∞ with the metric g of signature $(s, n-s)$, $0 \leqslant s \leqslant n$. The manifold (M, g) will be called a semi(pseudo)-Riemannian manifold. Clearly, if $s = 0$ or $s = n$ then (M, g) is a Riemannian manifold. If $s = 1$ or $s = n-1$, then (M, g) is a Lorentzian manifold. We define on M the endomorphisms $X \wedge_A Y$ and $\mathcal{R}(X, Y)$ of the Lie algebra $\Xi(M)$ by $(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y$ and $\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, respectively, where A is a symmetric $(0,2)$-tensor on M and $X, Y, Z \in \Xi(M)$. The Ricci tensor S, the Ricci operator $\mathcal{S}$, the scalar curvature κ and the endomorphism $\mathcal{C}(X, Y)$ are defined by $S(X, Y) = \operatorname{tr}\{Z \mapsto \mathcal{R}(Z, X)Y\}$, $g(\mathcal{S}X, Y) = S(X, Y)$, $\kappa = \operatorname{tr} \mathcal{S}$ and

$$\mathcal{C}(X, Y)Z = \mathcal{R}(X, Y)Z - \frac{1}{n-2}\Big(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y\Big)Z,$$

respectively. The $(0,4)$-tensors: G, R and C are defined by $G(X_1, \ldots, X_4) = g((X_1 \wedge_g X_2)X_3, X_4)$, $R(X_1, \ldots, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4)$, $C(X_1, \ldots, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4)$, respectively, where $X_1, X_2, \ldots \in \Xi(M)$. Further, we set $U_R = \{x \in M \mid R - (\kappa/((n-1)n))G \neq 0 \text{ at } x\}$, $U_S = \{x \in M \mid S - (\kappa/n)g \neq 0 \text{ at } x\}$ and $U_C = \{x \in M \mid C \neq 0 \text{ at } x\}$. We note that $U_S \cup U_C = U_R$.

Let $\mathcal{B}(X_1, X_2)$ be a skew-symmetric endomorphism of $\Xi(M)$ and B a $(0,4)$-tensor associated with $\mathcal{B}(X_1, X_2)$ by

$$B(X_1, X_2, X_3, X_4) = g(\mathcal{B}(X_1, X_2)X_3, X_4). \tag{2.1}$$

The tensor B is said to be a generalized curvature tensor if the following two conditions are fulfilled: $B(X_1, X_2, X_3, X_4) = B(X_3, X_4, X_1, X_2)$ and

$$B(X_1, X_2, X_3, X_4) + B(X_3, X_1, X_2, X_4) + B(X_2, X_3, X_1, X_4) = 0.$$

For the symmetric $(0,2)$-tensors E and F we define their Kulkarni–Nomizu product $E \wedge F$ (see, e.g., [**25**])

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4)\\ &\quad - E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3).\end{aligned}$$

The following tensors are generalized curvature tensors: R, C and $E \wedge F$, where E and F are symmetric $(0,2)$-tensors. We have $G = \frac{1}{2} g \wedge g$ and

$$C = R - \frac{1}{n-2}\, g \wedge S + \frac{\kappa}{(n-2)(n-1)}\, G. \tag{2.2}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of T_xM at a point $x \in M$ of a semi-Riemannian manifold (M, g), $n \geqslant 3$, and let $g(e_j, e_k) = \varepsilon_j \delta_{jk}$, $\varepsilon_j = \pm 1$, and $j, k \in \{1, 2, \ldots, n\}$. For a generalized curvature tensor B on M we denote by $\mathrm{Ric}(B)$, $\kappa(B)$ and $\mathrm{Weyl}(B)$ its scalar curvature, the Ricci tensor and the Weyl tensor, respectively. Thus at every $x \in M$ we have: $\mathrm{Ric}(B)(X, Y) = \sum_{j=1}^{n} \varepsilon_j\, B(e_j, X, Y, e_j)$, $\kappa(B) = \sum_{j=1}^{n} \varepsilon_j\, \mathrm{Ric}(B)(e_j, e_j)$ and

$$\mathrm{Weyl}(B) = B - \frac{1}{n-2}\, g \wedge \mathrm{Ric}(B) + \frac{\kappa(B)}{(n-2)(n-1)}\, G. \tag{2.3}$$

LEMMA 2.1. [**22**, Lemma 2(ii)]; cf. [**50**, p. 48]; *The Weyl tensor* $\mathrm{Weyl}(B)$ *of any generalized curvature tensor* B *on a 3-dimensional semi-Riemannian manifold* (M, g) *vanishes, i.e., on* M *we have* $B = g \wedge \mathrm{Ric}(B) - (\kappa(B)/2)\, G$.

Let $\mathcal{B}(X, Y)$ be a skew-symmetric endomorphism of $\Xi(M)$, and let B be the tensor defined by (2.1). We extend the endomorphism $\mathcal{B}(X, Y)$ to a derivation $\mathcal{B}(X, Y)\cdot$ of the algebra of tensor fields on M, assuming that it commutes with contractions and $\mathcal{B}(X, Y) \cdot f = 0$ for any smooth function f on M. Now for a $(0, k)$-tensor field T, $k \geqslant 1$, we can define the $(0, k+2)$-tensor $B \cdot T$ by

$$\begin{aligned}(B \cdot T)(X_1, \ldots, X_k, X, Y) &= (\mathcal{B}(X, Y) \cdot T)(X_1, \ldots, X_k)\\ &= -T(\mathcal{B}(X, Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, \mathcal{B}(X, Y)X_k).\end{aligned}$$

If A is a symmetric $(0,2)$-tensor, then we define the $(0, k+2)$-tensor $Q(A, T)$ by

$$\begin{aligned}Q(A, T)(X_1, \ldots, X_k, X, Y) &= (X \wedge_A Y \cdot T)(X_1, \ldots, X_k)\\ &= -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) - \cdots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).\end{aligned}$$

In this manner we obtain the $(0,6)$-tensors $B \cdot B$ and $Q(A, B)$. Substituting $\mathcal{B} = \mathcal{R}$ or $\mathcal{B} = \mathcal{C}$, $T = R$ or $T = C$ or $T = S$, $A = g$ or $A = S$ in the above formulas, we get the tensors $R \cdot R$, $R \cdot C$, $C \cdot R$, $R \cdot S$, $Q(g, R)$, $Q(S, R)$, $Q(g, C)$ and $Q(g, S)$.

Let A be a symmetric $(0, 2)$-tensor and T a $(0, k)$-tensor, $k \geqslant 2$. Following [**32**], we will call the tensor $Q(A, T)$ the Tachibana tensor of A and T, or the Tachibana tensor for short. We would like to point out that in some papers, the tensor $Q(g, R)$ is called the Tachibana tensor (see, e.g., [**57, 61, 62, 81**]).

Let B_{hijk}, T_{hijk}, A_{ij}, $(B \cdot T)_{hijklm}$ and $Q(A, T)_{hijklm}$, $h, i, \ldots, m \in \{1, \ldots, n\}$, be the local components of the generalized curvature tensors B and T, a symmetric $(0, 2)$-tensor A and the tensors $B \cdot T$ and $Q(A, T)$, respectively. We have [**32**]

$$\begin{aligned}
(B \cdot T)_{hijklm} &= g^{rs}(T_{rijk}B_{shlm} + T_{hrjk}B_{silm} \\
&\qquad + T_{hirk}B_{sjlm} + T_{hijr}B_{sklm}), \\
g^{rs}(B \cdot T)_{hrsklm} &= g^{rs}(\mathrm{Ric}(T)_{kr}B_{shlm} + \mathrm{Ric}(T)_{hr}B_{sklm}), \qquad (2.4)\\
Q(A, T)_{hijklm} &= A_{hl}T_{mijk} + A_{il}T_{hmjk} + A_{jl}T_{himk} + A_{kl}T_{hijm} \\
&\quad - A_{hm}T_{lijk} - A_{im}T_{hljk} - A_{jm}T_{hilk} - A_{km}T_{hijl}, \\
g^{rs}Q(A, T)_{hrsklm} &= A^s_l T_{skhm} - A^s_l T_{shmk} - A^s_m T_{skhl} + A^s_m T_{shlk} \qquad (2.5)\\
&\quad + Q(A, Ric(T))_{hklm}.
\end{aligned}$$

Let A be a symmetric $(0, 2)$-tensor on a semi-Riemannian manifold (M, g), $n \geqslant 3$. We define the tensors A^0, A^1, A^p, $p \geqslant 2$, and the endomorphisms (cf., [**82, 83**]) $\mathcal{A}^0$, $\mathcal{A}^1$, $\mathcal{A}^p$, $p \geqslant 2$, by $A^0 = g$, $A^1 = A$, $A^p(X, Y) = A^{p-1}(\mathcal{A}X, Y)$ and $\mathcal{A}^0 = Id$, $\mathcal{A}^1 = \mathcal{A}$, $\mathcal{A}^p X = \mathcal{A}^{p-1}(\mathcal{A}X)$, respectively, where $\mathcal{A}$ is the endomorphism related to A by (1.2) and Id the identity transformation of $\Xi(M)$.

Using the above presented definitions we can prove the following

Proposition 2.1. *If A is a symmetric $(0, 2)$-tensor and B a generalized curvature tensor on a semi-Riemannian manifold (M, g), $n \geqslant 3$, expressed by a linear combination of the tensors $A^{p_1} \wedge A^{p_2}$, $p_1, p_2 \geqslant 0$, then A^p, $p \geqslant 0$, are B-compatible.*

Let H be the second fundamental tensor of a hypersurface M, $\dim M \geqslant 3$, isometrically immersed in a conformally flat semi-Riemannian manifold N. Using Proposition 2.1 and identity (20) of [**47**] (cf. [**37**, Section 4]) we can easily prove that the tensors H^p, $p \geqslant 0$, are Weyl compatible.

Semi-Riemannian manifolds (M, g), $n \geqslant 4$, admitting generalized curvature tensors expressed by a linear combination of the tensors: $A \wedge A$, $g \wedge A$ and $g \wedge g$, where A is a symmetric $(0, 2)$-tensor on M, were investigated in [**65**]. In particular, [**65**] contains results on non-quasi Einstein and non-conformally flat manifolds having the Riemann–Christoffel curvature tensor expressed by a linear combination of the tensors $S \wedge S$, $g \wedge S$ and $g \wedge g$. Semi-Riemannian manifolds with this property are called Roter type manifolds, see [**27**] and [**53**] and references therein.

Example 2.1. We define on $M = \{(x, y, z, t) : x > 0, y > 0, z > 0, t > 0\} \subset \mathbb{R}^4$ the metric tensor g by $ds^2 = \exp(y)\, dx^2 + (x\, z)^2\, dy^2 + dz^2 - dt^2$. The Ricci tensor S of (M, g) is expressed by a linear combination of g and some other symmetric $(0, 2)$-tensors [**9**, Section 4]. Since g is a product metric of some 3-dimensional and an 1-dimensional metric, the equality $R \cdot R = Q(S, R)$ is satisfied on M [**11**, Corollary 3.2]. We also have on M: $\kappa = 1/(2\, x^2\, z^2)$, $\mathrm{rank}(S) = \cdots = \mathrm{rank}(S^4) = 3$, and

$$Q(S, S^2 \wedge S^2) = Q(S^3 - \exp(y)/(2xz^2)S^2, S \wedge S),$$
$$R = \phi_1 S \wedge S + \phi_2 S \wedge S^2 + \phi_3 S^2 \wedge S^2,$$
$$\omega(X)\mathcal{R}(Y,Z) + \omega(Y)\mathcal{R}(Z,X) + \omega(Z)\mathcal{R}(X,Y) = 0,$$
$$\phi_1 = (16x^2z^4 + z^2(4x^2+1)\exp(y))/(8z^2 + 2\exp(y)),$$
$$\phi_2 = -4x^2z^4\exp(y)/(4z^2 + \exp(y)), \quad \phi_3 = 8x^4z^6\exp(y)/(4z^2 + \exp(y)),$$

where the 1-form ω is defined by $\omega(\partial_x) = \omega(\partial_y) = 1$, $\omega(\partial_z) = \omega(\partial_t) = 0$. Finally, from Proposition 2.1 it follows that the tensors S^p, $p \geqslant 0$, are R-compatible.

3. Some special classes of semi-Riemannian manifolds

A semi-Riemannian manifold (M, g), $n \geqslant 2$, is said to be an Einstein manifold if its Ricci tensor S is proportional to g, i.e., on M we have $S = \frac{\kappa}{n}\, g$, where κ is the scalar curvature. It is well-known that the scalar curvature κ of an Einstein manifold of dimension $\geqslant 3$ is a constant. A semi-Riemannian manifold (M, g), $n \geqslant 3$, is called a quasi-Einstein manifold if at every $x \in M$ its Ricci tensor satisfies $\operatorname{rank}(S - \alpha g) \leqslant 1$, for some $\alpha \in \mathbb{R}$, i.e., the condition $S = \alpha g + \varepsilon w \otimes w$, for some $\alpha \in \mathbb{R}$, $\varepsilon = \pm 1$, $w \in T^*_x M$ holds at every $x \in U_S \subset M$ (see, e.g., [**39, 43, 54**]). Evidently, w is non-zero at every point of U_S. It is well-known that quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of conformally flat spaces. We refer to [**24, 27, 28, 37, 41, 42, 43, 54**] for results on quasi-Einstein hypersurfaces in spaces of constant curvature. Recently, quasi-Einstein manifolds were investigated amongst others in [**49, 63, 64, 68**].

An extension of the class of Einstein manifolds form Ricci-symmetric manifolds, i.e., manifolds of dimension $\geqslant 3$ with $\nabla S = 0$. An important subclass of the class of Ricci-symmetric manifolds form locally symmetric manifolds, i.e., manifolds with $\nabla R = 0$. The last two equations lead to the integrability conditions

$$\text{(3.1)} \qquad \text{(a)}\ \ R \cdot S = 0, \qquad \text{(b)}\ \ R \cdot R = 0,$$

respectively. Semi-Riemannian manifolds satisfying (3.1)(a) and (3.1)(b) are called Ricci-semisymmetric and semisymmetric [**84**], respectively. Any semisymmetric manifold is Ricci-semisymmetric. It is known that the converse statement is not true. Semisymmetric Riemannian manifolds were classified in [**84**]. Ricci-semisymmetric Riemannian manifolds were investigated, amongst others, in [**79**], see also [**69, 80**]. In those papers Ricci-semisymmetric manifolds (submanifolds) are called Ric-semisymmetric manifolds (submanifolds).

We consider now non-Riemannian semi-Riemannian manifolds (M, g), $n \geqslant 4$, with parallel Weyl tensor ($\nabla C = 0$), which are in addition non-locally symmetric ($\nabla R \neq 0$) and non-conformally flat ($C \neq 0$). Such manifolds are called essentially conformally symmetric manifolds, e.c.s. manifolds, in short (see e.g., [**15, 16**]). E.c.s. manifolds are semisymmetric manifolds satisfying $\kappa = 0$ and $Q(S, C) = 0$ [**15**, Theorems 7, 8 and 9]. In addition, on every e.c.s. manifold (M, g) we have [**16**] rank $S \leqslant 2$ and $F\,C = \frac{1}{2} S \wedge S$, where F is a function on M, called the fundamental function. Also the local structure of e.c.s. manifolds is determined [**17, 19**]. Certain e.c.s. metrics are realized on compact manifolds [**18, 20**]. E.c.s. warped products were investigated in [**59**].

A semi-Riemannian manifold (M, g), $n \geqslant 3$, is said to be pseudosymmetric [**33**] if the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent at every point of M. This is equivalent on $U_R \subset M$ to

$$R \cdot R = L_R Q(g, R), \tag{3.2}$$

where L_R is a function on this set. A pseudosymmetric manifold is called a pseudosymmetric space of constant type if the function L_R is constant [**4, 66**]. We mention that [**33**] is the first publication, in which a semi-Riemannian manifold satisfying (3.2) was called the pseudosymmetric manifold. However results on manifolds satisfying (3.2) also are contained in some papers published earlier than [**33**] (see, e.g., [**1, 55, 78**]). For instance, in [**55**, proof of Lemma 3] it was stated that fibres of semisymmetric warped products satisfy (3.2). We note that (3.2) is equivalent to $(R - L_R G) \cdot (R - L_R G) = 0$. Such expression of (3.2) was used in [**78**]. Evidently, any semisymmetric manifold is pseudosymmetric. The converse statement is not true. For instance, the Schwarzschild spacetime, the Kottler spacetime and the Reissner-Nordström spacetime satisfy (3.2) with non-zero function L_R [**48**] (see also [**34, 56**]). It is well-known that the Schwarzschild spacetime was discovered in 1916 by Schwarzschild, during his study on solutions of Einstein's equations. It seems that the Schwarzschild spacetime, the Reissner–Nordström spacetime, as well as some Friedmann–Lemaître–Robertson–Walker spacetimes are the "oldest" examples of a non-semisymmetric pseudosymmetric warped product manifolds (cf. [**35**]). We also mention that Roter type manifolds are non-quasi-Einstein and non-conformally flat pseudosymmetric (see, e.g., [**27, 53**]).

A semi-Riemannian manifold (M, g), $n \geqslant 3$, is said to be Ricci-pseudosymmetric [**21, 36**] if the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of M. This is equivalent on $U_S \subset M$ to

$$R \cdot S = L_S Q(g, S), \tag{3.3}$$

where L_S is some function on this set. A Ricci-pseudosymmetric manifold is called a Ricci-pseudosymmetric manifold of constant type if the function L_S is constant [**52**]. We note that (3.2) implies (3.3). The converse statement is not true, provided that $n \geqslant 4$, (see, e.g., [**27, 32**]). However, 3.2 and 3.3 are equivalent on every 3-dimensional manifold. Ricci-pseudosymmetric warped product manifolds were investigated, amongst others, in [**7, 21, 36, 46**]. An example of quasi-Einstein pseudosymmetric, resp. non-pseudosymmetric Ricci-pseudosymmetric, warped product manifold are given in [**37**], respectively [**43**]. Recently in [**60**] Ricci-semisymmetric and Ricci-pseudosymmetric Riemannian manifolds were called Riemannian manifolds having semi-parallel Ricci operator $\mathcal{S}$, $R(X, Y) \cdot \mathcal{S} = 0$, and pseudo-parallel Ricci operator $\mathcal{S}$, $R(X, Y) \cdot \mathcal{S} = L\,(X \wedge Y) \cdot \mathcal{S}$, respectively, where L is a function on M and $X, Y \in \Xi(M)$. Evidently, the last two conditions are equivalent to (3.1)(a) and (3.3), respectively.

We refer to [**23, 28, 35, 57, 58, 61, 62**] for further results related to those classes of manifolds. We mention only that a geometrical interpretation of (3.2) and (3.3), in the Riemannian case, is given in [**57**] and [**62**], respectively.

4. Riemann compatible tensors

Lemma 4.1. *Let A be a symmetric $(0,2)$-tensor and B, T and T_1 generalized curvature tensors on a semi-Riemannian manifold (M,g), $n \geqslant 4$, satisfying on M the condition $B \cdot T = Q(A,T) + L\,Q(g,T_1)$, where L is a function. Then*

$$\begin{aligned}(4.1)\quad & B(\mathcal{T}X,Y,Z,W) + B(\mathcal{T}Z,Y,W,X) + B(\mathcal{T}W,Y,X,Z)\\ &\quad + 3(T(\mathcal{A}X,Y,Z,W) + T(\mathcal{A}Z,Y,W,X) + T(\mathcal{A}W,Y,X,Z)) = 0\end{aligned}$$

holds on M, where $\mathcal{A}$ is defined by (1.2) *and $\mathcal{T}$ by $g(\mathcal{T}X,Y) = Ric(T)(X,Y)$.*

Proof. From the equation $(B\cdot T)_{hijklm} = Q(A,T)_{hijklm} + LQ(g,T_1)_{hijklm}$, by contraction with g^{ij} and making use of (2.4) and (2.5), we get

$$\begin{aligned}(4.2)\quad \mathcal{T}_h^s B_{sklm} + \mathcal{T}_k^s B_{shlm} &= Q(A,\mathrm{Ric}(T))_{hklm} + L\,Q(g,\mathrm{Ric}(T_1))_{hklm}\\ &\quad - \mathcal{A}_l^s R_{skmh} - \mathcal{A}_m^s R_{skhl} - \mathcal{A}_l^s R_{shmk} - \mathcal{A}_m^s R_{shkl},\end{aligned}$$

Summing (4.2) cyclically in h,l,m we obtain

$$\begin{aligned}&\mathcal{T}_h^s B_{sklm} + \mathcal{T}_l^s B_{skmh} + \mathcal{T}_m^s B_{skhl} + 2(\mathcal{A}_h^s T_{sklm} + \mathcal{A}_l^s T_{skmh} + \mathcal{A}_m^s T_{skhl})\\ &\qquad = \mathcal{A}_h^s(T_{smkl} + T_{slmk}) + \mathcal{A}_l^s(T_{shkm} + T_{smhk}) + \mathcal{A}_m^s(T_{slkh} + T_{shlk}),\\ &\mathcal{T}_h^s B_{sklm} + \mathcal{T}_l^s B_{skmh} + \mathcal{T}_m^s B_{skhl} + 3(\mathcal{A}_h^s T_{sklm} + \mathcal{A}_l^s T_{skmh} + \mathcal{A}_m^s T_{skhl}) = 0,\end{aligned}$$

completing the proof. □

Similarly, we also can prove the following

Lemma 4.2. *If A, A_1 and A_2 are symmetric $(0,2)$-tensors and B a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n \geqslant 4$, satisfying on M the condition $B\cdot A = Q(A_1,A_2)$, then* (1.1) *holds on M.*

As an immediate consequence of Lemma 2.1 we have

Lemma 4.3. *If A is a symmetric $(0,2)$-tensor and T a generalized curvature tensor on a semi-Riemannian manifold (M,g), $n = 3$, then we have on M*

$$\begin{aligned}(4.3)\quad & T(\mathcal{A}X,Y,Z,W) + T(\mathcal{A}Z,Y,W,X) + T(\mathcal{A}W,Y,X,Z)\\ &\qquad = g(X,Y)D(W,Z) + g(Z,Y)D(X,W) + g(W,Y)D(Z,X),\\ &\qquad\qquad T(\mathcal{T}X,Y,Z,W) + T(\mathcal{T}Z,Y,W,X) + T(\mathcal{T}W,Y,X,Z) = 0,\end{aligned}$$

where $D(X,Y) = \mathrm{Ric}(T)(\mathcal{A}X,Y) - \mathrm{Ric}(T)(\mathcal{A}Y,X)$, $\mathcal{T}$ is defined in Lemma 4.1 *and $\mathcal{A}$ by* (1.2).

From the above lemmas it follows

PROPOSITION 4.1. *Let (M,g), $n \geqslant 4$, be a semi-Riemannian manifold.*

(i) *If B and T are generalized curvature tensors on M satisfying on M*

$$B \cdot T = Q(\mathrm{Ric}(T), T) + LQ(g, \mathrm{Weyl}(T)), \tag{4.4}$$

where L is a function, then

$$\begin{aligned} B(\mathcal{T}X,Y,Z,W) + B(\mathcal{T}Z,Y,W,X) + B(\mathcal{T}W,Y,X,Z) \\ + 3(T(\mathcal{T}X,Y,Z,W) + T(\mathcal{T}Z,Y,W,X) + T(\mathcal{T}W,Y,X,Z)) = 0 \end{aligned} \tag{4.5}$$

holds on M, where $\mathcal{T}$ is defined in Lemma 4.1*.*

(ii) [**32**, Proposition 2.1] *If the following condition is satisfied on M*

$$R \cdot R = Q(S,R) + LQ(g,C), \tag{4.6}$$

where L is a function, then (1.5) *holds on M.*

(iii) [**10**, Lemma 2.2(i)] *If the following condition is satisfied on M*

$$R \cdot R = LQ(S,R), \tag{4.7}$$

where L is a function, then we have on M

$$(1+3L)(R(\mathcal{S}X,Y,Z,W) + R(\mathcal{S}Z,Y,W,X) + R(\mathcal{S}W,Y,X,Z)) = 0.$$

(iv) [**32**, Remark 2.1] (1.5) *is satisfied on any* 3*-dimensional manifold* (M,g)*.*

As it was shown in [**77**, Theorems 2.2 and 2.5], some curvature 2-forms on a Riemannian manifold (M,g) are closed if and only if (1.5) holds on M. For further results related to the questions related to the closedness of some forms and (1.5) see [**71**, Theorem 4.2], [**75**, Theorem 6.2] or [**76**, Theorem 3.4]. We mention that the result presented in Proposition 4.1(iii), i.e., Lemma 2.2(i) of [**10**], was also proved in [**71**, Theorem 4.17]. However, Lemma 2.2(i) of [**10**] is not cited in [**71**]. Similarly, the result presented in Proposition 4.1(iv), i.e., Remark 2.1 of [**32**], was also proved in [**73**,] (see Section 5.1). Unfortunately, [**32**] is not cited in [**73**].

Let (M,g), $n \geqslant 3$, be a semi-Riemannian manifold satisfying the condition

$$R \cdot R = LQ(S^p, R), \quad p \geqslant 0, \tag{4.8}$$

where L is a function on M. From Lemma 4.1 it follows that (4.8) implies (1.3), with $A = S + LS^p$. We mention that special para-Sasakian Riemannian manifolds satisfying (4.8) were investigated in [**82, 83**]. For instance, in [**82**] it was proved that such manifolds, under some additional assumptions, are the spaces of quasi constant curvature. Thus, in particular, they are quasi-Einstein manifolds.

Let M be a hypersurface isometrically immersed in a a semi-Riemannian space of constant curvature $N_s^{n+1}(c)$, with signature $(s, n+1-s)$, $n \geqslant 4$, where $c = \widetilde{\kappa}/(n(n+1))$ and $\widetilde{\kappa}$ are the sectional and the scalar curvature of the ambient space, respectively. It is known that $R \cdot R = Q(S,R) - ((n-2)\widetilde{\kappa})/(n(n+1))Q(g,C)$ holds on M [**47**]. Now Proposition 4.1(ii) implies (cf. [**41**, eq. (43)])

THEOREM 4.1. (1.5) *holds on every hypersurface M in $N_s^{n+1}(c)$, $n \geqslant 4$.*

Chen ideal submanifolds M isometrically immersed in Euclidean spaces [**5, 6**], satisfying some conditions of pseudosymmetry type, were investigated in [**31, 44, 45**]. Using equations (26.1)–(26.4) of [**31**] we can easily prove the following

THEOREM 4.2. (1.5) *holds on every Chen ideal submanifold M, of dimension $\geqslant 4$, isometrically immersed in a Euclidean space.*

For a $(0,6)$-tensor T on M we denote by

$$\sum_{(X_1,X_2),(X_3,X_4),(X_5,X_6)} T(X_1,X_2,X_3,X_4,X_5,X_6)$$

the sum $T(X_1,X_2,\ldots,X_6)+T(X_3,X_4,\ldots,X_6,X_1,X_2)+T(X_5,X_6,X_1,\ldots,X_4)$, where $X_1,\ldots,X_4,X,Y\in\Xi(M)$. It is well-known that on every semi-Riemannian manifold (M,g) the following identity, called the Walker identity, is satisfied

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R\cdot R)(X_1,X_2,X_3,X_4,X,Y)=0. \tag{4.9}$$

We can also investigate semi-Riemannian manifolds, of dimension $\geqslant 4$, satisfying:

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R\cdot C)(X_1,X_2,X_3,X_4,X,Y)=0, \tag{4.10}$$

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (C\cdot R)(X_1,X_2,X_3,X_4,X,Y)=0, \tag{4.11}$$

$$\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} (R\cdot C-C\cdot R)(X_1,X_2,X_3,X_4,X,Y)=0. \tag{4.12}$$

We mention that hypersurfaces in spaces of constant curvature satisfying (4.10)–(4.12) were investigated in [**29**], [**43**] and [**51**]. We also have

PROPOSITION 4.2. *Let (M,g), $n\geqslant 4$, be a semi-Riemannian manifold.*
(i) [**14**, Lemma 1] *For a symmetric $(0,2)$-tensor A and a generalized curvature tensor B on M we have $\sum_{(X_1,X_2),(X_3,X_4),(X,Y)} Q(A,B)(X_1,X_2,X_3,X_4,X,Y)=0$.*
(ii) [**29**, Proposition 4.1] *The conditions* (4.10)–(4.12) *are equivalent.*

It is easy to check that every pseudosymmetric manifold satisfies (4.10). More generally, in [**7**, Theorem 2.3] it was proved that (4.10) holds on any Ricci-pseudosymmetric manifold. In that paper it was proved that $R\cdot S=-(1/n)Q(g,A)$, holds on any semi-Riemannian manifold, of dimension $\geqslant 5$, satisfying (4.10), where A is the $(0,2)$-tensor with the local components $A_{ij}=g^{hk}(R\cdot S)_{hijk}$. Thus we have

THEOREM 4.3. (1.5) *holds on every manifold (M,g), $n\geqslant 5$, satisfying* (4.10).

In the next section we also prove that (1.5) holds on any 4-dimensional warped product satisfying (4.10). In addition, Proposition 4.2 and Theorem 4.3 imply

THEOREM 4.4. *Let (M,g), $n\geqslant 5$, be a semi-Riemannian manifold. If the tensor $R\cdot C$, or $C\cdot R$, or $R\cdot C-C\cdot R$, is expressed on M by a linear combination of the Tachibana tensors of the form $Q(A,B)$, where A is a symmetric $(0,2)$-tensor and B a generalized curvature tensor, then* (1.5) *holds on M.*

5. Warped products with Riemann compatible Ricci tensor

Let now $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, $\dim \overline{M} = p$, $\dim N = n - p$, $1 \leqslant p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let F be a positive smooth function on $\overline{M}$. The warped product $\overline{M} \times_F \widetilde{N}$ of $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g}$ defined by ([**3**], [**67**]) $\overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1)\, \pi_2^* \widetilde{g}$, where $\pi_1 : \overline{M} \times \widetilde{N} \longrightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \longrightarrow \widetilde{N}$ are the natural projections on $\overline{M}$ and $\widetilde{N}$, respectively. Let $\{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components g_{ij} of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart are the following $g_{ij} = \overline{g}_{ab}$ if $i = a$ and $j = b$, $g_{ij} = F \widetilde{g}_{\alpha\beta}$ if $i = \alpha$ and $j = \beta$, and $g_{ij} = 0$ otherwise, where $a, b, c, d, e, f \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \delta, \varepsilon, \mu \in \{p+1, \ldots, n\}$ and $h, i, j, k, l, m, r, s \in \{1, 2, \ldots, n\}$. We will denote by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$).

The local components of the Riemann–Christoffel curvature tensor R and the local components S_{ij} of the Ricci tensor S of the warped product $\overline{M} \times_F \widetilde{N}$ which may not vanish identically are the following (see, e.g., [**22, 25, 48**])

$$R_{abcd} = \overline{R}_{abcd}, \quad R_{\alpha ab\beta} = -\frac{1}{2} T_{ab} \widetilde{g}_{\alpha\beta}, \quad R_{\alpha\beta\gamma\delta} = F \widetilde{R}_{\alpha\beta\gamma\beta} - \frac{\Delta_1 F}{4} \widetilde{G}_{\alpha\beta\gamma\delta}, \tag{5.1}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}, \quad S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \Big(\frac{1}{2} \overline{g}^{ab} T_{ab} + \frac{n-p-1}{4F} \Delta_1 F \Big) \widetilde{g}_{\alpha\beta}, \tag{5.2}$$

where $\Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b$, $T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F} F_a F_b$ and $F_a = (\partial F)/(\partial x^a)$. Further, let $\mathrm{Hess}(\sqrt{F})$ be the Hessian of $\sqrt{F}$. We have $(\mathrm{Hess}(\sqrt{F}))_{ab} = 1/(2\sqrt{F})\, T_{ab}$. Using now (5.1) and (5.2) we can easily prove the following

PROPOSITION 5.1. *The manifold $\overline{M} \times_F \widetilde{N}$ satisfies (1.5) if and only if*

$$\begin{aligned} &\overline{g}^{ef} (\overline{S}_{ae} \overline{R}_{fbcd} + \overline{S}_{ce} \overline{R}_{fbda} + \overline{S}_{de} \overline{R}_{fbac}) \\ &\quad - \frac{n-p}{2F} \overline{g}^{ef} (\overline{T}_{ae} \overline{R}_{fbcd} + \overline{T}_{ce} \overline{R}_{fbda} + \overline{T}_{de} \overline{R}_{fbac}) = 0, \end{aligned} \tag{5.3}$$

$$\overline{g}^{ef} (\overline{S}_{de} (\mathrm{Hess}(\sqrt{F}))_{fa} - \overline{S}_{ae} (\mathrm{Hess}(\sqrt{F}))_{fd}) = 0, \tag{5.4}$$

$$\widetilde{g}^{\varepsilon\mu} (\widetilde{S}_{\alpha\varepsilon} \widetilde{R}_{\mu\beta\gamma\delta} + \widetilde{S}_{\gamma\varepsilon} \widetilde{R}_{\mu\beta\delta\alpha} + \widetilde{S}_{\delta\varepsilon} \widetilde{R}_{\mu\beta\alpha\gamma}) = 0. \tag{5.5}$$

As an immediate consequence of propositions 4.1 (iv) and 5.1 we get

THEOREM 5.1. (i) *The manifold $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = p \leqslant 2$, $\dim \widetilde{N} = n - p \leqslant 3$, satisfies* (1.5).

(ii) *If $(\overline{M}, \overline{g})$ is an p-dimensional manifold, $p \leqslant 2$, and $(\widetilde{N}, \widetilde{g})$ a manifold satisfying* (5.5)*, then* (1.5) *holds on $\overline{M} \times_F \widetilde{N}$.*

(iii) *If $(\overline{M}, \overline{g})$, $\dim \overline{M} = 3$, and $(\widetilde{N}, \widetilde{g})$ are manifolds satisfying* (5.4) *and* (5.5)*, respectively, then* (1.5) *holds on $\overline{M} \times_F \widetilde{N}$.*

(iv) *If $(\overline{M}, \overline{g})$, $\dim \overline{M} = p \geqslant 3$, is a space of constant curvature and $(\widetilde{N}, \widetilde{g})$ a manifold satisfying* (5.5)*, then* (1.5) *holds on $\overline{M} \times_F \widetilde{N}$.*

Remark 5.1. (i) From Theorem 5.1(i) it follows that any 4-dimensional warped product, with 1-dimensional base manifold $(\overline{M}, \overline{g})$, satisfies (1.5). In addition such warped product also satisfies (4.6) [**13**, Theorem 4.1]. Thus in particular, every generalized Robertson-Walker spacetime satisfies (1.5) and (4.6).

(ii) From Theorem 5.1(ii) it follows that if $(\overline{M}, \overline{g})$ is an 2-dimensional manifold and $(\widetilde{N}, \widetilde{g})$ an $(n-p)$-dimensional semi-Riemannian space of constant curvature, $n - p \geqslant 2$, then (1.5) holds on $\overline{M} \times_F \widetilde{N}$. Such warped product is a manifold with pseudosymmetric Weyl tensor [**30**], i.e., the condition $C \cdot C = L_C Q(g, C)$ is satisfied, where L_C is a function.

(iii) From Proposition 5.1 it follows that if $(\overline{M}, \overline{g})$, $\dim \overline{M} = p \geqslant 2$, and $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n-p \geqslant 2$, are Einstein manifolds and $\mathrm{Hess}(\sqrt{F})$ is proportional to $\overline{g}$, then (1.5) holds on $\overline{M} \times_F \widetilde{N}$.

(iv) In the previous section we proved that (1.5) holds on any manifold, of dimension $\geqslant 5$, satisfying (4.10). The condition (1.5) also holds on any 4-dimensional warped product satisfying (4.10). This is a consequence of Theorem 5.1 (i) and (iii) and the fact that (5.4) holds on any warped product satisfying (4.10) [**7**].

Example 5.1. We define on $M = \{(x, y, w, z) : x > 0, y > 0, w > 0, z > 0\} \subset \mathbb{R}^4$ the family of warped product metrics by

$$ds^2 = x^{\alpha_1} y^{\beta_1} w^{\gamma_1} dx^2 + x^{\alpha_2} y^{\beta_2} w^{\gamma_2} dy^2 + x^{\alpha_3} y^{\beta_3} w^{\gamma_3} dw^2 + x^{\alpha_4} y^{\beta_4} w^{\gamma_4} dz^2,$$

where $\alpha_1, \alpha_2, \dots, \gamma_4 \in \mathbb{R}$. Certain metrics of that family do not satisfy (1.5).

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Dept. of Math., Wrocław University of Environmental and Life Sci., Wrocław, Poland
Ryszard.Deszcz@up.wroc.pl
Malgorzata.Glogowska@up.wroc.pl
Jan.Jelowicki@up.wroc.pl

Dept. of Math., Faculty of Science, University of Kragujevac, Kragujevac, Serbia
mirapt@kg.ac.rs

Laboratoire de Mathématiques et leurs Applications de Valenciennes
Université de Valenciennes et du Hainaut-Cambrésis, Le Mont Houy, Lamath-ISTV2
59313 Valenciennes cedex 9, France
Georges.Zafindratafa@univ-valenciennes.fr