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ON RIEMANNIAN MANIFOLDS WITH A CERTAIN CLOSED GEODESIC

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1. Introduction. Y. Tsukamoto [7] proved a theorem concerning the characterization of complete and simply connected riemannian manifold with sectional curvature K which have a closed geodesic of length $2\pi/\sqrt{k}$ where $1/4 \leqq k \leqq K \leqq 1$. But the proof seems to have some gaps. Then we consider complete, connected and class C^∞-riemannian manifolds of positive curvature with a closed geodesic of length $2\pi/\sqrt{k}$ where k is the minimum of sectional curvature of the manifold we deal with. In this paper we give an affirmative solution to the theorem in the case of even dimension.

The results obtained in this paper are as follows.

THEOREM A. *Let M be an n-dimensional complete riemannian manifold with sectional curvature K, $0<k \leqq K$, where k is a constant. Then if M admits a closed geodesic c of length $2\pi/\sqrt{k}$ which can be decomposed into a quadrangle at least one pair of whose opposite sides have not same length, M is isometric to $S^n(k)$.*

THEOREM B. *Let M be an n-dimensional complete riemannian manifold with sectional curvature K, $1/4 \leqq k \leqq K \leqq 1$, where k is a constant. Then if M admits a simple closed geodesic c of length $2\pi/\sqrt{k}$ which can be decomposed into a quadrangle all of whose lengths of sides are not equal simultaneously, M is isometric to $S^n(k)$.*

REMARK 1. In Theorem A and Theorem B, we cannot improve the condition "All lengths of sides of the quadrangle are not equal simultaneously." In fact we consider G as a cyclic group of order 3 whose generator is

$$\begin{pmatrix} R(1/3) & \\ & R(1/3) \end{pmatrix}, \text{ where } R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Let us regard the lens space $S^3(k)/G$ as M. Let $\phi: S^3(k) \to M$ be the covering projection. The image of the great circle $t \to (1/\sqrt{k}\ \cos t,\ 0,\ 1\sqrt{k}\ \sin t,\ 0)$ of $S^3(k)$ under ϕ is a closed geodesic of M with length $2\pi/\sqrt{k}$ which can be decomposed into a quadrangle with vertices $\phi(0, 0, 0, 0)$, $\phi(\pi/2, 0, \pi/2, 0)$ $\phi(\pi, 0, \pi, 0)$ and $\phi(3\pi/2, 0, 3\pi/2, 0)$. Then all sides of the quadrangle have the same length $\pi/2\sqrt{k}$. But as it is well-known, M is not isometric to $S^3(k)$.

THEOREM C. *Let M be an even dimensional, complete and simply connected riemannian manifold with sectional curvature K, $1/4 \leqq k \leqq K \leqq 1$, where k is a constant. Then if there is a closed geodesic c of length $2\pi/\sqrt{k}$, M is isometric to the sphere with constant curvature k.*

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2. Notations and definitions. Let M be an n-dimensional complete, connected and class C^∞-riemannian manifold. We denote by $S^n(k)$ an n-dimensional sphere with constant curvature k. All the geodesics considered on M and $S^n(k)$ are parametrized by the arc length measured from their origin. Let $\Lambda = \{\lambda(s)\}(0 \leqq s \leqq l)$ be such a geodesic. Then $\dot{\lambda}(s)$ denotes its tangent vector at $\lambda(s)$ and $L(\Lambda)$ its length. If there exists a number $L > 0$ such that $\lambda(s+L) = \lambda(s)$, we call Λ a closed geodesic. The geodesic Λ is said to be simple closed if in addition $\lambda(s_1) \neq \lambda(s_2)$ for $0 < s_1 < s_2 \leqq L \leqq l$, and L is said the length of the closed geodesic Λ. For each point p of M we denote by $C(p)$ the cut locus of p. Sometimes we need the maximum of the sectional curvature of M. So we normalize it such that the maximum is 1. We denote by $d(p, q)$ (respectively $\hat{d}(\hat{p}, \hat{q})$) the distance between two points p and q of M (respectively $\hat{p}$ and $\hat{q}$ of $S^n(k)$) with respect to the riemannian metric of M (respectively canonical metric of $S^n(k)$). If M is compact, we denote by $d(M)$ its diameter. By a triangle (respectively quadrangle) we always mean a geodesic triangle (quadrangle) composed of three (four) shortest geodesic arcs each of which is not constant geodesic. $K(X, Y)$ stands for the sectional curvature of the plane section spanned by X and Y. We denote by M_p the tangent space at p of M.

3. Reviews of the known results.

THEOREM 1. (Myers [2]) *Let M be a complete riemannian manifold with sectional curvature K, $0 < k \leqq K$, where k is a constant. Then M is compact and we have the inequality $d(M) \leqq \pi/\sqrt{k}$.*

THEOREM 2. (Toponogov [6]) *Let M be a complete riemannian manifold* *with sectional curvature* K, $0<k\leqq K$ *where* k *is a constant. And let* $d(M)=\pi/\sqrt{k}$ *be satisfied. Then* M *is isometric to the sphere with constant curvature* k.

THEOREM 3. (Toponogov's basic theorem of triangles; see Toponogov [6]) *Let* M *be a complete riemannian manifold with sectional curvature* K, $0<k\leqq K$, *where* k *is a constant. We denote by* $\triangle(p,q,r)$ *a triangle on* M *and by* $\triangle(\hat{p},\hat{q},\hat{r})$ *a triangle on* $S^2(k)$ *such that* $\hat{d}(\hat{p},\hat{q})=d(p,q)$, $\hat{d}(\hat{q},\hat{r})=d(q,r)$ *and* $\hat{d}(\hat{r},\hat{p})=d(r,p)$. *Then each angle of* $\triangle(p,q,r)$ *is not less than the corresponding angle of* $\triangle(\hat{p},\hat{q},\hat{r})$.

THEOREM 4. (Convexity condition; see Toponogov [6]) *Let* M *be a complete riemannian manifold with sectional curvature* K, $0<k\leqq K$. *Consider any two shortest geodesic arcs* $\Phi=\{\phi(t)\}$ $(0\leqq t\leqq t_0)$ *and* $\Sigma=\{\sigma(s)\}$ $(0\leqq s\leqq s_0)$ *issuing from the same point* p *of* M. *Let* $\triangle(\hat{p},\widehat{\sigma(s)},\widehat{\phi(t)})$ *be a triangle on* $S^n(k)$ *of the same side lengths as* $\triangle(p,\sigma(s),\phi(t))$. *We denote by* $\gamma(s,t)$ *the angle* $\measuredangle(\widehat{\phi(t)},\hat{p},\widehat{\sigma(s)})$.

If the perimeter of the triangle $\triangle(p,\sigma(s_0),\phi(t_0))$ *is less than* $2\pi/\sqrt{k}$ *then we have* $\gamma(s_1,t_1)\geqq\gamma(s_2,t_2)$, *where* $0\leqq s_1\leqq s_2\leqq s_0$ *and* $0\leqq t_1\leqq t_2\leqq t_0$.

THEOREM 5. (Rauch's comparison theorem; see Rauch [3]) *Let* M *and* $\hat{M}$ *be complete riemannian manifolds of* $\dim M=\dim\hat{M}\geqq 2$. *Take* p *in* M *and* $\hat{p}$ *in* $\hat{M}$ *and fix. Let* $\iota: M_p\to\hat{M}_{\hat{p}}$ *be an isometric isomorphism and* $\varphi: I\to M_p$ *be a piecewise differentiable curve, where* I *is a finite interval of* $\boldsymbol{R}$.

Suppose the image of φ *and* $\iota\varphi$ *are contained in the domain of* $\exp_p$ *and* $\exp_{\hat{p}}$, *respectively. Put* $S=\{t\varphi(s)\,|\,t\in[0,1),s\in I\}$. *Suppose for all* v *in* S *the inequality*;

$$K\leqq\hat{K} \tag{1}$$

holds, where K (*resp.* $\hat{K}$) *is the sectional curvature of any plane section tangent to* $\exp_p(S)$ (*resp.* $\exp_{\hat{p}}(\iota S)$) *at* $\exp_p(v)$ (*resp.* $\exp_{\hat{p}}(\iota v)$).

If $\exp_{\hat{p}}$ *has the maximal rank on* ιS, *then we have the inequality*

$$L(\exp_p\cdot\varphi)\geqq L(\exp_{\hat{p}}\cdot\iota\varphi)\,. \tag{2}$$

In particular, the equality holds in (2) *if and only if the equality holds in* (1).

THEOREM 6. (Shiohama [4]) *Let* M *be a complete riemannian manifold of* $\dim M\geqq 2$. *Suppose that the sectional curvature* K *of* M *satisfies* $0<k\leqq K\leqq 1$, *where* k *is a constant. If we have the inequality* $d(M)>\pi/2\sqrt{k}$, *then* M

is simply connected.

REMARK 2. Owing to Theorem 6 we know the riemannian manifolds in Theorem A and Theorem B are simply connected.

THEOREM 7. (Shiohama [4]) *Let M be a complete riemannian manifold of* $\dim M \geqq 2$. *Suppose that the sectional curvature K of M satisfies the inequality $0<k\leqq K\leqq 1$, where k is a constant. If there exist two points x and y in M such that $d(x,y)>\pi/2\sqrt{k}$, then we have $d(x,C(x))\geqq\pi$ and $d(y,C(y))\geqq\pi$.*

4. Proof of theorems. For convenience' sake we put "$\wedge$" over the elements of $S^n(k)$ corresponding to the elements of M without any other statements. In particular we denote by $\triangle(\hat{p},\hat{q},\hat{r})$ a triangle on $S^n(k)$ with the same side lengths as $\triangle(p,q,r)$.

I. PROOF OF THEOREM A. Suppose c is decomposed into a quadrangle whose vertices are p, q, r and s. We denote by a_1, a_2, a_3 and a_4 the length of subarcs of c between p and q, q and r, r and s, s and p, respectively. We join p and r by a shortest geodesic arc Γ and denote by L its length. Owing to Theorem 1, we know a_i and L are not greater than $\pi/\sqrt{k}$.

If any one of a_i and L is equal to $\pi/\sqrt{k}$, our theorem is concluded from Theorem 2.

Hence we have only to consider the case all a_i and L are shorter than $\pi/\sqrt{k}$. Let us attach $\triangle(\hat{p},\hat{q},\hat{r})$ outside of $\triangle(\hat{p},\hat{r},\hat{s})$. We see the constructed quadrangle $(\hat{p},\hat{q},\hat{r},\hat{s})$ is convex and its perimeter is equal to $2\pi/\sqrt{k}$. From the assumption L being shorter than $\pi/\sqrt{k}$, the quadrangle $(\hat{p},\hat{q},\hat{r},\hat{s})$ becomes a lune. Therefore we have

$$a_1+a_4=\pi/\sqrt{k}=a_2+a_3\,. \tag{1}$$

On the other hand, let us join q and s by a shortest geodesic arc. By the same argument we have

$$a_1+a_2=\pi/\sqrt{k}=a_3+a_4\,. \tag{2}$$

From (1) and (2) we get

$$a_1=a_3 \quad \text{and} \quad a_2=a_4\,.$$

This contradicts to the hypothesis of Theorem A, Hence L must be equal to $\pi/\sqrt{k}$,

II. In order to prove Theorem B and Theorem C, we prepare the following two lemmas.

LEMMA 1. *Let M be a complete connected and class C^∞-riemannian manifold with sectional curvature K, $0<k\leqq K$, where k is a constant. Let $\triangle(p,q,r)$ be a triangle on it whose perimeter is less than $2\pi/\sqrt{k}$ and $\triangle(\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})$ a triangle on $S^n(k)$ with the same side lengths as $\triangle(p,q,r)$. Suppose $\sphericalangle(q,p,r)=\sphericalangle(\boldsymbol{q},\boldsymbol{p},\boldsymbol{r})$ and take any two points x and y on the sides between p and q, and p and r, respectively. Take $\boldsymbol{x}$ and $\boldsymbol{y}$ on the sides between $\boldsymbol{p}$ and $\boldsymbol{q}$, and $\boldsymbol{p}$ and $\boldsymbol{r}$, respectively, such that $\hat{d}(\boldsymbol{p},\boldsymbol{x})=d(p,x)$ and $\hat{d}(\boldsymbol{p},\boldsymbol{y})=d(p,y)$. Then we have $d(x,y)=\hat{d}(\boldsymbol{x},\boldsymbol{y})$.*

PROOF OF LEMMA 1. In virture of Theorem 3 we get

$$\begin{aligned}\sphericalangle(\boldsymbol{x},\boldsymbol{p},\boldsymbol{y}) &= \sphericalangle(x,p,y)\\ &\geqq \sphericalangle(\hat{x},\hat{p},\hat{y})\,.\end{aligned}$$

On the other hand, from Theorem 4 we get

$$\begin{aligned}\sphericalangle(\hat{x},\hat{p},\hat{y}) &= \gamma(d(p,x),d(p,y))\\ &\geqq \gamma(d(p,q),d(p,r))\\ &= \sphericalangle(q,p,r)\\ &= \sphericalangle(x,p,y)\\ &= \sphericalangle(\boldsymbol{x},\boldsymbol{p},\boldsymbol{y})\,.\end{aligned}$$

Hence we have $\sphericalangle(\hat{x},\hat{p},\hat{y})=\sphericalangle(\boldsymbol{x},\boldsymbol{p},\boldsymbol{y})$. Therefore $\triangle(\boldsymbol{p},\boldsymbol{x},\boldsymbol{y})$ is congruent with $\triangle(\hat{p},\hat{x},\hat{y})$ and $\hat{d}(\boldsymbol{x},\boldsymbol{y})=\hat{d}(\hat{x},\hat{y})$ holds. Thus we have $d(x,y)=\hat{d}(\boldsymbol{x},\boldsymbol{y})$.

LEMMA 2. *In addition to the assumption of Lemma 1, suppose $\sphericalangle(q,p,r)$ is equal to neither 0 nor π. Then there exists a totally geodesic surface of constant curvature k with boundaries the sides between p and q, p and r and some shortest geodesic arc joining q and r.*

PROOF OF LEMMA 2. Let $\Gamma=\{\gamma(s)\}(0\leqq s\leqq d(p,q))$, $\Sigma=\{\sigma(t)\}$ $(0\leqq t\leqq d(p,r))$ and $\Theta=\{\theta(w)\}$ $(0\leqq w\leqq d(q,r))$ be the sides between p and q, p and r, and q and r, respectively, of $\triangle(p,q,r)$ such that $\gamma(0)=\sigma(0)=p$ and $\theta(0)=r$.

An idea of the proof can be the following one. Γ is a shortest geodesic arc, but q may be conjugate to p along Γ. Then we consider a strongly convex normal neighborhood U of p in M. Any two points of U can be connected by a unique shortest geodesic arc. First, using Lemma 1 and Theorem 5, we make a small surface on a triangle $\triangle(p, x, y)$ in U, where x and y are any points on Γ and Σ, respectively, sufficiently near p. Next, since $\Gamma_{[d(p,x),d(p,q)]}$, the rest of Γ, is a shortest geodesic arc and q is not conjugate to x along Γ, we can take a suitable neighborhood V of $d(x, q)\dot{\gamma}(d(p, x))$ in M_x such that V is diffeomorphic to $\exp_x(V)$ and join z, a point on the side between x and y in $\exp_x(V)$ sufficiently near x, and q by a shortest geodesic arc in $\exp_x(V)$. Then we can make again a narrow surface on $\triangle(qzx)$ in virtue of Lemma 1 and Theorem 5. Moreover we see two surfaces lie on a common surface because they are totally geodesic and have the same tangent plane along a geodesic arc. By similar arguments we make surfaces one after another.

We shall now write down in a more precise way.

For the moment, we confine our attentions to a strongly convex normal neighborhood U of p. Take two points x and y in U sufficiently near p on Γ and Σ, respectively, and take $\hat{x}$ and $\hat{y}$ on $\hat{\Gamma}$ and $\hat{\Sigma}$, respectively, such that $\hat{d}(\hat{x}, \hat{p}) = d(x, p)$ and $\hat{d}(\hat{y}, \hat{p}) = d(y, p)$. Join $\hat{x}$ and $\hat{y}$ by a shortest geodesic arc $\hat{\Lambda} = \{\hat{\lambda}(u)\}$ $(0 \leqq u \leqq \hat{d}(\hat{x}, \hat{y}))$, $\hat{\lambda}(0) = \hat{x}$. We denote by $\hat{\Phi}_u = \{\hat{\phi}_u(v)\}$ $(0 \leqq v \leqq \hat{d}(\hat{p}, \hat{\lambda}(u)))$ the shortest geodesic arc starting from p and ending at $\hat{\lambda}(u)$ for $u \in [0, \hat{d}(\hat{x}, \hat{y})]$. In order to use Theorem 5, we define an isometric isomorphism $\iota_p : S^n(k)_{\hat{p}} \to M_p$ by $\iota_p(\dot{\hat{\gamma}}(0)) = \dot{\gamma}(0)$ and $\iota_p(\dot{\hat{\sigma}}(0)) = \dot{\sigma}(0)$. Put $\Phi_u = \{\phi_u(v) = f_p \cdot \hat{\phi}_u(v)\}$, where $f_p = \exp_p \cdot \iota_p \cdot \exp_{\hat{p}}^{-1}$. Since $\dot{\hat{\phi}}_u(0)$ for each $u \in [0, \hat{d}(\hat{x}, \hat{y})]$ belongs to the tangent plane spanned by $\dot{\hat{\gamma}}(0)$ and $\dot{\hat{\sigma}}(0)$, the image of the map $(u, v) \to \phi_u(v)$ makes a surface on M. Denote by $\Lambda = \{\lambda(u) = f_p \cdot \hat{\lambda}(u)\}$ the orbit of end points of curves Φ_u. We may consider that $\lambda([0, d(x, y)])$ lies in U. $d(x, y) \leqq L(\Lambda)$ holds in general. On the other hand, in virtue of Theorem 5 and Lemma 1 we have $L(\Lambda) \leqq L(\hat{\Lambda}) = \hat{d}(\hat{x}, \hat{y}) = d(x, y)$. Therefore we have $L(\Lambda) = d(x, y)$. Hence we see Λ is the unique shortest geodesic arc and the sectional curvature determined by the tangent plane at each point of the surface $(u, v) \to \phi_u(v)$ is equal to k.

Now we will show the surface is totally geodesic. For that purpose we have only to show that the shortest geodesic arc of M joining any two points on the surface lies again on it. Take any two points x' and y' on the surface. Then x' and y' are joined by a unique shortest geodesic arc Ψ in U. Put $\hat{x}' = f_p^{-1}(x')$ and $\hat{y}' = f_p^{-1}(y')$. Then there exist uniquely $u_1 \in [0, d(x, y)]$ and $v_1 \in [0, \hat{d}(\hat{p}, \hat{\lambda}(u_1))]$ such that $\hat{x}' = \hat{\phi}_{u_1}(v_1)$. Similarly there exist uniquely $u_2 \in [0, d(x, y)]$ and $v_2 \in [0, \hat{d}(\hat{p}, \hat{\lambda}(u_2))]$ such that $\hat{y}' = \phi_{u_2}(v_2)$. Join $\hat{x}'$ and $\hat{y}'$ by a shortest geodesic arc $\widehat{\Lambda}'$. Apply Theorem 5 to $\widehat{\Lambda}'$ and $\Lambda' = f_p\cdot\widehat{\Lambda}'$, we get $d(x',y') \leqq L(\Lambda') \leqq L(\widehat{\Lambda}') = \hat{d}(\hat{x}',\hat{y}')$. Making use of Theorem 5 again, we get $d(\lambda(u_1), \lambda(u_2)) = \hat{d}(\widehat{\lambda}(u_1), \widehat{\lambda}(u_2))$. We can apply Lemma 1 to $\triangle(\lambda(u_1), p, \lambda(u_2))$ and $\triangle(\widehat{\lambda}(u_1), \hat{p}, \widehat{\lambda}(u_2))$ and get $d(x',y') = \hat{d}(\hat{x}',\hat{y}')$. Therefore Λ' is a shortest geodesic arc and must coincice with Ψ. Thus we have shown the surface is totally geodesic.

Now we turn to $\triangle(q,x,y)$. Because Γ_1, the subarc of Γ between x and q, is a shortest geodesic arc and does not contain any conjugate point to x, there exists a neighborhood V of $d(x,q)\dot{\gamma}(d(p,x))$ in M_x such that V and $\exp_x(V)$ are diffieomorphic under $\exp_x|_V$. Take $\lambda(u_1)$ in $\exp_x(V)$ sufficiently near x and join $\widehat{\lambda}(u_1)$ and $\hat{q}$ by a shortest geodesic arc Ψ_{u_1}. The isometry of $\triangle(p,x,y)$ and $\triangle(\hat{p},\hat{x},\hat{y})$ implies $\sphericalangle(q,x,y) = \sphericalangle(\hat{q},\hat{x},\hat{y})$. Applying Lemma 1 to $\triangle(p,q,r)$ and $\triangle(\hat{p},\hat{q},\hat{r})$, we get $d(y,q) = \hat{d}(\hat{y},\hat{q})$. We can now use Lemma 1 to $\triangle(q,x,y)$ and $\triangle(\hat{q},\hat{x},\hat{y})$. We define $\iota_x: S^n(k)_{\hat{x}} \to M_x$ an isometric isomorphism by $\iota_x(\dot{\widehat{\lambda}}(0)) = \dot{\lambda}(0)$ and $\iota_x(\dot{\hat{\gamma}}(\hat{d}(\hat{p},\hat{x}))) = \dot{\gamma}(d(p,x))$ and put $f_x = \exp_x\cdot\iota_x\cdot\exp_{\hat{x}}^{-1}$. We may consider that u_1 is taken so small that $\Psi_{u_1} = f_x\cdot\widehat{\Psi}_{u_1}$ is contained in $\exp_x(V)$. In virtue of Theorem 5 and Lemma 1, Ψ_{u_1} becomes a shortest geodesic arc.

By the same argument as above, we get a totally geodesic surface of constant curvature k with boundaries Ψ_{u_1}, Γ_1 and $\Lambda|_{[0,u_1]}$ for $u_1 \in (0, d(x,y))$. Since this surface and the surface $\triangle(p,x,y)$ have the same tangent space at each point of $\Lambda|_{[0,u_1]}$, these surfaces lie on a common surface.

In this way for a sequence $u_1 < u_2 < \cdots\cdot < d(x,y)$, we get totally geodesic surfaces $\triangle(q, \lambda(u_i), \lambda(u_{i+1}))$ lying on a common surface of constant curvature k with boundaries $\Psi_{u_i}, \Psi_{u_{i+1}}$ and $\Lambda|_{[u_i, u_{i+1}]}$. Then we have $\sup u_i = d(x,y)$. In fact we assume $\sup u_i = a < d(x,y)$. Choosing a subsequence, if necessary, put $\Psi_0 = \lim_{i_k\to\infty} \Psi_{i_k}$. By the isometry of $\triangle(q, \lambda(u_{i_k}), \lambda(u_{i_k+1}))$ and $\triangle(\hat{q}, \widehat{\lambda}(u_{i_k}), \widehat{\lambda}(u_{i_k+1}))$, we have $\sphericalangle(q, \lambda(u_{i_k}), y) = \sphericalangle(\hat{q}, \widehat{\lambda}(u_{i_k}), \hat{y})$ and $d(q, \lambda(u_{i_k})) = d(\hat{q}, \widehat{\lambda}(u_{i_k}))$. Therefore we get $\sphericalangle(q, \lambda(a), y) = \sphericalangle(\hat{q}, \widehat{\lambda}(a), \hat{y})$, $d(q, \lambda(a)) = \hat{d}((\hat{q}, \widehat{\lambda}(a))$ and $d(\lambda(a), y) = \hat{d}(\widehat{\lambda}(a), \hat{y})$. Hence we can apply Lemma 1 to $\triangle(q, y, \lambda(a))$ and $\triangle(\hat{q}, \hat{y}, \widehat{\lambda}(a))$. Thus we can repeat the same argument as above by using Φ_0 for Φ_a. Therefore a must be equal to $d(x,y)$.

Now we consider about $\triangle(y,q,r)$. By $d(p,y) = \hat{d}(\hat{p},\hat{y})$, $d(y,r) = \hat{d}(\hat{y},\hat{r})$ holds. By the isometry of $\triangle(q,y,p)$ and $\triangle(\hat{q},\hat{y},\hat{p})$ we have $\sphericalangle(q,y,r) = \sphericalangle(\hat{q},\hat{y},\hat{r})$. We have already known $d(y,q) = \hat{d}(\hat{y},\hat{q})$. Hence $\triangle(y,q,r)$ is in the same situation as $\triangle(p,y,r)$ and we can apply Lemma 1 to $\triangle(y,q,r)$. Therefore we get a totally geodesic surface of constant curvature k with boundaries $\Psi_{d(x,y)}$, $\Sigma|_{[t_1,t_2]}$, $d(p,y) = t_1 < t_2$, and a shortest geodesic arc $\Psi_{\sigma(t_2)}$ joining q and $\sigma(t_2)$.

Similarly we can make surfaces for a sequence $t_1 < t_2 < \cdots < d(p,r)$ and we see $\sup t_k = d(p, r)$.

REMARK 3. By the method of making surfaces, the last geodesic are Φ_r does not coincide with Θ in general.

According to Remark 3 we get the following theorem easily.

THEOREM D. *Let M be a 2-dimensional complete riemannian manfiold with sectional curvature K, $0<k\leqq K$. If there exists a simple closed geodesic of M of perimeter $2\pi/\sqrt{k}$ which is decomposed into a quadrangle, then M is isometric to $S^2(k)$.*

Hitherto we have finished preparing the proof of Theorem B and Theorem C. We now prove them.

PROOF OF THEOREM B. Let c be decomposed into a quadrangle with vertices p, q, r and s. Denote by Γ_1, Γ_2, Γ_3 and Γ_4 the subarcs between p and q, q and r, r and s, s and p, respectively, and by a_i the length of Γ_i. Join r and p by a shortest geodesic arc Γ and denote by L its length.

Suppose L is less than $\pi/\sqrt{k}$. Then we can see the quadrangle $(\hat{p}, \hat{q}, \hat{r}, \hat{s})$ becomes a lune as well as in the proof of Theorem A. Hence we have $\sphericalangle(\hat{s}, \hat{r}, \hat{p}) = \sphericalangle(s, r, p)$, $a_1 = a_3$ and $a_2 = a_4$. We may only consider the case $a_1 > \pi/2\sqrt{k}$ and $a_4 < \pi/2\sqrt{k}$ without loss of generality. From Theorem 7, we know $d(p, C(p)) \geqq \pi > a_4$. On the other hand, applying Lemma 2 to $\triangle(p, r, s)$, we get a totally geodesic surface with boundaries Γ, Γ_3 and a shortest geodesic arc Ψ_s joining s and p. We have $\sphericalangle(\dot{\psi}_s(0), -\dot{\gamma}_3(a_3)) = \sphericalangle(\hat{r}, \hat{s}, \hat{p}) < \pi$ by $L < \pi/\sqrt{k}$. Hence we know Ψ_s does not coincide with Γ_4. This is a contradiction to $d(p, C(p)) > a_4$. Therefore L must be equal to $\pi/\sqrt{k}$.

PROOF OF THEOREM C. We can assume the closed geodesic c is simple. In fact under the assumption of Theorem C, we have $d(p, C(p)) \geqq \pi$ for any point p of M. Hence the length of any closed geodesic of M must be equal to or greater than 2π. If c has a self-intersection x, the subarcs become closed geodesics because $d(x, C(x)) \geqq \pi$. Hence they must be folded and each of them must be of length $2\pi/\sqrt{k}$. This is a contradiction. We may not consider such closed geodesic as c in Theorem C.

Let us divide c into four subarcs Γ_1, Γ_2, Γ_3 and Γ_4 of the same length $\pi/2\sqrt{k}$ such that $\gamma_1(0) = \gamma_4(\pi/2\sqrt{k}) = p$, $\gamma_1(\pi/2\sqrt{k}) = \gamma_2(0) = q$, $\gamma_2(\pi/2\sqrt{k}) = \gamma_3(0) = \gamma$ and $\gamma_3(\pi/2\sqrt{k}) = \gamma_4(0) = s$. Other notations are the same as those in the proof of Theorem B. Since $\pi/2\sqrt{k} \leqq \pi$, Γ_i, $i=1, 2, 3, 4$, are shortest geodesic arcs.

Suppose L is less than $\pi/\sqrt{k}$. Let us take t on Γ_2 such that $d(p, t) = \pi$. Then there exists a totally geodesic surface with curvature k whose boundaries are Γ, $\Gamma_2|_{[\pi-\pi/2\sqrt{k}, \pi/2\sqrt{k}]}$ and a shortest geodesic arc $\Phi_t = \{\phi_t(w)\}(0 \leqq w \leqq \pi)$, $\phi_t(0) = t$, joining t and p. We know $L(\Phi_t) = \pi$. By Berger [1] we know there exists a totally geodesic surface with constant curvature 1 whose boundaries are Φ_t and the shortest geodesic arc $\Gamma_1|_{[0,\pi]}$ extending Γ_1 till t. In the tangent space at t we have

$$1 = K(\dot{\phi}_t(0), -\dot{\gamma}_1(\pi)) = K(\dot{\phi}_t(0), \dot{\gamma}_1(\pi)) = k\,.$$

This is a contradiction. Hence L must be equal to $\pi/\sqrt{k}$.

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