Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 562494, 15 pages
doi:10.1155/2011/562494

Research Article

On Riemann-Liouville and Caputo Derivatives

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Received 27 June 2010; Accepted 25 January 2011

Academic Editor: Daniel Czamanski

Recently, many models are formulated in terms of fractional derivatives, such as in control processing, viscoelasticity, signal processing, and anomalous diffusion. In the present paper, we further study the important properties of the Riemann-Liouville (RL) derivative, one of mostly used fractional derivatives. Some important properties of the Caputo derivative which have not been discussed elsewhere are simultaneously mentioned. The partial fractional derivatives are also introduced. These discussions are beneficial in understanding fractional calculus and modeling fractional equations in science and engineering.

1. Introduction

Fractional calculus is not a new topic; in reality it has almost the same history as that of the classical calculus [1]. Since the occurrence of fractional (or fractional-order) derivative, the theories of fractional calculus (fractional derivative plus fractional integral) has undergone a significant and even heated development, which has been primarily contributed by pure but not applied mathematicians; the reader can refer to an encyclopedic book [2] and many references cited therein. In the last few decades, however, applied scientists and engineers realized that differential equations with fractional derivative provided a natural framework for the discussion of various kinds of real problems modeled by the aid of fractional derivative, such as viscoelastic systems, signal processing, diffusion processes, control processing, fractional stochastic systems, allometry in biology and ecology ([3–17] and huge cited references therein).

Different from classical (or integer-order) derivative, there are several kinds of definitions for fractional derivatives. These definitions are generally not equivalent with each other. In the following, we introduce several definitions [7, 14].

Definition 1.1. Υ_α, the convolution kernel of order $\alpha \in R^+$ for fractional integrals, is defined by

$$\Upsilon_\alpha(t) = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)} \in L^1_{\text{loc}}(R^+), \tag{1.1}$$

where Γ is the well-known Euler Gamma function and

$$t_+^{\alpha-1} = \begin{cases} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0. \end{cases} \tag{1.2}$$

Definition 1.2. The fractional integral (or the Riemann-Liouville integral) $D_{0,t}^{-\alpha}$ with fractional order $\alpha \in R^+$ of function $x(t)$ is defined as

$$D_{0,t}^{-\alpha} x(t) = \Upsilon_\alpha * x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau. \tag{1.3}$$

Υ_α has an important convolution property (or semigroup property), that is, $\Upsilon_\alpha * \Upsilon_\beta = \Upsilon_{\alpha+\beta}$ for arbitrary $\alpha > 0$ and $\beta > 0$. This implies that $D_{0,t}^{-\alpha} \cdot D_{0,t}^{-\beta} = D_{0,t}^{-\alpha-\beta}$.

Definition 1.3. The Grünwald-Letnikov fractional derivative with fractional order α is defined by, if $x(t) \in C^m[0,t]$,

$$\begin{aligned} {}_{\text{GL}}D_{0,t}^{\alpha} x(t) &= \sum_{k=0}^{m-1} \frac{x^{(k)}(0) t^{-\alpha+k}}{\Gamma(-\alpha+k+1)} \\ &\quad + \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau, \end{aligned} \tag{1.4}$$

where $m - 1 \leq \alpha < m \in Z^+$.

This is not the original definition. The initial definition is given by a limit, that is,

$$ {}_{\text{GL}}D_{0,t}^{\alpha} x(t) = \lim_{h \to 0,\, nh=t} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{p}{k} x(t-kh). \tag{1.5}$$

The limit expression is not convenient for analysis but often used for numerical approximation.

Definition 1.4. The Riemann-Liouville derivative of fractional order α of function $x(t)$ is given as

$$\begin{aligned} {}_{\mathrm{RL}}D_{0,t}^{\alpha}x(t) &= \frac{d^m}{dt^m}D_{0,t}^{-(m-\alpha)}x(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_0^t (t-\tau)^{m-\alpha-1}x(\tau)d\tau, \end{aligned} \tag{1.6}$$

where $m-1 \le \alpha < m \in Z^{+}$.

From Definitions 1.3 and 1.4, one can see that ${}_{\mathrm{RL}}D_{0,t}^{\alpha}x(t) = {}_{\mathrm{GL}}D_{0,t}^{\alpha}x(t)$ if $x(t) \in C^m[0,t]$ which can be verified via integration by parts. This fact and the original definition of ${}_{\mathrm{GL}}D_{0,t}^{\alpha}$ provide a numerical method for fractional differential equation with Riemann-Liouville derivative [18].

Definition 1.5. The Riesz fractional derivative of fractional order α of function $x(t)$ is given as

$$\begin{aligned} {}_{R}D^{\alpha}x(t) = &-\frac{1}{2\cos(\pi\alpha/2)\Gamma(m-\alpha)} \\ &\cdot \frac{d^m}{dt^m}\left(\int_{-\infty}^{t}(t-\tau)^{m-\alpha-1}x(\tau)d\tau + (-1)^m\int_{t}^{\infty}(t-\tau)^{m-\alpha-1}x(\tau)d\tau\right), \end{aligned} \tag{1.7}$$

in which $m-1 \le \alpha < m \in Z^{+}$.

This derivative was induced by the Riemann-Liouville derivative and is useful in physics.

Definition 1.6. The Caputo derivative of fractional order α of function $x(t)$ is defined as

$$\begin{aligned} {}_{C}D_{0,t}^{\alpha}x(t) &= D_{0,t}^{-(m-\alpha)}\frac{d^m}{dt^m}x(t) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau, \end{aligned} \tag{1.8}$$

in which $m-1 < \alpha < m \in Z^{+}$.

From this definition, one can see that ${}_{C}D_{0,t}^{\alpha}x^{(n)}(t) = {}_{C}D_{0,t}^{\alpha+n}x(t)$.

Comparing this definition with the Riemann-Liouville one, functions which are derivable in the Caputo sense are much "fewer" than those which are derivable in the Riemann-Liouville sense.

The following definition is also used in mathematical analysis.

Definition 1.7. $Y_{-\alpha}$ $(\alpha \in R^{+})$ is the generalized function in the sense of Schwartz, as the unique convolution inverse of $Y_{+\alpha}$ in the convolution algebra $D'_{+}(R)$: with the use of the Dirac distribution, which is the neutral element of convolution; this reads $Y_{+\alpha} * Y_{-\alpha} = \delta$.

With the notation, the generalized fractional derivative of order α of a casual function or distribution is ${}_G D_{0,t}^{\alpha} x(t) = Y_{-\alpha} * x(t)$.

From this definition and the semigroup property of Y_α, one has ${}_G D_{0,t}^{\alpha} \cdot {}_G D_{0,t}^{\beta} = {}_G D_{0,t}^{\alpha+\beta}$, where $\alpha > 0$, $\beta > 0$. These definitions for fractional derivatives are not equivalent. There are some discussions available, say, in [9, 14].

In the realm of the fractional differential equations, Caputo derivative and Riemann-Liouville ones are mostly used. It seems that the former is more welcome since the initial value of fractional differential equation with Caputo derivative is the same as that of integer differential equation; for example, the initial value condition of fractional differential equation ${}_C D_{0,t}^{\alpha} x(t) = f(t,x)$ with $\alpha \in (0,1)$, $t > 0$ is posed as $x(0) = x_0$. But for the fractional differential equation ${}_{\mathrm{RL}} D_{0,t}^{\alpha} x(t) = f(t,x)$ with $\alpha \in (0,1)$, $t > 0$, its initial value condition involves fractional integral (and/or derivative), its initial value condition is given as $[{}_{\mathrm{RL}} D_{0,t}^{\alpha-1} x(t)]_{t=0} = x_0$ (if $\alpha \in (1,2)$, then its initial value conditions are given as $[{}_{\mathrm{RL}} D_{0,t}^{\alpha-2} x(t)]_{t=0} = x_0$, $[{}_{\mathrm{RL}} D_{0,t}^{\alpha-1} x(t)]_{t=0} = x_0^{(1)}$). Most people think that these fractional-order initial values are not easy to measure. This makes an illusion; that is, RL derivative seems to be used in less situations. But in reality, this is not the case. Physical and geometric interpretations for RL derivative can be found in [19]. It makes it possible to observe and/or measure values of RL integral and derivative(s).

On the other hand, besides the smooth requirement, Caputo derivative does not coincide with the classical derivative [9], say, for $\alpha \in (m-1, m)$, $m \in Z^+$,

$$\begin{gathered} \lim_{\alpha \to (m-1)^+} {}_C D_{0,t}^{\alpha} x(t) = x^{(m-1)}(t) - x^{(m-1)}(0), \\ \lim_{\alpha \to m^+} {}_C D_{0,t}^{\alpha} x(t) = x^{m}(t), \end{gathered} \tag{1.9}$$

while RL derivative is in-line with the classical derivative, this can be seen from the following equations for $\alpha \in (m-1, m)$, $m \in Z^+$, for $\alpha \in (m-1, m)$, $m \in Z^+$:

$$\begin{gathered} \lim_{\alpha \to (m-1)^+} {}_{\mathrm{RL}} D_{0,t}^{\alpha} x(t) = x^{(m-1)}(t), \\ \lim_{\alpha \to m^+} {}_{\mathrm{RL}} D_{0,t}^{\alpha} x(t) = x^{(m)}(t). \end{gathered} \tag{1.10}$$

Furthermore, fractional-order initial value condition(s) for RL-type differential equation can be given as usual. For example, the initial value condition $[{}_{\mathrm{RL}} D_{0,t}^{\alpha-1} x(t)]_{t=0} = x_0$ for equation ${}_{\mathrm{RL}} D_{0,t}^{\alpha} x(t) = f(t,x)$ with $\alpha \in (0,1)$, $t > 0$ can be replaced by $[t^{1-\alpha} x(t)]_{t=0} = x_0/\Gamma(\alpha)$ [20]. Of course, for $1 < \alpha \in (m-1, m)$, $m \in Z^+$, we can use the formula ${}_C D_{0,t}^{\alpha} x(t) = {}_{\mathrm{RL}} D_{0,t}^{\alpha}(x(t) - \sum_{k=0}^{m-1} t^k x^{(k)}(0)/k!)$ [9] to change corresponding fractional-order initial values into integer-order initial values. It has been found that RL derivative is very useful to characterize anomalous diffusion, Lévy flights and traps [21, 22], and so forth.

Here, we have no intention of mentioning which derivative is more widely utilized, but we must stress that every derivative has its own serviceable range. Since there are much more studies on properties of Caputo derivative [9, 10, 14], in this paper we focus on further studying the properties of RL derivative, which is helpful in understanding RL derivative and modeling fractional equations by the aid of RL derivative. And some extra properties of

Caputo derivative are also introduced. The outline of the rest paper is organized as follows. In Section 2, we further study the important properties of RL derivative which have not appeared elsewhere. In the following section, we generalize the RL derivative to the RL partial derivative. The last section includes conclusions.

2. Further Properties of RL and Caputo Derivatives

We first list the known properties [9, 10, 14] just for reference.

Property 1.

(1) For $\alpha \in (0,1)$, $x(t) \in L^1_{\text{loc}}(R^+)$,

$$\begin{aligned} {}_{\text{RL}}D^{\alpha}_{0,t}x(t) &= {}_{G}D^{\alpha}_{0,t}x(t) - \delta(t)\left[{}_{\text{RL}}D^{\alpha-1}_{0,t}x(t)\right]_{t=0} \\ &= \frac{d}{dt}\int_0^t Y_{1-\alpha}(t-\tau)x(\tau)d\tau. \end{aligned} \tag{2.1}$$

(2) For $n-1<\alpha<n\in Z^+$, $x(t) \in L^1_{\text{loc}}(R^+)$,

$$\begin{aligned} {}_{\text{RL}}D^{\alpha}_{0,t}x(t) &= {}_{G}D^{\alpha}_{0,t}\,x(t) - \delta(t)\left[{}_{\text{RL}}D^{\alpha-1}_{0,t}x(t)\right]_{t=0} - \sum_{k=1}^{n-1}Y_{-k}\left[{}_{\text{RL}}D^{\alpha-k-1}_{0,t}x(t)\right]_{t=0} \\ &= \frac{d^n}{dt^n}\int_0^t Y_{n-\alpha}(t-\tau)x(\tau)d\tau. \end{aligned} \tag{2.2}$$

(3) Assume $\alpha>0$, then $(d^n/dt^n)\,{}_{\text{RL}}D^{\alpha}_{0,t}x(t) = {}_{\text{RL}}D^{n+\alpha}_{0,t}x(t)$; $(d^n/dt^n)\,{}_{\text{RL}}D^{-\alpha}_{0,t}x(t) = {}_{\text{RL}}D^{n-\alpha}_{0,t}x(t)$ if $n-\alpha>0$, and $(d^n/dt^n)\,{}_{\text{RL}}D^{-\alpha}_{0,t}x(t) = D^{n-\alpha}_{0,t}x(t)$ if $n-\alpha<0$, hold for any $n \in Z^+$.

(4) ${}_{\text{RL}}D^{\alpha}_{0,t}\cdot D^{-\alpha}_{0,t}x(t) = x(t)$ for all $\alpha>0$. More generally, ${}_{\text{RL}}D^{\alpha}_{0,t}\cdot D^{-\beta}_{0,t}x(t) = {}_{\text{RL}}D^{\alpha-\beta}_{0,t}x(t)$ for all $\beta>0$. If $\alpha<\beta$, then ${}_{\text{RL}}D^{\alpha-\beta}_{0,t}x(t) = D^{\alpha-\beta}_{0,t}x(t)$.

(5) $D^{-\alpha}_{0,t}\cdot {}_{\text{RL}}D^{\alpha}_{0,t}x(t) = x(t) - \sum_{k=1}^{n}(t^{\alpha-k}[{}_{\text{RL}}D^{\alpha-k}_{0,t}x(t)]_{t=0}/\Gamma(\alpha-k+1))$, where $n-1<\alpha<n\in Z^+$.

(6) $D^{-n}_{0,t}\cdot {}_{\text{RL}}D^{n}_{0,t}x(t) = x(t) - \sum_{k=0}^{n-1}(t^k/k!)x^{(k)}(0)$.

(7) ${}_{\text{RL}}D^{\alpha}_{0,t}c = (t^{-\alpha}/\Gamma(1-\alpha))c$, where $\alpha>0$ and c is an arbitrary constant.

(8) $\mathcal{L}[{}_{\text{RL}}D^{\alpha}_{0,t}x(t)] = s^{\alpha}X(s) - \sum_{k=0}^{n-1}s^k[{}_{\text{RL}}D^{\alpha-k-1}_{0,t}x(t)]_{t=0}$, in which $\mathcal{L}$ is the Laplace transform, and $\mathcal{L}[x(t)] = X(s)$.

(9) ${}_{C}D^{\alpha}_{0,t}\,x(t) = {}_{\text{RL}}D^{\alpha}_{0,t}(x(t) - \sum_{k=0}^{n-1}(t^k/k!)x^{(k)}(0))$, where $n-1<\alpha<n\in Z^+$.

For Caputo derivative, ${}_{C}D_{0,t}^{\alpha}\cdot D_{0,t}^{-\beta} = {}_{C}D_{0,t}^{\alpha-\beta}$ generally does not hold for all $\alpha, \beta > 0$.
From (4) in Property 1, one has very interesting conclusions as follows.

Conclusion 1. If $x(t)$ is defined in the interval $[a, b]$ and $(1/\Gamma(\alpha)) \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau = 0$ for $\alpha > 0$ and for all $t \in [a, b]$, then $x(t) \equiv 0$.

Proof. The condition implies that $D_{0,t}^{-\alpha} x(t) = 0$. Taking the RL derivative operator in both sides and applying Property 1(4) yields $x(t) \equiv 0$ in $[a, b]$. □

Conclusion 2. The following equation

$$
\begin{gathered}
{}_{C}D_{0,t}^{\alpha} x(t) = f(x), \quad \alpha \in (0,1),\ x \in R, \\
x(0) = x_0,
\end{gathered}
\tag{2.3}
$$

does not have a periodic solution if x_0 does not solve $f(x) = 0$, where $f(x)$ is continuous.

Proof. The above equation is equivalent to the following Volterra integral equation [23]:

$$
x(t) = x_0 + D_{0,t}^{-\alpha} f(x(t)). \tag{2.4}
$$

If $x(t)$ has a periodic solution with period $T > 0$, then setting $t = T$ in the above formula and using Conclusion 1 lead to $f(x(T)) \equiv 0$; that is, x_0 solves $f(x) = 0$ due to $x(T) \equiv x_0$, which is contradictory to the assumption. So the result holds. □

But the above conclusion is not suitable for the nonautonomous fractional system with the Caputo derivative. The counterexample is constructed as follows:

$$
\begin{gathered}
{}_{C}D_{0,t}^{\alpha} x(t) = x(t) + \sum_{k=0}^{+\infty} (-1)^{k} t^{2k+1} \left[\frac{(2k+1)!!}{\Gamma(2k+2-\alpha)} t^{-\alpha} + \frac{1}{(2k+1)!!} \right], \quad \alpha \in (0,1), \\
x(0) = 0,
\end{gathered}
\tag{2.5}
$$

and has a periodic solution $x(t) = \sin t$.

Some discussions on the periodic solution of the Caputo-type fractional differential equation can be referred to [24].

For the RL derivative case, the corresponding equation does not have the integer-order initial value condition(s). Its Cauchy problem is often posed as follows [2, 20]:

$$
\begin{gathered}
{}_{\mathrm{RL}}D_{0,t}^{\alpha} x(t) = f(x), \quad \alpha \in (0,1),\ x \in R,\ t > 0, \\
{}_{\mathrm{RL}}D_{0,t}^{\alpha-1} x(t)\Big|_{t=0} = x_0 \quad \left(\text{or } t^{1-\alpha} x(t)\Big|_{t=0} = \frac{x_0}{\Gamma(\alpha)} \right).
\end{gathered}
\tag{2.6}
$$

Conclusion 3. Assume that $f(x)$ is continuous, x is a function of $t > 0$ and that $\lim_{t\to 0^+}$ is not bounded, but $\lim_{t\to 0^+} f(x(t))$ exists. Then (2.6) does not have a periodic solution.

Proof. Equation (2.6) is equivalent to the following integral equation [2, 20]:

$$x(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1} + D_{0,t}^{-\alpha} f(x(t)). \tag{2.7}$$

If $\lim_{t\to 0^+}$ is bounded, then the case is trivial so it is omitted here. We only show interests in the case that $\lim_{t\to 0^+}$ is not bounded. Suppose that (2.6) has a periodic solution with period $T > 0$, then, for arbitrary small $\delta > 0$, one has $x(\delta) = x(\delta+T)$. From (2.7), $|x(\delta+T)|$ has a bound independent of δ for arbitrary $\delta > 0$ due to the assumption of $f(x)$, but $|x(t)|$ approaches to $+\infty$ as $\delta \to 0^+$. This completes Conclusion 3. □

The previous conclusion can be very smoothly generalized to the higher-dimensional case. In the following, we further study the important nature of RL derivative.

Property 2.

(1) Composition with the integral operator: for $\alpha > 0$, $\beta > 0$, then ${}_{\mathrm{RL}}D_{0,t}^{\alpha-\beta} = {}_{\mathrm{RL}}D_{0,t}^{\alpha} \cdot D_{0,t}^{-\beta} \neq D_{0,t}^{-\beta} \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha}$.

(2) Composition with the integer derivative operator: for $\alpha \in (n-1, n)$, $n \in Z^+$, $m \in Z^+$, then $(d^m/dt^m) \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha} = {}_{\mathrm{RL}}D_{0,t}^{\alpha+m} \neq {}_{\mathrm{RL}}D_{0,t}^{\alpha} \cdot (d^m/dt^m)$.

(3) Composition with Caputo operator: for $\alpha \in (n-1, n)$, $n \in Z^+$, $(\alpha \neq)\beta \in (m-1, m)$, $m \in Z^+$, then ${}_{\mathrm{RL}}D_{0,t}^{\alpha+\beta-m}(d^m/dt^m) = {}_{\mathrm{RL}}D_{0,t}^{\alpha} \cdot {}_{C}D_{0,t}^{\beta} \neq {}_{C}D_{0,t}^{\beta} \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha} = D_{0,t}^{-(m-\beta)} \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha+m}$.

(4) Composition with the generalized fractional derivative operator: for $\alpha \in (n-1, n)$, $n \in Z^+$, $\beta > 0$, then $(d^n/dt^n)[\Upsilon_{n-\alpha-\beta} *] = {}_{\mathrm{RL}}D_{0,t}^{\alpha} \cdot {}_{G}D_{0,t}^{\beta} \neq {}_{G}D_{0,t}^{\beta} \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha} = \Upsilon_{-\beta} * {}_{\mathrm{RL}}D_{0,t}^{\alpha}$.

Proof. (1) Can be regarded as the direction conclusion of Property 1(4) and (5).

(2) Can be derived by the direct computation.

(3) Means that the RL derivative operators cannot commute with each other unless the involved initial value conditions are homogeneous [14].

(4) Can be proved by Property 1(2) and corresponding definitions. □

Although the Riemann-Liouville integral operator $D_{0,t}^{-\alpha}$ $(\alpha \in R^+)$ has the semigroup property, that is, $D_{0,t}^{-\alpha} \cdot D_{0,t}^{-\beta} = D_{0,t}^{-\alpha-\beta}$ $(\alpha > 0, \beta > 0)$, RL derivative operator ${}_{\mathrm{RL}}D_{0,t}^{\alpha}$ does not have this character, that is, ${}_{\mathrm{RL}}D_{0,t}^{\alpha} \cdot {}_{\mathrm{RL}}D_{0,t}^{\beta} \neq {}_{\mathrm{RL}}D_{0,t}^{\alpha+\beta}$ and ${}_{\mathrm{RL}}D_{0,t}^{\beta} \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha} \neq {}_{\mathrm{RL}}D_{0,t}^{\alpha+\beta}$ [14]. However, we have following interesting result.

Property 3. If $x(t) \in C^1[0,T]$, $\alpha_i \in (0,1)$ $(i = 1,2)$ (the trivial case $\alpha_i = 0$ or 1 is simple and removed here), and $\alpha_1 + \alpha_2 \in (0,1]$, then ${}_{\mathrm{RL}}D_{0,t}^{\alpha_1} \cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha_2} x(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_1+\alpha_2} x(t)$.

Proof. According to Property 1(3), one gets

$$\begin{aligned} {}_{C}D_{0,t}^{\alpha_2} \cdot {}_{C}D_{0,t}^{\alpha_1} x(t) &= {}_{C}D_{0,t}^{\alpha_2}\left\{ {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}[x(t) - x(0)] \right\} \\ &= {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}\left\{ {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}[x(t) - x(0)] - {}_{C}D_{0,t}^{\alpha_1} x(t)\Big|_{t=0} \right\} \end{aligned}$$

$$= {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}\left\{ {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}[x(t)-x(0)]\right\}$$

$$= {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}\cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x(t)-\frac{t^{-\alpha_1-\alpha_2}}{\Gamma(1-\alpha_1-\alpha_2)}x(0). \tag{2.8}$$

Similarly,

$${}_{C}D_{0,t}^{\alpha_2}\cdot {}_{C}D_{0,t}^{\alpha_1}\,x(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}\cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x(t)-\frac{t^{-\alpha_1-\alpha_2}x(0)}{\Gamma(1-\alpha_1-\alpha_2)}. \tag{2.9}$$

On the other hand,

$${}_{C}D_{0,t}^{\alpha_1+\alpha_2}\,x(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_1+\alpha_2}x(t)-\frac{t^{-\alpha_1-\alpha_2}}{\Gamma(1-\alpha_1-\alpha_2)}x(0). \tag{2.10}$$

If $\alpha_1+\alpha_2 = 1$, then $(t^{-\alpha_1-\alpha_2}/\Gamma(1-\alpha_1-\alpha_2))\,x(0)$ is automatically equal to zero because $\Gamma(0) = \infty$. By using Theorem 3.3 of [9], one obtains that ${}_{\mathrm{RL}}D_{0,t}^{\alpha_2}\cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}x(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_1}\cdot {}_{\mathrm{RL}}D_{0,t}^{\alpha_2}x(t) = {}_{\mathrm{RL}}D_{0,t}^{\alpha_1+\alpha_2}x(t)$.

The following result is for comparison nature of fractional derivatives. □

Property 4. (1) If $\alpha\in(n-1,n)$, $n\in Z^+$, and $x^{(k)}(0)\geq 0$ $(k=0,1,\ldots,n-1)$, then ${}_{C}D_{0,t}^{\alpha}\geq {}_{\mathrm{RL}}D_{0,t}^{\alpha}$.

(2) If $\alpha\in(n-1,n)$, $n\in Z^+$, ${}_{\mathrm{RL}}D_{0,t}^{\alpha}x(t)\geq {}_{\mathrm{RL}}D_{0,t}^{\alpha}y(t)$, and $[{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}x(t)]_{t=0}\geq [{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}x(t)]_{t=0}$ $(k=0,1,\ldots,n-1)$, then $x(t)\geq y(t)$. Parallelly, if $\alpha\in(n-1,n)$, $n\in Z^+$, ${}_{C}D_{0,t}^{\alpha}x(t)\geq {}_{C}D_{0,t}^{\alpha}y(t)$, and $x^{(k)}(0)\geq y^{(k)}(0)$ $(k=0,1,\ldots,n-1)$, then $x(t)\geq y(t)$.

Proof. (1) It is just the direction conclusion of Property 1(3).

(2) We only show the first part. The proof of the second part (the general case of Lemma 10 [25]) can be similarly given.

Setting ${}_{\mathrm{RL}}D_{0,t}^{\alpha}x(t)=\xi(t)+{}_{\mathrm{RL}}D_{0,t}^{\alpha}y(t)$ and taking the Laplace transform in both sides, one has

$$\begin{aligned} s^{\alpha}X(s) &= s^{\alpha}Y(s)+\mathcal{L}[\xi(t)] \\ &\quad+\sum_{k=0}^{n-1}s^{k}\left\{\left[{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}x(t)\right]_{t=0}-{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}y(t)\right]_{t=0}\right\}. \end{aligned} \tag{2.11}$$

It immediately follows from dividing by s^{α} and taking the inverse Laplace transform in both sides that

$$\begin{aligned} x(t) &= y(t)+D_{0,t}^{-\alpha}\xi(t) \\ &\quad+\sum_{k=0}^{n-1}\left\{\left[{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}x(t)\right]_{t=0}-{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}y(t)\right]_{t=0}\right\}Y_{\alpha-k}. \end{aligned} \tag{2.12}$$

The last two addends in the right side of the above equality are nonnegative. This completes the proof. □

Property 5. Let $\mathcal{A} = \{x(t) \in R, t \geq 0, x(t)$ is analytical for any $t \geq 0\}$. If $\alpha \in (0,1)$, then RL derivative operator ${}_{\mathrm{RL}}D_{0,t}^{\alpha}$ defined in $\mathcal{A}$ can be expressed as

$$
{}_{\mathrm{RL}}D_{0,t}^{\alpha} = \sum_{k=0}^{\infty} \frac{\left[d^k \cdot /dt^k\right]_{t=0}}{\Gamma(k-\alpha+1)} t^{k-\alpha}. \tag{2.13}
$$

More generally, if $n-1 < \alpha < n \in Z^{+}$, then ${}_{RL}D_{0,t}^{\alpha}$ defined in $\mathcal{A}$ has also the following form:

$$
{}_{\mathrm{RL}}D_{0,t}^{\alpha} = \sum_{k=0}^{\infty} \frac{\left[d^k \cdot /dt^k\right]_{t=0}}{\Gamma(k-\alpha+1)} t^{k-\alpha}. \tag{2.14}
$$

The proof is easy so it is left out here.

Remark 2.1. (1) For an arbitrary function $x(t) \in \mathcal{A}$, according to the expressions of Caputo differential operator [10] and RL differential operator, one can also easy get Property 1(3).

(2) Even if $x(t) \in \mathcal{A}$ (it implies that $x(0)$ exists), $[{}_{\mathrm{RL}}D_{0,t}^{\alpha} x(t)]_{t=0} (\alpha > 0)$ may not exist unless the initial value $x(0) = 0$.

The following example shows that a function is not derivable at one point in the classical sense but is derivable at the same point in RL sense.

Example 2.2. Consider

$$
x(t) = \begin{cases} 1-t, & 0 < t \leq 1, \\ t-1, & 1 < t < 1+\varepsilon,\ \varepsilon > 0. \end{cases} \tag{2.15}
$$

$x(t)$ exists (right) derivative in the classical sense at $t = 0$ but does not exist derivative in the same sense at $t = 1$. By simple calculation, one has

$$
{}_{\mathrm{RL}}D_{0,t}^{\alpha} x(t) = \begin{cases} \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} - \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)}, & 0 < t \leq 1, \\[2ex] \dfrac{t^{-\alpha}}{\Gamma(1-\alpha)} - \dfrac{t^{1-\alpha} - 2(t-1)^{1-\alpha}}{\Gamma(2-\alpha)}, & t \in (1, 1+\varepsilon), \end{cases} \tag{2.16}
$$

where $\alpha \in (0,1)$.

From the above example, $x'(0+)$ exists, but $x'(1)$ does not exist; it is quite the reverse for the RL derivative, that is, ${}_{\mathrm{RL}}D_{0,t}^{\alpha} x(t)$ does not exist at $t = 0$ but exists at $t = 1$. So we cannot in general terms say that RL derivative is more general than the classical derivative unless the initial time (or the origin) is excluded. From the above example, we also see that ${}_{\mathrm{RL}}D_{0,t}^{\alpha} x(t) > 0$ if $t < 1-\alpha$, but $x(t)$ is not monotonously increasing for $t \in (0, 1-\alpha)$. The RL derivative ${}_{\mathrm{RL}}D_{0,t}^{\alpha} x(t) > 0$ only means that $D_{0,t}^{-(1-\alpha)} x(t)$ is monotonously increasing with respect to t but does not imply that $x(t)$ is monotonously increasing. Geometrically speaking, the value ${}_{RL}D_{0,t}^{\alpha} x(t)$ at point t relates to an "area." On the other hand, its Caputo derivative exists in the whole interval $(0, 1+\varepsilon)$, although its classical derivative does not at $t = 1$. *So we cannot regard RL and Caputo derivatives as the generalization of the typical derivative in rigorous mathematical meaning.*

Definition 1.4 is sometimes called the "left RL fractional derivative." Correspondingly, the right RL fractional derivative with α order ($\alpha \in (m-1, m)$, $m \in Z^+$) is defined as

$$_{\mathrm{RL}}D_{t,b}^{\alpha}x(t) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_t^b (\tau - t)^{m-\alpha-1}x(\tau)d\tau, \tag{2.17}$$

in which $t \in (0, b)$.

The Riesz fractional derivative (Definition 1.5) is actually induced by left and right RL derivatives. The properties of right RL derivative can be similarly given.

3. Partial RL Derivative

Present studies on the anomalous diffusion are often restricted in one space dimension, say [22, 26–28] and references cited therein, where the involved RL derivative is defined with order $\alpha \in (0, 1)$ in one spatial dimension. If the anomalous diffusion phenomenon appears in R^2 or in higher spatial dimensions, how do we model it? In another words, how do we define the partial RL derivative? In this section, we first introduce the partial RL derivatives which were mentioned in [2], and then we define the partial Caputo derivatives in a similar manner.

Suppose $\alpha_i \in (0, 1)$, $i = 1, 2$, $\alpha = \alpha_1 + \alpha_2$. If we define

$$\begin{aligned}
{}_{\mathrm{RL}}\partial_{x_1^{\alpha_1}\,x_2^{\alpha_2}}^{\alpha_1+\alpha_2}u(x_1, x_2) &= \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}u(x_1, x_2)\right) \\
&= \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\left(\frac{1}{\Gamma(1-\alpha_1)}\frac{\partial}{\partial x_1}\int_0^{x_1}(x_1 - \xi)^{-\alpha_1}u(\xi, x_2)d\xi\right) \\
&= \frac{1}{\Gamma(1-\alpha_2)}\frac{\partial}{\partial x_2}\int_0^{x_2}\frac{(x_2 - \tau)^{-\alpha_2}}{\Gamma(1-\alpha_1)}\frac{\partial}{\partial x_1}\int_0^{x_1}(x_1 - \xi)^{-\alpha_1}u(\xi, \tau)d\xi\, d\tau,
\end{aligned} \tag{3.1}$$

then

$$_{\mathrm{RL}}\partial_{x_1^{\alpha_1}\,x_2^{\alpha_2}}^{\alpha_1+\alpha_2}u(x_1, x_2) = \frac{1}{\Gamma(1-\alpha_1)\cdot\Gamma(1-\alpha_2)}\frac{\partial^2}{\partial x_1 \partial x_2}\int_0^{x_2}\int_0^{x_1}(x_2 - \tau)^{-\alpha_2}(x_1 - \xi)^{-\alpha_1}u(\xi, \tau)d\xi\, d\tau. \tag{3.2}$$

According to the classical calculus, if

$$\begin{aligned}
&\frac{\partial^2}{\partial x_1 \partial x_2}\int_0^{x_2}\int_0^{x_1}(x_2 - \tau)^{-\alpha_2}(x_1 - \xi)^{-\alpha_1}u(\xi, \tau)d\xi\, d\tau, \\
&\frac{\partial^2}{\partial x_2 \partial x_1}\int_0^{x_1}\int_0^{x_2}(x_2 - \tau)^{-\alpha_2}(x_1 - \xi)^{-\alpha_1}u(\xi, \tau)d\tau\, d\xi
\end{aligned} \tag{3.3}$$

exist in a neighborhood of (x_1, x_2) and are continuous at this point (x_1, x_2), then

$$ {}_{\mathrm{RL}}\partial^{\alpha_1+\alpha_2}_{x_1^{\alpha_1}\, x_2^{\alpha_2}} u(x_1, x_2) = {}_{\mathrm{RL}}\partial^{\alpha_1+\alpha_2}_{x_2^{\alpha_2}\, x_1^{\alpha_1}} u(x_1, x_2). \tag{3.4} $$

If $\alpha = \alpha_1 + \alpha_2 \in (0, 1)$, then the above partial RL derivative can characterize subdiffusion in R^2.

The case with $\alpha_1 = 0$ or $\alpha_2 = 0$ was simply mentioned in [7],

$$ \begin{aligned} {}_{\mathrm{RL}}\partial^{\alpha_2}_{x_2^{\alpha_2}} u(x_1, x_2) &= \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} u(x_1, x_2) \\ &= \frac{1}{\Gamma(1-\alpha_2)} \frac{\partial}{\partial x_2} \int_0^{x_2} (x_2 - \tau)^{-\alpha_2} u(x_1, \tau) d\tau, \\ {}_{\mathrm{RL}}\partial^{\alpha_1}_{x_1^{\alpha_1}} u(x_1, x_2) &= \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} u(x_1, x_2) \\ &= \frac{1}{\Gamma(1-\alpha_1)} \frac{\partial}{\partial x_1} \int_0^{x_1} (x_1 - \xi)^{-\alpha_1} u(\xi, x_2) d\xi. \end{aligned} \tag{3.5} $$

Now we give the definition of the partial RL derivative as follows.

Definition 3.1. The partial RL derivative with order $\alpha_1+\alpha_2$ (α_1th order in x_1-direction and α_2th order in x_2 direction) is defined as follows:

$$ \begin{aligned} &{}_{\mathrm{RL}}\partial^{\alpha_1+\alpha_2}_{x_1^{\alpha_1}\, x_2^{\alpha_2}} u(x_1, x_2) \\ &\quad = \frac{1}{\Gamma(m-\alpha_1)\cdot\Gamma(n-\alpha_2)} \frac{\partial^{m+n}}{\partial x_1^m \partial x_2^n} \cdot \int_0^{x_2}\int_0^{x_1} (x_2-\tau)^{n-\alpha_2-1}(x_1-\xi)^{m-\alpha_1-1} u(\xi,\tau) d\xi d\tau, \end{aligned} \tag{3.6} $$

where $\alpha_1 \in (m-1, m)$, $\alpha_2 \in (n-1, n)$, $m, n \in Z^+$.

In the right side of the above equality, if the derivative value of the integral has no relation to partial differential sequence, then the value of the left side of the above equation does not either.

The definition in more higher-dimensional space is given in the following.

Definition 3.2. The partial RL derivative with order $\sum_{i=1}^{\ell} \alpha_i$ (α_i th order in x_i-direction, $i = 1, \ldots, \ell$) is defined as follows:

$$ \begin{aligned} &{}_{\mathrm{RL}}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots\, x_\ell^{\alpha_\ell}} u(x_1, \ldots, x_\ell) \\ &\quad = \frac{(\partial^{m_1} + \cdots + m_\ell)/\left(\partial x_1^{m_1} \cdots \partial x_\ell^{m_\ell}\right)}{\prod_{i=1}^{\ell}\Gamma(m_i-\alpha_i)} \int_0^{x_\ell} \cdots \int_0^{x_1} (x_\ell - \xi_\ell)^{m_\ell-\alpha_\ell-1} \cdots (x_1-\xi_1)^{m_1-\alpha_1-1} u d\xi_1 \cdots d\xi_\ell, \end{aligned} \tag{3.7} $$

where $\alpha_i \in (m_i - 1, m_i)$, $m_i \in Z^+$, $i = 1, \ldots, \ell$.

It is easy to show that (may refer to [9])

$$\begin{aligned}
&\lim_{\alpha_i \to m_i^-} {}_{\mathrm{RL}}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u(x_1,\ldots,x_\ell) \\
&\quad = {}_{\mathrm{RL}}\partial^{\alpha_1+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}\, x_{i+1}^{\alpha_{i+1}}\cdots x_\ell^{\alpha_\ell}} \frac{\partial^{m_i}}{\partial x_i^{m_i}} u(x_1,\ldots,x_\ell), \\
&\lim_{\alpha_i \to (m_i-1)^+} {}_{\mathrm{RL}}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u(x_1,\ldots,x_\ell) \\
&\quad = {}_{\mathrm{RL}}\partial^{\alpha_1+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}\, x_{i+1}^{\alpha_{i+1}}\cdots x_\ell^{\alpha_\ell}} \frac{\partial^{m_i-1} u(x_1,\ldots,x_\ell)}{\partial x_i^{m_i-1}}.
\end{aligned} \tag{3.8}$$

We can similarly define the partial Caputo derivative.

Definition 3.3. A two-dimensional case: the partial Caputo derivative with order $\alpha_1+\alpha_2$ (α_1th order in x_1-direction and α_2th order in x_2-direction) is defined as follows:

$$\begin{aligned}
{}_C\partial^{\alpha_1+\alpha_2}_{x_1^{\alpha_1} x_2^{\alpha_2}} u(x_1,x_2) &= \frac{1}{\Gamma(m-\alpha_1)\cdot\Gamma(n-\alpha_2)} \\
&\quad \cdot \int_0^{x_2}\int_0^{x_1} (x_2-\tau)^{n-\alpha_2-1}(x_1-\xi)^{m-\alpha_1-1} \frac{\partial^{m+n}}{\partial\xi^m \partial\tau^n} u(\xi,\tau)\, d\xi\, d\tau,
\end{aligned} \tag{3.9}$$

where $\alpha_1 \in (m-1,m)$, $\alpha_2 \in (n-1,n)$, $m, n \in Z^+$.

Definition 3.4. A higher-dimensional case: the partial Caputo derivative with order $\sum_{i=1}^{\ell}\alpha_i$ (α_ith order in x_i-direction,$i=1,\ldots,\ell$) is defined as follows:

$$\begin{aligned}
&{}_{\mathrm{RL}}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u(x_1,\ldots,x_\ell) \\
&= \frac{1}{\prod_{i=1}^{\ell}\Gamma(m_i-\alpha_i)} \\
&\quad \cdot \int_0^{x_\ell}\cdots\int_0^{x_1} (x_\ell-\xi_\ell)^{m_\ell-\alpha_\ell-1}\cdots(x_1-\xi_1)^{m_1-\alpha_1-1} \frac{\partial^{m_1}+\cdots+m_\ell}{\partial\xi_1^{m_1}\cdots\partial\xi_\ell^{m_\ell}} u(\xi_1,\ldots,\xi_\ell)\, d\xi_1\cdots d\xi_\ell,
\end{aligned} \tag{3.10}$$

where $\alpha_i \in (m_i-1,m_i)$, $m_i \in Z^+$, $i=1,\ldots,\ell$.

In the right sides of the above equalities of Definitions 3.2–3.4, if the derivative values of the integrals do not relate to partial differential sequences, then the values of the left sides of the above equations do not either.

One can also get

$$\lim_{\alpha_i \to m_i^-} {}_C\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u(x_1,\ldots,x_\ell) = {}_C\partial^{\alpha_1+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}} x_{i+1}^{\alpha_{i+1}}\cdots x_\ell^{\alpha_\ell}} \frac{\partial^{m_i}}{\partial x_i^{m_i}} u(x_1,\ldots,x_\ell),$$
$$\lim_{\alpha_i \to (m_i-1)^+} {}_C\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u(x_1,\ldots,x_\ell)$$
$$= {}_C\partial^{\alpha_1+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}} x_{i+1}^{\alpha_{i+1}}\cdots x_\ell^{\alpha_\ell}} \frac{\partial^{m_i-1} u(x_1,\ldots,x_\ell)}{\partial x_i^{m_i-1}}$$
$$- {}_C\partial^{\alpha_1+\cdots+\alpha_{i-1}+\alpha_{i+1}+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}} x_{i+1}^{\alpha_{i+1}}\cdots x_\ell^{\alpha_\ell}} \frac{\partial^{m_i-1} u(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_\ell)}{\partial x_i^{m_i-1}}. \tag{3.11}$$

By the way, the partial (RL) fractional is also defined here.

Definition 3.5. The partial (RL) integral with order $\sum_{i=1}^{\ell} \alpha_i$ (α_i th order in x_i-direction, $i = 1,\ldots,\ell$) is defined as follows:

$$_{RL}\partial^{-(\alpha_1+\cdots+\alpha_\ell)}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u(x_1,\ldots,x_\ell) = \frac{1}{\prod_{i=1}^{\ell}\Gamma(\alpha_i)} \cdot \int_0^{x_\ell}\cdots\int_0^{x_1} (x_\ell-\xi_\ell)^{\alpha_\ell-1}\cdots(x_1-\xi_1)^{\alpha_1-1} u\, d\xi_1\cdots d\xi_\ell, \tag{3.12}$$

where $\alpha_i \in R^+$, $i = 1,\ldots,\ell$.

Example 3.6. Let $u = u(x_1,\ldots,x_\ell) = x_1\cdots x_\ell$.

(1) By simple calculation, one has $_{RL}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u = \prod_{i=1}^{\ell} x_i^{1-\alpha_i}/\Gamma(2-\alpha_i)$, in which $\alpha_i > 0$, $i = 1,2,\ldots,\ell$. If there exists an $(2 \le)\alpha_i \in Z^+$, then $_{RL}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u = 0$ because $\Gamma(-k) = \infty$, $k = 0,1,2,\ldots$. This coincides with the property of classical derivative.

(2) By almost the same calculation, one has $_C\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u = \prod_{i=1}^{\ell}(x_i^{1-\alpha_i}/\Gamma(2-\alpha_i))$, in which $1 \ge \alpha_i > 0$, $i = 1,2,\ldots,\ell$. For same α_i's values, $_C\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u = {}_{RL}\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u$ due to the zero initial value condition. If there exists an i such that $\alpha_i > 1$, then $_C\partial^{\alpha_1+\cdots+\alpha_\ell}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u = 0$.

(3) One also has $\partial^{-(\alpha_1+\cdots+\alpha_\ell)}_{x_1^{\alpha_1}\cdots x_\ell^{\alpha_\ell}} u = \prod_{i=1}^{\ell}(x_i^{1+\alpha_i}/\Gamma(2+\alpha_i))$, in which $\alpha_i > 0$, $i = 1,2,\ldots,\ell$.

4. Conclusions

In this paper, we further studied the important properties of the RL derivatives. We also discussed some properties of the Caputo derivative which have not been studied elsewhere. And we generalized the fractional derivative defined in the real line to the partial fractional derivatives in higher space dimensions. How to generalize the fractional derivatives in the real plane to those in the complex plane is our future work.

Acknowledgments

The present work was supported in part by the National Natural Science Foundation of China under Grant no. 10872119 and the Key Disciplines of Shanghai Municipality under Grant no. S30104.

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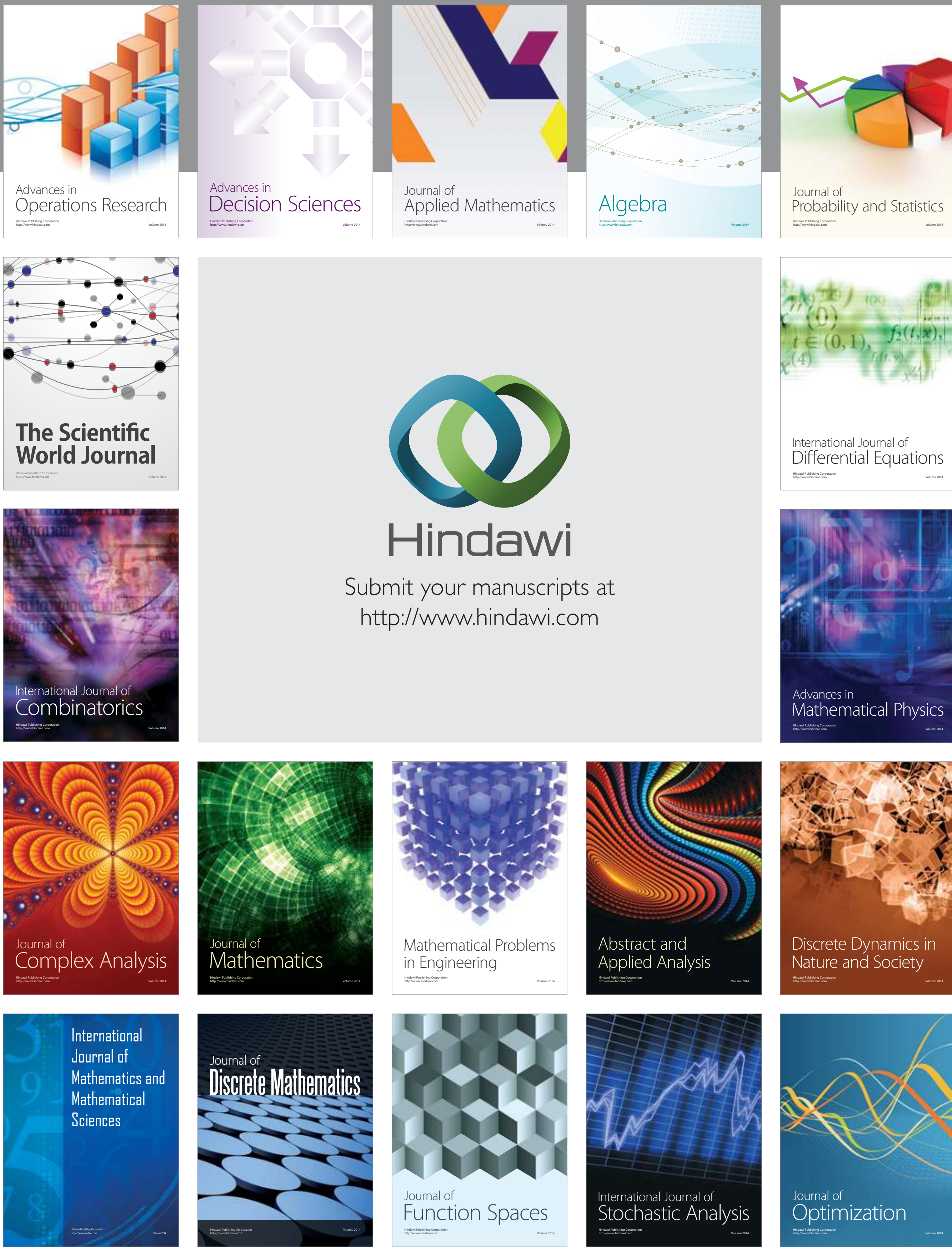

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