On Riesz summability of a series of Bessel functions

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1. Introduction

Let $c_\nu(\alpha, \beta) \equiv J_\nu(\alpha) Y_\nu(\beta) - Y_\nu(\alpha) J_\nu(\beta)$, where $J_\nu(z)$ and $Y_\nu(z)$ denote the Bessel functions of first and second kind of order $\nu > 0$.

Let γ_m denote the mth positive root of the equation

$$c_\nu(az, bz) = 0.$$

An arbitrary function $f(x)$ defined for $0 < a < x < b$ can be expanded in a series of the form

$$f(x) \sim \sum_{m=1}^{\infty} b_m c_\nu(\gamma_m x, \gamma_m b), \tag{1.1}$$

where

$$b_m = \frac{\pi^2 \gamma_m^2}{2} \frac{J_\nu^2(\gamma_m a)}{\{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)\}} \int_a^b t f(t) c_\nu(\gamma_m t, \gamma_m b)\, dt. \tag{1.2}$$

E. C. TITCHMARSH ([3]) studied the convergence of the series (1. 1) and gave the following theorem:

Theorem A. *If $f(t)$ be Lebesgue integrable in the range (a, b) and has bounded variation in the neighbourhood of the point $t = x$, then the series* (1. 1) *converges to the sum* $\frac{1}{2}\{f(x+0) + f(x-0)\}$.

The purpose of this paper is to study the Riesz summability of the series (1. 1).

G. N. WATSON ([5], pp. 606—607) has studied the Riesz summability of the ordinary Fourier-Bessel series

$$f(x) \sim \sum_{m=1}^{\infty} a_m J_\nu(j_m x), \tag{1.3}$$

where $j_1 < j_2 < \dots j_m < \dots$ are the positive zeros of the function $J_\nu(z)$ and

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t f(t) J_\nu(j_m t)\, dt. \tag{1.4}$$

He has taken the Riesz mean as the sum

$$\text{(1. 5)} \qquad \lim_{n\to\infty} \sum_{m=1}^{n} \left(1-\frac{j_m}{A_n}\right) a_m J_\nu(j_m x),$$

where A_n stands for $(n+\frac{1}{2}\nu+\frac{1}{4})\pi$ and $j_n < A_n < j_{n+1}$. The asymptotic value of j_m is known to be ([5], p. 618) given as

$$\text{(1. 6)} \qquad j_m = \left(m+\frac{1}{2}\nu-\frac{1}{4}\right)\pi + O(m^{-1}).$$

Analogous to above, we define the Riesz sum for the series (1. 1) by

$$\text{(1. 7)} \qquad R_n(x,f) \sim \lim_{n\to\infty} \sum_{m=1}^{n} \left(1-\frac{\gamma_m}{\beta_n}\right) b_m c_\nu(\gamma_m x, \gamma_m b),$$

where β_n lies between γ_n and γ_{n+1} and*)

$$\text{(1. 8)} \qquad \gamma_m = \frac{m\pi}{b-a} + \frac{4(\nu^2-1)(b-a)}{8m\pi ab} + O(m^{-3}).$$

The series (1. 1) is said to be summable-(R) if the limit in (1. 7) exists.

In § 3 we shall prove the following theorem.

Theorem: *If* $\nu > 0$ *and*

$$\text{(1. 9)} \qquad \int_a^b t^{1/2} |f(t)|\, dt < \infty,$$

and if the limits $f(x \pm 0)$ *exist, then the series* (1. 1) *is summable-*(R) *at all points of the open interval* (a, b) *to the sum*

$$\tfrac{1}{2}\{f(x+0)+f(x-0)\}.$$

2. The following lemmas will be needed during the course of the proof of the theorem:

Lemma 1. *On the rectangle whose vertices are* $\pm\beta i$, $\beta_n \pm \beta i$ *in the* ω *plane, where* β *will be made to tend to infinity, we have*

$$\text{(2. 1)} \qquad \left| \frac{\omega c_\nu(\omega x, a\omega) c_\nu(\omega t, b\omega)}{c_\nu(a\omega, b\omega)} \right| = O\left(\frac{e^{-v(t-x)}}{\sqrt{xt}}\right)$$

for $t > x$ *and* $\omega = u + iv$.

Proof. Titchmarsh ([4], p. 73) has obtained the relation

$$\text{(2. 2)} \qquad J_\nu(bs) Y_\nu(xs) - Y_\nu(bs) J_\nu(xs) = -\frac{2\sin\{s(b-x)\}}{\sqrt{bx}\,\pi s} + O\left(\frac{e^{|t|(b-x)}}{|s|^2}\right),$$

where $s = \sigma + it$ and $0 < x < b$.

*) See Naylor [2].

Using the above relation we easily have

$$(2.3)\qquad c_\nu(x\omega, a\omega) = -\frac{2\sin\{\omega(x-a)\}}{\sqrt{ax}\,\pi\omega} + O\left\{\frac{e^{|v|(x-a)}}{|\omega|^2}\right\}, \quad \text{for} \quad 0<a<x;$$

$$(2.4)\qquad c_\nu(\omega t, \omega b) = \frac{2\sin\{\omega(b-t)\}}{\sqrt{bt}\,\pi\omega} + O\left\{\frac{e^{|v|(b-t)}}{|\omega|^2}\right\}, \quad \text{for} \quad 0<t<b;$$

and

$$(2.5)\qquad c_\nu(a\omega, b\omega) = \frac{2\sin\{\omega(b-a)\}}{\sqrt{ab}\,\pi\omega} + O\left\{\frac{e^{|v|(b-a)}}{|\omega|^2}\right\}, \quad \text{for} \quad 0<a<b.$$

Also, since on the above mentioned rectangle in the ω plane [4, pp. 13—14]

$$|\sin\{\omega(b-a)\}| > Ae^{v(b-a)},$$

we have

$$\left|\omega\frac{c_\nu(x\omega, a\omega)\,c_\nu(t\omega, b\omega)}{c_\nu(a\omega, b\omega)}\right| < |\omega|\left[\left|\frac{\sin\{\omega(x-a)\}\sin\{\omega(b-t)\}}{\pi\sqrt{xt}\,\omega\sin\{\omega(b-a)\}}\right| + O\left\{\frac{e^{|v|(x-t)}}{|\omega|^2}\right\}\right] =$$

$$(2.6)\qquad = O\left(\frac{e^{-v(t-x)}}{\sqrt{xt}\,\pi}\right) + O\left(\frac{e^{-v(t-x)}}{|\omega|}\right) = O\left(\frac{e^{-v(t-x)}}{\sqrt{xt}}\right).$$

Thus the proof of the lemma is complete.

Riesz sum: The Riesz sum of the series (1. 1) is given by

$$(2.7)\qquad R_n(x,f) = \sum_{m=1}^{n}\left(1-\frac{\gamma_m}{\beta_n}\right) b_m c_\nu(\gamma_m x, \gamma_m b)$$

$$= \sum_{m=1}^{n}\left(1-\frac{\gamma_m}{\beta_n}\right)\frac{\pi^2\gamma_m^2 J_\nu^2(\gamma_m a)\,c_\nu(\gamma_m x, \gamma_m b)}{2\{J_\nu^2(\gamma_m a)-J_\nu^2(\gamma_m b)\}}\int_a^b tf(t)\,c_\nu(\gamma_m t, \gamma_m b)\,dt$$

$$(2.8)\qquad = \int_a^b tf(t)\,R_n(t, x/R)\,dt,$$

where

$$(2.9)\qquad R_n(t, x/R) = \sum_{m=1}^{n}\left(1-\frac{\gamma_m}{\beta_n}\right)\frac{\pi^2\gamma_m^2 J_\nu^2(\gamma_m a)\,c_\nu(\gamma_m x, \gamma_m b)\,c_\nu(\gamma_m t, \gamma_m b)}{2\{J_\nu^2(\gamma_m a)-J_\nu^2(\gamma_m b)\}}.$$

In order to study the relevant properties of the kernel function $R_n(t, x|R)$ we observe that it is a symmetrical function in t and x and hence we first suppose that $a<x<t<b$ and subsequently the results for $a<t<x<b$ can be written by an interchange of t and x.

Lemma 2. *The following inequalities hold true*):*

$$|R_n(t,x|R)| < \frac{2K}{\beta_n}\frac{1}{\sqrt{xt}\,(t-x)^2}, \quad \textit{if} \quad a<x<t<b; \tag{2.10}$$

$$|R_n(t,x|R)| < \frac{2K}{\beta_n}\frac{1}{\sqrt{xt}\,(x-t)^2}, \quad \textit{if} \quad a<t<x<b; \tag{2.11}$$

$$|R_n(t,x|R)| < \frac{3\beta_n K}{\sqrt{\tfrac{1}{2}xt}}, \quad \textit{if} \quad a<x\leqq t<b \quad \textit{or} \quad a<t\leqq x<b. \tag{2.12}$$

PROOF. Consider the function

$$\Phi(\omega) = \pi\omega\left(1-\frac{\omega}{\beta_n}\right)\frac{c_\nu(x\omega,a\omega)\,c_\nu(t\omega,b\omega)}{c_\nu(a\omega,b\omega)}. \tag{2.13}$$

This function has simple poles at $\gamma_1,\gamma_2,\gamma_3,\dots,\gamma_n$. Residue at $\omega=\gamma_m$ is given by

$$\operatorname*{Lim}_{\omega\to\gamma_m}\pi\omega\left(1-\frac{\omega}{\beta_n}\right)(\omega-\gamma_m)\frac{c_\nu(x\omega,a\omega)\,c_\nu(t\omega,b\omega)}{c_\nu(a\omega,b\omega)} = \tag{2.14}$$

$$= \lim_{\omega\to\gamma_m} \times$$

$$\times\frac{\pi\omega\left(1-\frac{\omega}{\beta_n}\right)c_\nu(x\omega,a\omega)\,c_\nu(t\omega,b\omega)+(\omega-\gamma_m)\frac{d}{d\omega}\left[\pi\omega\left(1-\frac{\omega}{\beta_n}\right)c_\nu(x\omega,a\omega)\,c_\nu(t\omega,b\omega)\right]}{c_\nu'(a\omega,b\omega)} =$$

$$= \frac{\pi\gamma_m\left(1-\frac{\gamma_m}{\beta_n}\right)c_\nu(\gamma_m x,\gamma_m a)\,c_\nu(\gamma_m t,\gamma_m b)}{a\{J_\nu'(\gamma_m a)\,Y_\nu(\gamma_m b)-J_\nu(\gamma_m b)\,Y_\nu'(\gamma_m a)\}+b\{J_\nu(\gamma_m a)\,Y_\nu'(\gamma_m b)-J_\nu'(\gamma_m b)\,Y_\nu(\gamma_m a)\}} =$$

$$= \frac{\pi\gamma_m\left(1-\frac{\gamma_m}{\beta_n}\right)c_\nu(\gamma_m x,\gamma_m a)\,c_\nu(\gamma_m t,\gamma_m b)}{a\left\{J_\nu'(\gamma_m a)\frac{Y_\nu(\gamma_m a)}{\eta}-\frac{J_\nu(\gamma_m a)}{\eta}Y_\nu'(\gamma_m a)\right\}+b\{\eta J_\nu(\gamma_m b)\,Y_\nu'(\gamma_m b)-J_\nu'(\gamma_m b)\,\eta Y_\nu(\gamma_m b)\}}$$

where

$$\eta = \frac{J_\nu(\gamma_m a)}{J_\nu(\gamma_m b)} = \frac{Y_\nu(\gamma_m a)}{Y_\nu(\gamma_m b)}, \tag{2.15}$$

$$\frac{\pi\gamma_m\left(1-\frac{\gamma_m}{\beta_n}\right)c_\nu(\gamma_m x,\gamma_m a)\,c_\nu(\gamma_m t,\gamma_m b)}{\frac{1}{\eta}\left(-\frac{2}{\pi\gamma_m}\right)+\eta\left(\frac{2}{\pi\gamma_m}\right)} = \frac{\pi^2\gamma_m^2\left(1-\frac{\gamma_m}{\beta_n}\right)c_\nu(\gamma_m x,\gamma_m a)\,c_\nu(\gamma_m t,\gamma_m b)}{2\left(\eta-\frac{1}{\eta}\right)},$$

*) K will denote a constant not necessarily the same on each occasion.

since

$$J_\nu(x)Y'_\nu(x)-Y_\nu(x)J'_\nu(x)=\frac{2}{\pi x}. \tag{2.16}$$

Using (2.15) again and after a little simplification we find that the residue at $\omega=\gamma_m$ is

(2.17)

$$\frac{\pi^2\gamma_m^2\left(1-\frac{\gamma_m}{\beta_n}\right)J_\nu(\gamma_m a)J_\nu(\gamma_m b)\left[J_\nu(\gamma_m x)Y_\nu(\gamma_m a)-J_\nu(\gamma_m a)Y_\nu(\gamma_m x)\right]c_\nu(\gamma_m t,\gamma_m b)}{2\{J_\nu^2(\gamma_m a)-J_\nu^2(\gamma_m b)\}}=$$

$$=\frac{\pi^2\gamma_m^2\left(1-\frac{\gamma_m}{\beta_n}\right)J_\nu^2(\gamma_m a)c_\nu(\gamma_m x,\gamma_m b)c_\nu(\gamma_m t,\gamma_m b)}{2\{J_\nu^2(\gamma_m a)-J_\nu^2(\gamma_m b)\}}$$

Therefore $R_n(t,x|R)$ is the sum of the residues of the function $\Phi(\omega)$ at γ_1, $\gamma_2,\dots,\gamma_n$.

Proof of inequality (2.10). We take the contour of integration as the rectangle with the vertices at $\pm\beta i$, $\beta_n\pm\beta i$ in the ω plane, where β will be made to tend to ∞. Hence

$$R_n(t,x|R)=\frac{1}{2\pi i}\left[\int_{\beta_n-i\infty}^{\beta_n+i\infty}-\int_{-i\infty}^{i\infty}\right]\frac{\pi\omega\left(1-\frac{\omega}{\beta_n}\right)c_\nu(\omega x,\omega a)c_\nu(\omega t,\omega b)}{c_\nu(a\omega,b\omega)}\,d\omega, \tag{2.18}$$

as it can be easily shown in view of Lemma 1 that the other parts of the integral tend to zero.

Now

$$\begin{aligned} R_n(t,x|R)&=\frac{1}{2i}\int_{\beta_n-i\infty}^{\beta_n+i\infty}\omega\left(1-\frac{\omega}{\beta_n}\right)\frac{c_\nu(\omega x,a\omega)c_\nu(t\omega,b\omega)}{c_\nu(a\omega,b\omega)}\,d\omega+\\ &+\frac{1}{2i\beta_n}\int_{-i\infty}^{i\infty}\frac{\omega^2c_\nu(\omega x,a\omega)c_\nu(t\omega,b\omega)}{c_\nu(a\omega,b\omega)}\,d\omega, \end{aligned} \tag{2.19}$$

since $\dfrac{\omega c_\nu(\omega x,a\omega)c_\nu(t\omega,b\omega)}{c_\nu(a\omega,b\omega)}$ is an odd function of ω.

Using Lemma 1, and replacing ω by $\beta_n\pm iv$ and $\pm iv$ in the first and second integrals respectively, we obtain

$$|R_n(t,x|R)|<\frac{2k}{\sqrt{xt}}\frac{1}{\beta_n}\int_0^\infty ve^{-(t-x)v}\,dv=\frac{2k}{\beta_n}\frac{1}{\sqrt{xt}}\frac{1}{(t-x)^2}, \tag{2.20}$$

which completes the proof of inequality (2. 10). The inequality (2. 11) follows by an interchange of x and t.

Proof of inequality (2. 12). In this case we take the contour as the rectangle whose vertices are $\pm i\beta_n$ and $\beta_n \pm i\beta_n$ in the ω-plane.

Using the result of Lemma 1, we find that

$$\text{(2. 21)} \qquad \left| \left(1-\frac{\omega}{\beta_n}\right) \frac{\omega c_\nu(x\omega, a\omega) c_\nu(t\omega, b\omega)}{c_\nu(a\omega, b\omega)} \right| \leqq \frac{k}{\sqrt{xt}} \sqrt{2},$$

when $a < x \leqq t < b$ and the same inequality will hold for $a < t \leqq x < b$ as can be seen by interchanging x and t.

Consequently

$$\text{(2. 22)} \qquad |R_n(t, x|R)| < \frac{3k\beta_n}{\sqrt{\frac{1}{2}xt}}.$$

Lemma 3.

$$\lim_{n\to\infty} \int_a^b t^{\nu+1} R_n(t, x|R)\, dt = x^\nu \qquad (a < x < b).$$

PROOF. From (2. 9) we have

(2. 23)

$$\int_a^b t^{\nu+1} R_n(t, x|R)\, dt = \sum_{m=1}^{n} \left(1-\frac{\gamma_m}{\beta_n}\right) \frac{\pi^2 \gamma_m^2 J_\nu^2(\gamma_m a) c_\nu(\gamma_m x, \gamma_m b)}{2[J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)]} \int_a^b t^{\nu+1} c_\nu(\gamma_m t, \gamma_m b)\, dt.$$

Using (2. 9) again it can be seen easily that

$$= \sum_{m=1}^{n} \left(1-\frac{\gamma_m}{\beta_n}\right) \frac{\pi^2 \gamma_m^2 J_\nu^2(\gamma_m a) c_\nu(\gamma_m x, \gamma_m b)}{2[J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)]} \left[\frac{b^{\nu+1}}{\gamma_m}\left(\frac{2}{\pi\gamma_m b}\right) - \frac{a^{\nu+1}}{\eta\gamma_m}\left(\frac{2}{\pi\gamma_m a}\right)\right] =$$

$$= \sum_{m=1}^{n} \frac{\pi\left(1-\frac{\gamma_m}{\beta_n}\right)\left\{b^\nu J_\nu^2(\gamma_m a) - \frac{J_\nu(\gamma_m b)}{J_\nu(\gamma_m a)} J_\nu^2(\gamma_m a) a^\nu\right\} c_\nu(\gamma_m x, \gamma_m b)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)} =$$

$$= \sum_{m=1}^{n} \pi\left(1-\frac{\gamma_m}{\beta_n}\right) J_\nu(\gamma_m a) \frac{b^\nu J_\nu(\gamma_m a) - a^\nu J_\nu(\gamma_m b)}{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)} c_\nu(\gamma_m x, \gamma_m b),$$

and this is evidently the sum of the residues of the function

$$\left(1-\frac{\omega}{\beta_n}\right) \frac{2b^\nu c_\nu(\omega x, a\omega) - 2a^\nu c_\nu(x\omega, b\omega)}{\omega c_\nu(a\omega, b\omega)}$$

at $\gamma_1, \gamma_2, \ldots, \gamma_n$. Therefore transforming the sum into a contour integral as in Lemma 2, we have

$$\text{(2. 24)} \qquad \int_a^b t^{\nu+1} R_n(t, x|R)\, dt = \frac{1}{2\pi i} \int_{c'} \left(1-\frac{\omega}{\beta_n}\right) \frac{2b^\nu c_\nu(x\omega, a\omega) - 2a^\nu c_\nu(x\omega, b\omega)}{\omega c_\nu(a\omega, b\omega)}\, d\omega$$

where c' is the contour indented at the origin in the usual manner by a semi-circle to the right of the imaginary axis.

In view of the Lemma 1, we obtain that

$$(2.25)\quad \int_a^b t^{\nu+1} R_n(t, x|R)\,dt = \frac{1}{2\pi i}\left[-\int_{-\infty i}^{\infty i}\left(1-\frac{\omega}{\beta_n}\right)\frac{2b^\nu c_\nu(x\omega, a\omega)-2a^\nu c_\nu(\omega x, b\omega)}{\omega c_\nu(a\omega, b\omega)}\,d\omega + \right.$$

$$\left. + \int_{\beta_n - i\infty}^{\beta_n + i\infty}\left(1-\frac{\omega}{\beta_n}\right)\frac{2b^\nu c_\nu(x\omega, a\omega)-2a^\nu c_\nu(\omega x, b\omega)}{\omega c_\nu(a\omega, b\omega)}\,d\omega\right].$$

Since the function $\dfrac{2b^\nu c_\nu(\omega x, a\omega)-2a^\nu c^\nu(\omega x, b\omega)}{\omega c_\nu(a\omega, b\omega)}$ is an odd function of ω, the value of the integral on the right of (2. 25) reduces to πi times the residue of the integrand at the origin as $n\to\infty$.

Now, residue of the integrand at the simple pole $w=0$ is

$$\lim_{\omega\to 0}\frac{2b^\nu c_\nu(\omega x, a\,\omega)-2a^\nu c_\nu(\omega x, \omega b)}{c_\nu(a\omega, b\omega)},$$

which gives after some simplification

$$(2.26)\quad \lim_{\omega\to 0}\frac{2b^\nu \dfrac{J_\nu(a\omega)}{J_\nu(b\omega)}\dfrac{Y_\nu(a\omega)}{Y_\nu(b\omega)}\left\{\dfrac{J_\nu(x\omega)}{J_\nu(a\omega)}-\dfrac{Y_\nu(x\omega)}{Y_\nu(a\omega)}\right\}-2a^\nu\left\{\dfrac{J_\nu(x\omega)}{J_\nu(b\omega)}-\dfrac{Y_\nu(x\omega)}{Y_\nu(b\omega)}\right\}}{\dfrac{J_\nu(a\omega)}{J_\nu(b\omega)}-\dfrac{Y_\nu(a\omega)}{Y_\nu(b\omega)}}.$$

Using now the asymptotic values ([1], p. 135)

$$J_\nu(xt)\sim\frac{\left(\dfrac{xt}{2}\right)^\nu}{\sqrt{\nu+1}},\qquad \nu\geqq 0;$$

and

$$Y_\nu(xt)\sim-\frac{2^\nu\sqrt{\nu}}{\pi x^\nu},\qquad \nu>0;$$

as $x\to 0$ and t fixed, we obtain

$$(2.27)\qquad \lim_{\omega\to 0}\frac{J_\nu(a\omega)}{J_\nu(b\omega)}=\frac{a^\nu}{b^\nu},$$

and

$$(2.28)\qquad \lim_{\omega\to 0}\frac{Y_\nu(a\omega)}{Y_\nu(b\omega)}=\frac{b^\nu}{a^\nu}$$

Consequently (2. 26) becomes

$$\frac{2b^{\nu}\dfrac{a^{\nu}}{b^{\nu}}\dfrac{b^{\nu}}{a^{\nu}}\left(\dfrac{x^{\nu}}{a^{\nu}}-\dfrac{a^{\nu}}{x^{\nu}}\right)-2a^{\nu}\left(\dfrac{x^{\nu}}{b^{\nu}}-\dfrac{b^{\nu}}{x^{\nu}}\right)}{\dfrac{a^{\nu}}{b^{\nu}}-\dfrac{b^{\nu}}{a^{\nu}}}$$

which ultimately reduces to the value $-2x^{\nu}$.

Coming back to (2. 25) we find that

$$\int_a^b t^{\nu+1}R_n(t,x|R)\,dt = x^{\nu}+\frac{1}{\pi i}\int_{\beta_n-i\infty}^{\beta_n+i\infty}\left(1-\frac{\omega}{\beta_n}\right)\frac{b^{\nu}c_{\nu}(\omega x,a\omega)-a^{\nu}c_{\nu}(\omega x,b\omega)}{\omega c_{\nu}(a\omega,b\omega)}\,d\omega+$$

(2. 29)

$$+\frac{1}{\beta_n\pi^i}\int_{-i\infty}^{+i\infty}\frac{b_{\nu}c_{\nu}(x\omega,a\omega)-a^{\nu}c_{\nu}(x\omega,b\omega)}{c_{\nu}(a\omega,b\omega)}\,d\omega = x^{\nu}+L_1,$$

say.

Now by Lemma 1

$$|L_1|<\frac{2kb^{\nu+1/2}}{\beta_n^2\sqrt{x}}\int_0^{\infty}ve^{-(b-x)v}\,dv+\frac{2ka^{\nu+1/2}}{\beta_n^2\sqrt{x}}\int_0^{\infty}ve^{-(x-a)v}\,dv+$$

(2. 30)

$$=\frac{2kb^{\nu+1/2}}{\beta_n\sqrt{x}}\int_0^{\infty}e^{-(b-x)v}dv+\frac{2ka^{\nu+1/2}}{\beta_n\sqrt{x}}\int_0^{\infty}e^{-(x-a)v}dv=$$

$$=\frac{2kb^{\nu+1/2}}{\beta_n^2\sqrt{x}\,(b-x)}+\frac{2ka^{\nu+1/2}}{\beta_n^2\sqrt{x}\,(x-a)}+\frac{2kb^{\nu+1/2}}{\beta_n\sqrt{x}\,(b+x)}+\frac{2ka^{\nu+1/2}}{\beta_n\sqrt{x}\,(x-a)}.$$

From (2. 29) and (2. 30) we get the result of the lemma.

Lemma 4. *For any* $x\in(a,b)$

$$\lim_{n\to\infty}\int_a^x t^{\nu+1}R_n(t,x|R)\,dt=\tfrac{1}{2}x^{\nu}$$

$$\lim_{n\to\infty}\int_x^b t^{\nu+1}R_n(t,x|R)\,dt=\tfrac{1}{2}x^{\nu}$$

PROOF. TITCHMARSH ([3]) has shown that

$$\lim_{n\to\infty}\int_a^x t^{\nu+1}R_n(t,x)\,dt=\tfrac{1}{2}x^{\nu}\qquad(a<x<b),$$

where $R_n(t, x)$ denotes the convergence kernel of the series (1. 1). The result of the lemma follows from the fact that convergence implies Riesz summability.

3. Proof of the theorem. In order to prove the theorem it is sufficient to show that

$$S_n(x|R) \equiv \sum_{m=1}^{n} \left(1-\frac{\gamma_m}{\beta_n}\right) b_m c_\nu(\gamma_m x, \gamma_m b) - \tag{3. 1}$$
$$- \int_a^x t^{\nu+1} R_n(t, x|R) x^{-\nu} f(x-0)\, dt - \int_x^b t^{\nu+1} R_n(t, x|R) x^{-\nu} f(x+0)\, dt = 0(1).$$

Now

$$S_n(x|R) = \int_a^b t f(t) R_n(t, x|R)\, dt -$$
$$- \int_a^x t^{\nu+1} R_n(t, x|R) x^{-\nu} f(x-0)\, dt - \int_x^b t^{\nu+1} R_n(t, x|R) x^{-\nu} f(x+0)\, dt =$$
$$= \int_a^x t^{\nu+1} \{t^{-\nu} f(t) - x^{-\nu} f(x-0)\} R_n(t, x|R)\, dt + \tag{3. 2}$$
$$+ \int_x^b t^{\nu+1} \{t^{-\nu} f(t) - x^{-\nu} f(x+0)\} R_n(t, x|R)\, dt = I_1 + I_2,$$

say, where

$$I_1 = \int_a^{x-\delta} + \int_{x-\delta}^{x-\frac{1}{\beta_n}} + \int_{x-\frac{1}{\beta_n}}^{x} \equiv P_1 + P_2 + P_3, \tag{3. 3}$$

say where $\delta > \dfrac{1}{\beta_n}$ for sufficiently large n,

and

$$I_2 = \int_x^{x+\frac{1}{\beta_n}} + \int_{x+\frac{1}{\beta_n}}^{x+\delta} + \int_{x+\delta}^{b} = Q_1 + Q_2 + Q_3, \tag{3. 4}$$

say. Using (2. 11) we have

$$|P_1| = \left| \int_a^{x-\delta} t^{\nu+1} \{t^{-\nu} f(t) - x^{-\nu} f(x-0)\} R_n(t, x|R)\, dt \right| \leqq$$
$$\leqq \int_a^{x-\delta} \left| t^{\nu+1/2} \{t^{-\nu} f(t) - x^{-\nu} f(x-0)\} \right| \frac{2k}{\beta_n} \frac{1}{\sqrt{x}(x-t)^2}\, dt < \tag{3. 5}$$
$$< \frac{2k}{\beta_n \delta^2 \sqrt{x}} \int_a^{x-\delta} |t^{1/2} f(t) - t^{\nu+1/2} x^{-\nu} f(x-0)|\, dt = o(1).$$

Similarly

(3. 6) $$|Q_3| = o(1).$$

Now

(3. 7)

$$|P_2| = \left| \int_{x-\delta}^{x-\frac{1}{\beta_n}} t^{\nu+1}[t^{-\nu}f(t) - x^{-\nu}f(x-0)]\, R_n(t, x|R)\, dt \right| \leqq \int_{x-\delta}^{x-\frac{1}{\beta_n}} |t^{\nu+1/2}|\, \varepsilon \frac{2k}{\beta_n \sqrt{x}(x-t)^2}\, dt,$$

where $\varepsilon > 0$ is an arbitrarily small number and $\delta > 0$ has been chosen such that

$$|t^{-\nu}f(t) - x^{-\nu}f(x-0)| < \varepsilon, \qquad x - \delta \leqq t \leqq x,$$

and

$$|t^{-\nu}f(t) - x^{-\nu}f(x+0)| < \varepsilon, \quad \text{if} \quad x \leqq t \leqq x + \delta.$$

Therefore

(3. 8) $$|P_2| < \frac{2k\varepsilon}{\sqrt{x}\beta_n} \int_{x-\delta}^{x-\frac{1}{\beta_n}} \frac{t^{\nu+1/2}}{(x-t)^2} = 2kx^{\nu}\varepsilon = o(1).$$

Similarly

(3. 9) $|Q_2| = o(1)$, if we use the inequality (2. 10).

Now using (2. 12) we find that

(3. 10) $$|P_3| = \left| \int_{x-\frac{1}{\beta_n}}^{x} t^{\nu+1}[t^{-\nu}f(t) - x^{-\nu}f(x-0)]\, R_n(t, x)\, dt \right| \leqq$$

$$\leqq \int_{x-\frac{1}{\beta_n}}^{x} t^{\nu+1} \varepsilon \frac{3k\beta_n}{\sqrt{\frac{1}{2}xt}}\, dt < k\beta_n \varepsilon \int_{x-\frac{1}{\beta_n}}^{x} dt = o(1).$$

Similarly

(3. 11) $$|Q_1| = o(1).$$

Combining (3. 5), (3. 6), (3. 8), (3. 9), (3. 10) and (3. 11) we get the result. *This completes the proof of the theorem.*

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(Received December 29, 1968.)