# ON RIEMANNIAN MANIFOLDS OF SEPARATED CURVATURE

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**1. Introduction.** We consider an $n$-dimensional Riemannian manifold whose curvature tensor satisfies

$$R_{ijkl} = \sigma S_{ij} S_{kl} \qquad (i, j, k, l = 1, 2, \ldots\ldots\ldots\ldots, n), \tag{1.1}$$

where $\sigma$ is a non-zero scalar. We shall call such a manifold a *manifold of separated curvature*, and denote it by $M_n$. There have been published several papers dealt with the manifold in consideration, though it was not their main subjects, for example, A. J. Walker [1], S. Sasaki [2], T. Ōtsuki [3], and M. Matsumoto [4]. The purpose of the present note is to investigate exclusively such a manifold.

Furthermore, providing a convenience for our discussions, we write down the following definitions due to S. S. Chern and H. Kuiper [5]:

(a) If, in an $n$-dimensional Riemannian manifold, an integer $n-\mu$ is the minimum number of linearly independent linear differential forms in terms of which the curvature forms $\Omega_{ij} = R_{ijkl}\, dx^k \wedge dx^l$ can be expressed, then we call $\mu$ the *index of nullity*.

(b) The above integer $n-\mu$ is also the maximum number of linearly independent equations in the system

$$R_{ijkl}\, dx^l = 0.$$

They define a linear subspace of dimension $\mu$ in the tangent space by the last equation. We call the subspace the *space of nullity*.

**2.** Now, by (1.1) the curvature forms $\Omega_{ij}$ in $M_n$ are expressed by the following equations:

$$\Omega_{ij} = R_{ijkl}\, dx^k \wedge dx^l = \sigma S_{ij} S_{kl}\, dx^k \wedge dx^l, \tag{2.1}$$

where $x^i$'s are local coordinates in $M_n$. Substituting (1.1) into the third identity of the curvature tensor, we get

$$S_{ij} S_{kl} + S_{ik} S_{lj} + S_{il} S_{jk} = 0. \tag{2.2}$$

In order that the quadratic differential form $S_{kl}\, dx^k \wedge dx^l$ be monomial, it is necessary and sufficient that the tensor $S_{ij}$ satisfies (2.2) [6]. So we obtain

$$S_{kl}\, dx^k \wedge dx^l = u \wedge v,$$

where $u$ and $v$ are Pfaffian forms with respect to $dx^i$. Accordingly, from (2.1) we get

$$\Omega_{ij} = \sigma S_{ij}\, u \wedge v.$$

If we denote the index of nullity at every point by $\mu$, by its definition and the above expression we have

$$\mu = n - 2.$$

Conversely, suppose that in an $n$-dimensional Riemannian manifold the index of nullity at every point equals to $n-2$. Then $n-(n-2)=2$ is the minimum number of linearly independent linear differential forms in terms of which $\Omega_{ij}$ can be expressed. Accordingly, $\Omega_{ij}$ can be expressed by

$$\Omega_{ij} = \rho\, S_{ij}\, u \wedge v. \tag{2.3}$$

Putting

$$u = a_k\, dx^k, \quad v = b_l\, dx^l, \quad S'_{kl} = (a_k b_l - a_l b_k)/2,$$

(2.3) gives us the following

$$R_{ijkl} = \rho\, S_{ij}\, S'_{kl}. \tag{2.4}$$

If we use (2.4) and $R_{ijkl} = R_{klij}$, we can easily see that the following relation holds good:

$$R_{ijkl} = \sigma\, S_{ij}\, S_{kl}.$$

These considerations imply the following:

THEOREM 1. *In order that an n-dimensional Riemannian manifold is of separated curvature, it is necessary and sufficient that the index of nullity at every point equals to* $n-2$.

Because of Theorem 3 in [5] and Theorem 1, we have

THEOREM 2. *A compact Riemannian manifold of separated curvature of dimension n cannot be isometrically imbedded in a Euclidean space of dimension* $2n-3$.

Next, since in $M_n$ the index of nullity equals to $n-2$, the space of nullity is $(n-2)$-dimensional at every point. Then by Theorem 6 in [5], $M_n$ can be locally sliced into submanifolds of dimension $n-2$ which are everywhere tangent to the space of nullity and are locally flat in the induced metric. Because of this fact and [7], if the (homogeneous) holonomy group of $M_n$ fixes the space of nullity, then $M_n$ can be locally expressed as a direct product $V_2 \times E_{n-2}$, where $V_2$ is a 2-dimensional Riemannian space and $E_{n-2}$ is an $(n-2)$-dimensional Euclidean space. On the other hand, a Riemannian manifold can be locally expressed as a direct product $V_2 \times E_{n-2}$, if and only if the manifold admits one parametric holonomy group [2]. These considerations give us the following:

THEOREM 3. *If in a Riemannian manifold of separated curvature its holonomy group fixes the space of nullity, then the manifold admits one parametric holonomy group.*

Furthermore, we remark that

$$\Omega_{i\alpha} = 0 \qquad (\alpha = 1, 2, \ldots\ldots\ldots\ldots, n-2), \tag{2.5}$$

[5, (44)] and hence that

(2.6) $$S_{i\alpha} = 0.$$

Using (2.5), Gauss-Bonnet's formula gives us the following:

THEOREM 4. *In a compact orientable Riemannian manifold of separated curvature the Euler-Poincaré characteristic vanishes, in other words, by Hopf's Theorem, there exists a continuous vector field throughout.*

3. The relation (2.2) in section 2 moreover shows that $S_{ij}$ is a simple bivector. We denote the measure of simple bivector $S_{ij}$ by $S$ and the angle defined by the two simple bivectors $S_{ij}$ and $T_{ij}$ by $\theta$. Then the well-known relations $S_{ij}S^{ij} = S^2$ and $S_{ij}T^{ij} = ST\cos\theta$ hold good. We consider the sectional curvature determined by a simple bivector $T_{ij}$ and denote it by $K(T)$. $K(T)$ is given by

(3.1) $$K(T) = -\frac{\frac{1}{4}R_{ijkl}T^{ij}T^{kl}}{\frac{1}{2}T_{ij}T^{ij}} = -\frac{\sigma(S_{ij}T^{ij})^2}{2T^2} = -\frac{1}{2}\sigma S^2\cos^2\theta,$$

and in particular

(3.2) $$K(S) = -\frac{1}{2}\sigma S^2 = \frac{1}{2}R,$$

where $R$ is the scalar curvature. Consequently from (3.1) and (3.2), we find

(3.3) $$K(T) = K(S)\cos^2\theta = (R\cos^2\theta)/2.$$

Thus we have

LEMMA *In a Riemannian manifold of separated curvature with positive (negative) scalar curvature throughout, we have*

$$\frac{1}{2}R \geqq K(T) \geqq 0 \quad \left(\frac{1}{2}R \leqq K(T) \leqq 0\right).$$

From now on, we shall describe the applications of the above Lemma. First, by Theorem [8. p. 348] we get

THEOREM 5. *A complete simply connected Riemannian manifold of separated curvature with negative scalar curvature is homeomorphic to Euclidean space.*

Returning to a general Riemannian manifold, we consider $n$ mutually orthogonal unit vectors $\xi, \xi_2, \ldots, \xi_n$. Then the Ricci curvature with respect to the unit vector $\xi$ is given by

$$R_{ij}\xi^i\xi^j = \sum_{\lambda=2}^{n} K_a$$

where $K_a$ is the sectional curvature determined by a two-dimensional plane spanned by $\xi$ and $\xi_a$. If $R$ is positive (negative) everywhere, then we get

$$R_{ij}\xi^i\xi^j > 0\,(<0),$$

since by Lemma there is at least one non-zero $K_a$. Consequently, by virtue of Theorems in [9. Chap. II] we attain the following conclusions.

THEOREM 6. *In a compact Riemannian manifold of separated curvature with positive scalar curvature throughout, there exists no harmonic vector other than zero vector, and moreover, if the manifold is orientable, then the first Betti number vanishes.*

THEOREM 7. *In a compact Riemannian manifold of separated curvature with negative scalar curvature throughout, there exists neither one-parametric group of motions nor one-parametric group of affine transformations, and moreover, if the manifold is orientable, then there exists no one-parametric group of conformal transformations.*

**4.** In this section, we assume that $\sigma$ in (1.1) is $\pm 1$ without any loss of generality. In a general Riemannian manifold we consider a two-dimensional plane spanned by two vectors $\xi_1$ and $\xi_2$ at every point and denote its current Plücker coordinates by $\xi^{ij}$. Then Ruse's Riemannian complex is given by

$$R_{ijkl}\,\xi^{ij}\,\xi^{kl} = 0.$$

Returning to a manifold $M_n$ of separated curvature, the above complex turns to

$$S_{ij}\,\xi^{ij} = 0,$$

which we shall call the *special Riemannian complex.*

The necessary and sufficient condition that the holonomy group of $M_n$ fixes the special Riemannian complex is that we have

$$S_{ij\,;\,k} = S_{ij}\,\theta_k, \tag{4.1}$$

which is derived from $d(S_{ij}\,\xi^{ij}) = \theta\, S_{ij}\,\xi^{ij}$, the operator $d$ being the ordinary differentiation. By virtue of (4.1), it is easily seen that if in $M_n$ the holonomy group fixes the special Riemannian complex, then $M_n$ is of recurrent curvature.

Conversely, suppose that $M_n$ of separated curvature is of recurrent curvature. Then we see

$$(S_{ij}\,S_{kl})_{;\,m} = S_{ij}\,S_{kl}\cdot 2\,\theta_m, \tag{4.2}$$

which can be written also as

$$S_{ij}(S_{kl\,;\,m} - S_{kl}\,\theta_m) + S_{kl}(S_{ij\,;\,m} - S_{ij}\,\theta_m) = 0. \tag{4.3}$$

Since the rank of the matrix $(S_{ij})$ is equal to two by (2.6), there exists a coordinate system such that at the origin only one component $S_{n-1n}$ of $S_{ij}$ does not vanish. Referring to such a coordinate, (4.3) with $i = k = n-1$, $j = l = n$ gives

$$S_{n-1n\,;\,m} - S_{n-1n}\,\theta_m = 0.$$

Putting $i = n-1$, $j = n$; $k$, $l \neq n-1$, $n$ in (4.3), we have

$$S_{kl\,;\,m} - S_{kl}\,\theta_m = 0,$$

that is, we obtain (4,1). Consequently, by means of (4.1) the holonomy group of $M_n$ fixes the special Riemannian complex.

Differentiating (3.2) we get

$$\frac{\partial R}{\partial x^m} = -2\sigma S \frac{\partial S}{\partial x^m}.$$

On the other hand, we have

$$\frac{\partial R}{\partial x^m} = 2\sigma R \theta_m.$$

By above expressions we see that $\theta_m$ vanishes if and only if the measure of $S_{ij}$ is always constant. Thus we have the following:

THEOREM 8. *In order that a manifold of separated curvature is of recurrent curvature, it is necessary and sufficient that the holonomy group fixes the special Riemannian complex. Moreover, if and only if the measure of $S_{ij}$ is constant, a manifold in consideration is symmetric in the sense of Cartan.*

Secondly, by virtue of Ricci's formula we have

$$\begin{aligned} S_{ij;k;l} - S_{ij;l;k} &= -S_{aj} R^a{}_{ikl} - S_{ia} R^a{}_{jkl} \\ &= -\sigma g^{ab} S_{kl}(S_{aj} S_{bi} + S_{ia} S_{bj}) \\ &= -\sigma g^{ab} S_{kl}(-S_{ba} S_{ji}) = \sigma S_{kl} S_{ji} S_{ba} g^{ba} = 0. \end{aligned}$$

On the other hand we get

$$S_{ij;k;l} - S_{ij;l;k} = S_{ij}(\theta_{k;l} - \theta_{l;k}).$$

Consequently we have

$$\theta_{k;l} - \theta_{l;k} = 0,$$

that is, $\theta_k$ is a gradient vector.

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