Tôhoku Math. Journ.
22(1970), 14-23.

ON RIEMANNIAN MANIFOLDS WITH CERTAIN CUT LOCI

HISAO NAKAGAWA AND KATSUHIRO SHIOHAMA

(Received March 15, 1969)

Introduction. In this paper we deal with an n-dimensional $(n \geqq 2)$ connected and compact Riemannian manifold M of class C^∞ whose sectional curvatures take the maximal value 1 with respect to the Riemannian metric of M. It has been studied by L. W. Green [3]*, S. Kobayashi [4], T. Ōtsuki [8] and F. W. Warner [11] to investigate the manifold structure of M with the first conjugate locus $Q(p)$ of an arbitrary point p in M satisfying suitable conditions. In particular, F. W. Warner [11] has shown that if there exists a point p in a compact and simply connected Riemannian manifold M for which each point of the spherical conjugate locus in M_p is regular, then that has the same multiplicity as conjugate points which is greater than or equal to 1, and M is homeomorphic to a sphere or has the integral cohomology ring of one of compact irreducible symmetric spaces of rank 1. For a submanifold N of M, the cut locus N' of N is by definition the set of minimal points of each point q in N along every geodesic which starts from q and whose initial tangent vector is orthogonal to N. Recently, H. Ōmori [7] has proved that if a real analytic M has a real analytic submanifold N such that the cut locus N' of N has the constant distance from N, then N' is a real analytic submanifold of M and M has a decomposition $M = D_N \cup \phi D_{N'}$, where D_N and $D_{N'}$ are normal disc bundles of N and N' respectively. Since it is well known that the cut locus $C(p)$ of a point p is not necessarily closely related to the first conjugate locus $Q(p)$, it might be significant to investigate the manifold structure of M having a point p in such a way that the cut locus of p is spherical.

In §1, we prepare the notations and definitions. In §2, we study the general properties of M with a spherical cut locus. Further additional conditions for M with a spherical cut locus are stated in §3 and §4.

1. Preliminaries. Let there be given an $n(n \geqq 2)$-dimensional connected and compact Riemannian manifold M of class C^∞ whose sectional curvature takes maximal value 1 with the metric of M. For a point p in M we denote the cut locus and the first conjugate locus of p in M by $C(p)$ and $Q(p)$ respectively. Let M_p be the tangent space at p and $\exp_p$ the exponential map of

*) Numbers in brackets refer to the bibliography at the end of this paper.

M_p onto M. We denote by C_p the set of all tangent vectors X in M_p such that the point $\exp_p X$ is the cut point to p along the geodesic $\exp_p \frac{tX}{\|X\|}$, where $\|X\|$ is the norm of X and $t \geqq 0$. C_p is called the cut locus of p in M_p. We denote also by Q_p the set of all tangent vectors Y in M_p such that $\exp_p Y$ is the first conjugate point to p along the geodesic $\exp_p \frac{tY}{\|Y\|}$, $t \geqq 0$. Throughout this paper let a geodesic be parametrized by its arc length, unless otherwise stated.

For two points p and q in M, let $\Gamma(p, q)$ be the set of all shortest geodesic segments which start from p and end at q. A geodesic loop γ at p is by definition a closed geodesic segment having the same end points as p without self intersection except p. The geodesic sphere in M with a center at x and of radius r is denoted by $S(x, r)$, and the sphere of dimension m in M_x with a center at the origin and of radius r is denoted by $S_x^m(r)$.

We denote by $P = P(X, Y)$ the plane section spanned by two vectors X and Y linearly independent on each other in M_p, and by $K(P) = K(X, Y)$ the sectional curvature corresponding to a plane section $P = P(X, Y)$, which is given by $K(X, Y) = -<R(X,Y)X, Y>/(<X, X> <Y, Y> - <X, Y>^2)$ where $<,>$ is the inner product with respect to the Riemannian metric on M and R is the Riemannian curvature tensor on M.

2. Spherical cut locus. We assume now that there exists a point p in M in such a way that each point of the cut locus $C(p)$ of p has the constant distance, say l, with respect to the Riemannian metric mentioned above. Under the condition the cut locus $C(p)$ in M is the image of an $(n-1)$-dimensional sphere $S_p^{n-1}(l)$ with a center at the origin and of radius l in M_p under the exponential map, that is, the cut locus C_p in M_p is $S_p^{n-1}(l)$. From the assumption above it follows that

$$(1) \qquad d(p, q) = d(p, C(p)) = l,$$

for any point q of $C(p)$, where d denotes the distance function on M. First of all, we prove the following ;

LEMMA 2.1. *If there exists a point p for which the cut locus C_p in M_p is an $(n-1)$-dimensional sphere with a center at the origin and of radius l, i.e., $C_p = S_p^{n-1}(l)$, and that $l < \pi$, then all the geodesic segments starting from p and of length $2l$ are geodesic loops at p.*

PROOF. By virtue of the hypothesis of the metric on M, we have $\|Y\| \geqq \pi$

for any $Y \in Q_p$, which implies together with the assumption $l<\pi$ that $C_p \cap Q_p = \emptyset$. For an arbitrary geodesic segment γ starting from p and of length $2l$, $\gamma(l)$ is a point in $C(p)$, say q. Taking account of an elementary property of the cut locus we see that there exists a geodesic segment γ^* in $\Gamma(p, q)$ different from $\gamma|[0, l]$ such that the angle $\measuredangle(\gamma'(l), \gamma^{*\prime}(l))$ at q is equal to π, where $\gamma'(l)$ denotes the tangent vector to γ at $\gamma(l)$. This implies that $\gamma|[l, 2l]$ coincides with the inverse geodesic segment γ^{*-1} of γ^*. Thus γ is a geodesic loop at p which does not intersect itself except p. Q. E. D.

PROPOSITION 2.2. *If there exists a point p for which $C_p = S_p^{n-1}(l)$, then l is greater than or equal to $\pi/2$.*

PROOF. If $l<\pi/2$, the assumption of Lemma 2.1 is satisfied, from which it follows that all the geodesic segments γ starting from p and of length $2l$ is geodesic loops at p. For such a structure of geodesic segments it is seen [5] that $\gamma(2l)$ is conjugate to $\gamma(0)$ along γ with multiplicity $n-1$. But this is a contradiction. Q. E. D.

In the case $l<\pi$, taking account of the property of Lemma 2.1 and developing the similar discussion to that of the proof of Proposition 2.2, we see that for any geodesic segment γ starting from p and of length $2l$, $p=\gamma(2l)$ is the first conjugate point to $p=\gamma(0)$ along γ with multiplicity $n-1$. Making use of the result obtained in [6], we have immediately,

THEOREM 2.3. *If there exists a point p for which $C_p = S_p^{n-1}(\pi/2)$, then M is isometric to an n-dimensional real projective space $PR^n(1)$ with constant curvature* 1.

THEOREM 2.4. *If there exists a point p for which $C_p = S_p^{n-1}(l)$ such that $\pi/2 < l < \pi$, then M has the same (co)homology group as that of PR^n and the universal covering manifold $\tilde{M}$ of M is homeomorphic to S^n.*

By virtue of Lemma 1.4 in [5], we have

COROLLARY 2.5. *If M is simply connected and there exists a point p for which $C_p = S_p^{n-1}(l)$, then l is greater than or equal to π.*

THEOREM 2.6. *If there exists a point p in M for which $C_p = S_p^{n-1}(l)$ and the cut locus $C(p)$ in M is not contained entirely in $Q(p)$ in M, then $C_p \cap Q_p = \emptyset$ and M has the same (co)homology group as that of PR^n and $\tilde{M}$ is homeomorphic to S^n.*

PROOF. Let $\widetilde{Q}_p$ be the set of all points in M_p for each point of which $\exp_p$ has not maximal rank. It is evident that $\widetilde{Q}_p$ is closed and we have $\widetilde{Q}_p \cap S_p^{n-1}(l) = Q_p \cap S_p^{n-1}(l) \subset Q_p$. This fact means that $Q_p \cap S_p^{n-1}(l)$ is a closed subset in $S_p^{n-1}(l)$. By the assumption $C(p) \not\subset Q(p)$ there exists a point $q \in C(p) \cap Q(p)^c$, that is, for any $\gamma \in \Gamma(p, q)$, q is not conjugate to p along γ. Then we have X_1 and X_2 in M_p such that $\|X_1\| = \|X_2\| = l$, $X_1 \neq X_2$ and $\exp_p X_1 = \exp_p X_2 = q$. Putting $\gamma_i(t) = \exp_p(tX_i/l)$ $(i = 1,2)$, we have $\sphericalangle(\gamma_1'(l), \gamma_2'(l)) = \pi$. By the closedness of $Q_p \cap S_p^{n-1}(l)$ there exists neighborhoods U_i of X_i in $S_p^{n-1}(l)$ such that $U_i \cap Q_p = \emptyset$ for $i=1,2$, and $\exp_p U_1 = \exp_p U_2$, where $\exp_p$ restricted to U_i is a diffeomorphism of U_i onto the image $\exp_p U_i$. Hence for any $Y_1 \in U_1 \cap C_p$, there exists $Y_2 \in U_2 \cap C_p$ such that $\exp_p Y_1 = \exp_p Y_2 \in C(p)$ and the geodesic segment σ defined by $\sigma(t) = \exp_p(tY_i/l)$ is a geodesic loop at p of length $2l$, along which $p = \sigma(2l)$ is the first conjugate point to $p = \sigma(0)$ along σ with multiplicity $n-1$.

On the other hand, we suppose that there were a point Z in $C_p \cap Q_p$. We denote the great circle of $S_p^{n-1}(l)$ connecting X and Y by $[X, Y]$. Then there exists a point X in Q_p on the great circle $[X_1, Z]$ (or $[X_2, Z]$) in such a way that X is nearest to X_1 (or X_2) on $[X_1, Z]$ (or $[X_2, Z]$) and for any interior point Y_1 of $[X_1, X]$, the vector Y_2 and the neighborhoods U_1 and U_2 mentioned above exist. By virtue of the hypothesis of X, $\exp_p X$ is the first conjugate point to p along the geodesic γ_X defined by $\gamma_X(t) = \exp_p(tX/l)$. We have the Jacobi field J_X along γ_X such that $J_X(0) = J_X(l) = 0$, which is orthogonal to γ_X. For any $Y \in [X_1, X]$ we have a family of Jacobi fields J_Y along γ_Y defined by $\gamma_Y(t) = \exp_p(tY/l)$ such that $J_Y(0) = 0$, $J_Y'(0) = J_X'(0)$ and J_Y is orthogonal to γ_Y, where $J_Y'(t)$ is the covariant derivation of $J_Y(t)$ with respect to $\gamma_Y'(t)$. Since p itself is the first conjugate point to p with multiplicity $n-1$ along γ_Y for any interior point Y of $[X_1, X]$ because of the choice of X, we have $J_Y(l) \neq 0$ and $J_Y(2l) = 0$. But the first conjugate point of p along γ_Y depends continuously on the initial condition of γ_Y. This is a contradiction. Consequently the first assertion of the theorem holds, and furthermore all geodesic segments starting from p of length $2l$ are geodesic loops at p with index 0. This implies that M is not simply connected by Lemma 1. 4 in [5] and then the proof is completed.

As a direct consequence of the theorem above, we have the following

COROLLARY 2. 7. *If there exists a point p in a simply connected M where $C_p = S_p^{n-1}(l)$ is satisfied, then the cut locus $C(p)$ in M is contained in the first conjugate locus $Q(p)$ in M.*

REMARK. It is not certain whether the following statement is true or not : If there is a point p in M for which the cut locus C_p in M_p is a sphere and $C(p)$ coincides with $Q(p)$, then M is simply connected. Recently A. D.

Weinstein [12] has shown that the following conjecture given by Rauch [9] is false in general : In a compact and simply connected Riemannian manifold, C_p and Q_p will have a common point. Corollary 2.7 shows that in our case the conjecture is affirmative.

3. Spherical cut locus of positive curvature. In this section we consider the additional condition that M has the positive curvature such that

$$0 < k \leqq K(P) \leqq 1\,, \tag{2}$$

where $K(P)$ denotes the sectional curvature of an arbitrary plane section P. By virture of the theorem due to Myers we have $d(M) \leqq \pi/\sqrt{k}$, where $d(M)$ is the diameter of M. By the assumption that there exists a point p for which $C_p = S_p^{n-1}(l)$, it is evident that $l \leqq d(M)$. The theorem of Morse-Schoenberg shows that along any geodesic γ the first conjugate point to $\gamma(0)$, say $\gamma(t_0)$, satisfies the inequality $\pi \leqq t_0 \leqq \pi/\sqrt{k}$. Making use of Corollary 2.5, we have

LEMMA 3.1. *If there is a point p for which $C_p = S_p^{n-1}(l)$ hold, then we have $\pi \leqq l \leqq \pi/\sqrt{k}$ if M is simply connected and we have $\pi/2 \leqq l \leqq \pi/2\sqrt{k}$ if M is not simply connected.*

PROOF. The case where M is simply connected is trivial. We suppose that M is not simply connected. There are at least two different points $\tilde{p}_1$ and $\tilde{p}_2$ on the universal covering manifold $\tilde{M}$ such that $\pi(\tilde{p}_1) = \pi(\tilde{p}_2) = p$, where π is the covering map. By means of the properties of the universal covering manifold we have $\tilde{\gamma} \in \Gamma(\tilde{p}_1, \tilde{p}_2)$ such that $L(\tilde{\gamma}) = d(\tilde{p}_1, \tilde{p}_2)$ and $L(\tilde{\gamma}) \leqq d(\tilde{M}) \leqq \pi/\sqrt{k}$, where $L(\tilde{\gamma})$ denotes the length of $\tilde{\gamma}$. The projection γ of $\tilde{\gamma}$ under the covering map π is a closed geodesic segment with the same extremals as p, because $\tilde{p}_1$ and $\tilde{p}_2$ are contained in the inverse image of p under π. Hence we have $2l \leqq L(\gamma) = L(\tilde{\gamma}) \leqq \pi/\sqrt{k}$. Q. E. D.

Now by the theorem of Toponogov [10], we have

THEOREM 3.2. *If there exists a point p in M of positive curvature satisfying (2) for which C_p is a sphere with radius $\pi/\sqrt{k}$, then M is isometric to $S^n(k)$ with constant curvature k.*

THEOREM 3.3. *If there exists a point p in M of positive curvature satisfying (2) at which $C_p = S_p^{n-1}(l)$ and l satisfies $\pi/2\sqrt{k} < l < \pi/\sqrt{k}$, then M is homeomorphic to S^n.*

PROOF. It suffices to show that the cut locus $C(p)$ in M consists of only one point. Take a point q in $C(p)$ and let $B(q,\varepsilon)$ be the open ball in M with a center at q and of radius ε, where $\varepsilon=l-\pi/2\sqrt{k}$. Suppose that there were a point r in $C(p)\cap B(q,\varepsilon)$ different from q. For any geodesic segment σ in $\Gamma(q,r)$, we take a point y on σ such that $d(p,y)=d(p,\sigma)$. We may assume that y lies in the interior of the segment σ because of $d(p,\sigma)\leqq l$. Making use of the triangle inequality for p,q and y, we get $l=d(p,q)\leqq d(p,y)+d(y,q)<d(p,y)+\varepsilon$, and hence we have $d(p,y)>l-\varepsilon=\pi/2\sqrt{k}$. By virtue of Proposition 3 in Berger [1], there is a point z on σ such that $d(p,y)>d(p,z)$. This is a contradiction. By the connectedness of $C(p)$ the theorem is proved completely. Q. E. D.

4. Spherical cut locus of positive curvature with $l=\pi/2\sqrt{k}$. In this section we assume now that M is a compact and connected Riemannian manifold of positive curvature satisfying (2) and there exists a point p in M for which the cut locus C_p in M_p is a sphere of radius $l=\pi/2\sqrt{k}$. In the rest of this section we develop the similar discussion to that of Berger [2], who has showed the following important theorem : If an even dimensional compact and simply connected Riemannian manifold N of (1/4)-pinching is not homeomorphic to a sphere of the same dimension as N, then N is isometric to a compact symmetric space of rank 1. We shall prove that all of the geodesic segments starting from p with length $\pi/\sqrt{k}$ are geodesic loops at p. If $C(p)$ consists of only one point, the statement above is trivial. Then we shall consider the case $C(p)\neq\{q\}$. At first we prove the following ;

LEMMA 4.1. *For any two points q and r in $C(p)$, the geodesic segment σ in $\Gamma(q,r)$ lies entirely in $C(p)$.*

PROOF. Let σ be a shortest geodesic in $\Gamma(q,r)$ such that $\sigma(0)=q$ and $\sigma(a)=r$. Then we have $a\leqq\pi/\sqrt{k}=2l$ because of the Myers' theorem. When $q=r$, the proof is trivial and hence we suppose that q is different from r. In the case $a=\pi/\sqrt{k}$, M is isometric to $S^n(k)$ by the Toponogov's theorem [10], which contradicts our assumption $C_p=S_p^{n-1}(\pi/2\sqrt{k})$. We have therefore $a<\pi/\sqrt{k}$. Suppose that there were a point x on σ lying in the interior of the geodesic segment σ such that $d(p,x)=d(p,\sigma)<\pi/2\sqrt{k}$. Without loss of generality, we may consider that $d(q,x)\leqq a/2<\pi/2\sqrt{k}$. Making use of the basic theorem on triangles of Toponogov [10], we must have $d(p,q)<l$ because the angle of segments at x is equal to $\pi/2$. Then $d(p,q)=l$ implies $d(p,x)=l$. This shows that $d(p,\sigma(t))=l$ for all $t\in[0,a]$. Q. E. D.

For any two points q and r in $C(p)$ and any $\sigma\in\Gamma(q,r)$ such that $0<L(\sigma)=a<\pi/\sqrt{k}$ and for any fixed $t\in(0,a)$ we have $\gamma_t\in\Gamma(p,\sigma(t))$ such that $\gamma_t(0)=p$, $\gamma_t(l)=\sigma(t)$ and $<\gamma_t'(l),\sigma'(t)>=0$. Let X_t be a unit parallel vector field along γ_t defined by $X_t(l)=\sigma'(t)$, then we get $<X_t(s), \gamma_t'(s)> = 0$ for all $s \in [0, l]$. Putting $Y_t(s)=X_t(s) \sin \pi s/2l$, we have a 1-parameter variation $V(s, u)$ of γ_t defined by $V(s, u)=\exp_{\gamma_t(s)}(uY_t(s))$ for all $u \in (-\varepsilon, \varepsilon)$, where ε is a sufficiently small positive number. Taking account of the fact that the variation vector field $Y_t(s)$ of the variation $V(s,u)$ is orthogonal to $\gamma'(s)$, we see that the first variation formula with respect to the variation shows that $L'(0)=0$. For the second variation $L''(0)$ we have

$$(3) \qquad L''(0)=\int_0^l (<Y_t'(s), Y_t'(s)> - K(Y_t(s), \gamma_t'(s)) < Y_t(s), Y_t(s) >)ds$$
$$\leqq \int_0^l \left(\frac{\pi^2}{4l^2} \cos^2 \frac{\pi}{2l} s - k \sin^2 \frac{\pi}{2l} s\right) ds = 0,$$

where $K(Y_t(s), \gamma_t'(s))$ is the sectional curvature of the plane section spanned by $Y_t(s)$ and $\gamma_t'(s)$.

On the other hand, $V(l, u)$ is contained entirely in $C(p)$ because of the construction of the variation, and we have therefore $L''(0) \geqq 0$. This shows that the equality of (3) holds, i. e., we have $K(Y_t(s), \gamma_t'(s))=k$ for all $s \in [0, l]$. Since k is an eigenvalue of the quadratic form $X \to <R(X, \gamma_t')\gamma_t', X>$, it follows that $R(Y_t(s), \gamma_t'(s))\gamma_t'(s)=kY_t(s)$ for all $s \in [0, l]$. This implies that $Y_t(s)$ is a Jacobi field along γ_t and $Y_t/\|Y_t\|$ is parallel along γ_t.

As t_n tends to 0, we can choose a subsequence of a sequence $\{\gamma_{t_n}'(0)\}$ converging to a unit vector V in M_p. Putting $\gamma_0(t)=\exp_p tV$, we have $\gamma_0(l)=q$. Now let X_0 be a unit parallel vector field along γ_0 defined by $X_0(l)=\sigma'(0)$, and put $Y_0(s)=X_0(s)\sin\pi s/2l$. Because of $\lim_{n\to\infty} X_{t_n}=X_0$ it follows that Y_0 is a Jacobi field along γ_0 and $K(X_0(x), \gamma_0'(s))=k$ for all $s \in [0, l]$. Then we shall prove the following ;

LEMMA 4. 2. *For any* $\gamma \in \Gamma(p, q)$, *we have*

$$(4) \qquad <\gamma'(l), \sigma'(0)> = 0\,.$$

(5) *Let* X *be a unit parallel vector field along* γ *defined by* $X(l)=\sigma'(0)$, *then we get* $K(X(s), \gamma'(s))=k$ *for all* $s \in [0, l]$.

(6) $Y(s)=X(s) \sin \pi s/2l$ *is a Jacobi field along* γ.

PROOF. Suppose that there were a geodesic segment γ in $\Gamma(p, q)$ such that $<\gamma'(l), \sigma'(0)> \neq 0$. We shall derive a contradiction. Let θ be the angle between $\gamma'(l)$ and $\sigma'(0)$ at q. Since it follows from Toponogov's theorem that θ is equal to or less than $\pi/2$, we suppose that θ is less than $\pi/2$. For the geodesic segment γ_0 and X_0, Y_0 stated above we may consider that the length of all variational curves $V_0(s, u)$ whose variation vector field is Y_0 are just equal to l. We may also consider that the variational curve $V_0(s, u)$ is a geodesic segment for all $u \in (-\varepsilon, \varepsilon)$, that is to say, $V_0(s, u)$ is defined by $V_0(s, u) = \exp_p s\left(\gamma_0'(0)\cos\frac{u}{l} + X_0(0)\sin\frac{u}{l}\right)$. Because of $\exp_{p*}\mathcal{F}_{l\gamma'(0)}X_0(0) \neq 0$, there is a small number $\varepsilon > 0$ such that for every $u \in (-\varepsilon, \varepsilon)$, $\exp_{p*}\mathcal{F}_{l(\gamma_0'(0)\cos\frac{u}{l}+X_0(0)\sin\frac{u}{l})}\left(-\gamma_0'(0)\sin\frac{u}{l} + X_0(0)\cos\frac{u}{l}\right) \neq 0$ is satisfied, where $\mathcal{F}_z$ means the parallel translation along z from M_p to $(M_p)_z$, $z \in M_p$. The curve $u \to V_0(l, u)$ can be considered as a regular curve and putting this curve $\tau(u) = V_0(l, u)$, we get $\tau(u) = \sigma(u)$ for $u \in [0, \varepsilon)$, $\tau'(0) = \sigma'(0)$ and $\tau(u)$ is contained in $C(p)$ for all $u \in (-\varepsilon, \varepsilon)$. Let $B(q, \delta)$ be a normal convex ball with a center at q and of radius δ. Take a point y on γ such that $y = \gamma(l-a) \in B(q, \delta)$ and take a point z on τ such that $z \in B(q, \delta)$ and $d(y, z) = d(y, \tau)$. Then we can consider that $z = \tau(-b)$, $\varepsilon > b > 0$ and as y tends to q, z also tends to q. Consider the triangle $\tilde{\triangle}_a(\tilde{q}, \tilde{z}, \tilde{y})$ in R^2 such that $d(\tilde{q}, \tilde{z}) = d(q, z)$, $d(\tilde{z}, \tilde{y}) = d(z, y)$, and $d(\tilde{y}, \tilde{q}) = d(y, q) = a$. For a sequence of geodesic triangles $\triangle_a = (q, z, y)$ in M shrinking to q as a tends to 0 in such a way that the angles of $\triangle_a$ approach limits equal to neither 0 nor π, we have $\lim_{a\to0}(\measuredangle(q,z,y) - \measuredangle(\tilde{q},\tilde{z},\tilde{y})) = \lim_{a\to0}(\measuredangle(z,y,q) - \measuredangle(\tilde{z},\tilde{y},\tilde{q})) = \lim_{a\to0}(\measuredangle(y,q,z) - \measuredangle(\tilde{y},\tilde{q},\tilde{z})) = 0$, by virtue of an elementary property of Riemannian manifolds. Hence we have $\lim_{a\to0}\measuredangle(y, q, z) = \theta$, $\lim_{a\to0}\measuredangle(q, z, y) = \pi/2$ and $\lim_{a\to0}\measuredangle(z, y, q) = \pi/2 - \theta$, from which it follows that for sufficiently small $\eta > 0$ there exist $C_0 > 0$ and $C_1 > 0$ such that $C_0 \leqq \sin\measuredangle(y, q, z) \leqq C_1 < 1$ for all $a \in (0, \eta)$. Then we have $d(p, z) \leqq d(p, y) + d(y, z) \leqq (l-a) + C_1 \cdot a = l - (1 - C_1)a$. This is a contradiction. Then the first assertion (4) is proved.

By means of the discussion above, it can be shown that $\tau|(-\varepsilon, 0]$ coincides with $\sigma|(-\varepsilon, 0]$ and is also contained in $C(p)$ and $<\sigma'(0), \gamma'(l)> = 0$. Thus an analogous argument for γ leads the other assertions (5) and (6). Q.E.D.

Taking account of two lemmas obtained above and developing the same discussion as that of Lemmas 6 and 8 in [2], where we replace 1/4 in [2] by k, we can prove the following two lemmas:

LEMMA 4. 3. *For an arbitrary fixed point q in $C(p)$, let M_q^0 be a subset of M_q consisting of all tangent vectors at q of curves in $C(p)$ passing through q. Then M_q^0 is a subspace of M_q.*

LEMMA 4. 4. *Let $M_q^\perp$ be the orthogonal complement of M_q^0 in M_q. Then we have $\exp_q lX = p$ for any $X \in M_q^\perp$ and $\|X\| = 1$.*

As a direct consequence of Lemma 4. 4, we have

LEMMA 4. 5. *All of the geodesic segments starting from p and of length $\pi/\sqrt{k}$ are geodesic loops at p.*

This lemma shows that by virtue of the results obtained in [5], all geodesic loops at p are of the same index λ, where $\lambda=0, 1, 3, 7, n-1$, and λ is equal to 0 if and only if M is not simply connected and λ is positive if and only if M is simply connected. Consequently if $\lambda>0$, then $C(p)$ coincides with $Q(p)$. For any $q \in Q(p)$ the multiplicity of p and q as conjugate points must be equal to λ. By means of the main theorem in [5] we have

THEOREM 4. 6. *Let there be given a point in M satisfying the condition that $C_p=S_p^{n-1}(\pi/2\sqrt{k})$ and (2). Then we have*

(a) *For any $q \in Q(p)$, the multiplicity of p and q as conjugate points is constant λ, where $\lambda=0, 1, 3, 7, n-1$.*

(b) *If M is simply connected, then the integral cohomology ring $H^*(M, Z)$ is a truncated polynomial ring generated by an element. In particular when λ is equal to $n-1$, M is homeomorphic to S^n.*

(c) *If M is not simply connected, then M is isometric to a real projective space $PR^n(k)$ with constant curvature k.*

The case $\lambda=n-1$ in the assertion (b) is obtained by the Warner's theorem and the assertion (c) is due to the Toponogov's maximal diameter theorem. As a straightforward consequence of the theorem above, we have

COROLLARY 4. 7. *$C(p)$ is a totally geodesic submanifold.*

BIBLIOGRAPHY

[1] M. BERGER, Les variétés riemanniennes à courbure positive, Bull. Soc. Math. Belg., 10(1958), 89-104.

[2] M. BERGER, Sur quelques variété riemanniennes suffisamment pincees, Bull. Soc. Math. France, 88(1960), 57-71.

[3] L. W. GREEN, Auf Wiedersehensflächen, Ann. of Math., 78(1963), 289-299.

[4] S. KOBAYASHI, Riemannian manifolds without conjugate points, Ann. Math. Pure. Appl., 59(1961), 149-158.

[5] H. NAKAGAWA, A note on theorems of Bott and Samelson, J. Math. Kyoto Univ., 7(1967), 205-220.

[6] H. NAKAGAWA, Riemannian manifolds with many geodesic loops, J. Math. Soc. Japan., 20(1968), 648-654.

[7] H. ŌMORI, A class of riemannian metrics on a manifold, J. Diff. Geom., 2(1968), 233-252.

[8] T. ŌTSUKI, On focal elements and the spheres, Tôhoku Math. J., 17(1965), 285-304.
[9] H. E. RAUCH, Geodesics and Curvatures in Differential Geometry in the large, Yeshiva Univ. Press, New York, 1959.
[10] V. A. TOPONOGOV, Riemannian spaces having their curvature bounded below by a positive number, Amer. Math. Soc. Transl., 18(1964), 291-336. (Uspehi Math. Soc. 14(1959), 87-130).
[11] F. W. WARNER, Conjugate loci of constant order, Ann. of Math., 86(1967), 192-212.
[12] A. D. WEINSTEIN, The cut locus and conjugate locus of a Riemannian manifold, Ann. of Math., 87(1968), 29-41.

DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF AGRICULTURE AND TECHNOLOGY
TOKYO, JAPAN

AND

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
TOKYO, JAPAN