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On KD-null Sets in N-dimensional Euclidean Space

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Introduction

Ahlfors and Beurling [1] introduced the notion of a null set of class N_D in the complex plane: A compact set E is a null set of class N_D if and only if every analytic function in $D(\Omega-E)$ can be extended to a function in $D(\Omega)$ for a domain Ω containing E, where $D(\Omega)$ is the class of single-valued analytic functions in Ω with finite Dirichlet integrals. They characterized a null set of class N_D by means of the span, the extremal length and the others. On the other hand, the class KD, which consists of all harmonic functions u with finite Dirichlet integrals such that $^*du$ is semiexact, was considered on Riemann surfaces and various characterizations of the class O_{KD} were given by many authors; see, for example, Rodin [5], Royden [7], Sario [8]. We can consider the class KD also on an N-dimensional euclidean space R^N ($N \geqq 3$) and define KD-null sets as a compact set E such that any function in $KD(\Omega-\mathrm{E})$ can be extended to a function in $KD(\Omega)$ for a bounded domain Ω containing E.

In the present paper, we shall prove some theorems on KD-null sets analogous to those on null sets of class N_D. In §3, we observe some relations between KD-null set and the span, which was introduced by Rodin and Sario [6] in Riemannian manifolds. Moreover we show that the N-dimensional Lebesgue measure of a KD-null set is equal to zero. In §4, we shall give a necessary condition for a set to be KD-null in terms of the extremal length.

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§1. Preliminaries

We shall denote by $x=(x_1, x_2,\cdots, x_N)$ a point in R^N, and set $|x|=\sqrt{x_1^2+x_2^2+\cdots+x_N^2}$. By an unbounded domain in R^N we shall mean a domain which is equal to the complement of a compact set. A harmonic function u defined in an unbounded domain is called regular at infinity if $\lim_{|x|\to\infty} u(x)=0$. Consider a C^1-surface τ which divides R^N into a bounded domain and an unbounded domain. When we consider the normal derivative $\frac{\partial}{\partial n}$ at a point of τ, the normal is drawn in the direction of the unbounded domain.

Let G be an open set. Denote by $\{\tau\}_G$ be the class of surfaces τ in G each of which is a compact C^1-surface and divides R^N into a bounded domain and an unbounded domain. Let $KD(G)$ be the class of harmonic functions u defined in G satisfying the following conditions:

(1) the Dirichlet integral $D_G(u)=\int_G |\mathrm{grad}\ u|^2 dV$ is finite, where dV is the volume element,

(2) $\int_\tau \frac{\partial u}{\partial n} dS=0$ for all τ in $\{\tau\}_G$, where dS is the surface element on τ,

(3) in the case that G contains an unbounded domain, u is regular at infinity.

Let E be a compact set in R^N and Ω be a bounded domain which contains E. If every harmonic function u in $KD(\Omega-E)$ is continued to a harmonic function belonging to $KD(\Omega)$, then E is called a KD-null set with respect to Ω. The class of KD-null sets with respect to Ω is denoted by N_{KD}^{Ω}.

§2. Properties of *KD*-null sets

Let Ω be a bounded domain which contains a compact set E. Generally $R^N-E(=E^c)$ is an open set which consists of an unbounded domain and a bounded open set. First we shall show

THEOREM 1. *If E^c contains a bounded component, then E does not belong to N_{KD}^{Ω}.*

PROOF. Suppose E^c contains a bounded component D. Take two mutually disjoint closed balls e_0, e_1 in D with the same radius. Since the Newtonian capacity of e_i $(i=0, 1)$ is positive, there exists an equilibrium mass-distribution of unit mass on each of e_0 and e_1. Let μ be the measure which consists of the equilibrium mass-distributions on e_0 and e_1, and set

$$U^\mu(x)=\int_{e_0}\frac{d\mu(y)}{|x-y|^{N-2}}-\int_{e_1}\frac{d\mu(y)}{|x-y|^{N-2}}\ .$$

Then $U^\mu(x)$ is a harmonic function with finite Dirichlet integral in $\Omega-e_0\cup e_1$. Using Green's formula, we have

$$\int_\tau \frac{\partial U^\mu}{\partial n} dS=0 \qquad \text{for all } \tau \text{ in } \{\tau\}_{\Omega-\bar{D}}.$$

Therefore U^μ belongs to $KD(\Omega-\bar{D})$. Let $\tilde{U}^\mu$ equal U^μ in $\Omega-\bar{D}$ and 0 in D. Obviously $\tilde{U}^\mu$ belongs to $KD(\Omega-E)$ but cannot be continued to a function in $KD(\Omega)$. Accordingly we conclude $E \notin N_{KD}^{\Omega}$.

By virtue of Theorem 1 we shall be concerned with the compact set E such that E^c is an unbounded domain throughout the rest of this paper.

THEOREM 2. *A compact set E is a KD-null set with respect to Ω if and only if $KD(E^c)$ contains only the constant function* 0.

PROOF. First we assume $E \in N_{KD}^{\Omega}$ and let u be a harmonic function in $KD(E^c)$. Let h be the restriction of u to $\Omega - E$. Obviously $h \in KD(\Omega - E)$. By assumption there exists a harmonic function $\hat{h}$ in $KD(\Omega)$ such that $h = \hat{h}$ in $\Omega - E$. Hence u is continued to a harmonic function in R^N which is regular at infinity. Therefore u is equal to the constant 0.

Conversely assume that $KD(E^c) = \{0\}$. Now we take three domains Ω_0, Ω^* and Ω_1 such that $E \subset \Omega_0 \subset \bar{\Omega}_0 \subset \Omega^* \subset \bar{\Omega}^* \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$ hold and each of $\partial\Omega_0$, $\partial\Omega^*$ and $\partial\Omega_1$ consists of one compact C^1-surface. For any u in $KD(\Omega - E)$, we set

$$h_i(x) = \frac{1}{\sigma_N}\int_{\partial\Omega_i}\left(\frac{1}{r^{N-2}}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{1}{r^{N-2}}\right)\right) dS, \qquad (i=0, 1),$$

where r denotes the distance from a point x to the variable on $\partial\Omega_i$ and σ_N is the surface area of the unit sphere in R^N. Then $h_0(x)$ is harmonic in $R^N - \bar{\Omega}_0$ and regular at infinity and $h_1(x)$ is harmonic in Ω_1. When x lies in the domain $\Omega_1 - \bar{\Omega}_0$, the equality

$$u(x) = h_1(x) - h_0(x)$$

holds. Let $\tilde{h}$ equal $h_0(x)$ in $R^N - \bar{\Omega}_0$ and $h_1(x) - u(x)$ in $\bar{\Omega}_0 - E$. It is easy to see that $\tilde{h}$ is harmonic in $R^N - E$ and regular at infinity. In $\Omega^* - E$ both h_1 and u have finite Dirichlet integrals so that $\tilde{h} = h_1 - u$ has finite Dirichlet integral there. On the other hand, Green's formula gives

$$D_{R^N - \bar{\Omega}^*}(h_0) \leqq \int_{\partial\bar{\Omega}^*} |h_0| \left|\frac{\partial h_0}{\partial n}\right| dS < \infty.$$

It follows that

$$D_{R^N - E}(\tilde{h}) = D_{\Omega^* - E}(h_1 - u) + D_{R^N - \bar{\Omega}^*}(h_0) < \infty.$$

Take any τ in $\{\tau\}_{E^c}$ such that the interior of τ contains non-empty compact subset of E. Since τ is homologous in E^c to some τ^* consisting of a finite number of elements in $\{\tau\}_{\Omega_0 - E}$, we have

$$\int_{\tau}\frac{\partial\tilde{h}}{\partial n} dS = \int_{\tau^*}\frac{\partial\tilde{h}}{\partial n} dS.$$

In view of Green's formula and the fact $u \in KD(\Omega - E)$, we have

$$\int_{\tau^*}\frac{\partial\tilde{h}}{\partial n} dS = \int_{\tau^*}\frac{\partial h_1}{\partial n} dS - \int_{\tau^*}\frac{\partial u}{\partial n} dS = 0.$$

From these facts we conclude $\tilde{h} \in KD(E^c)$. Therefore we have $\tilde{h}=0$ by assumption. It follows that $h_1=u$ in Ω_0-E. On account of harmonicity of h_1 in Ω_1, u can be continued to a harmonic function in Ω. Since u is arbitrary, we have $E \in N_{KD}^{\Omega}$.

Theorem 2 implies the following corollary.

COROLLARY 1. *The property $E \in N_{KD}^{\Omega}$ does not depend on the choice of Ω.*

We shall omit the suffix Ω in the notation N_{KD}^{Ω} throughout the rest of this paper.

§3. Principal functions

Let E be a compact set such that E^c is a domain. Let $\{\Omega_n\}_{n=1}^{\infty}$ be an exhaustion of E^c with the following properties:

(1) Ω_n is a bounded subdomain of E^c,

(2) $\partial\Omega_n$ consists of a finite number of C^1-surfaces, denoted by $\partial\Omega_n^j$, $j=1,\cdots,j(n)$,

(3) $\bar{\Omega}_n \subset \Omega_{n+1} (n=1, 2,\cdots)$ and $\bigcup_{n=1}^{\infty} \Omega_n = E^c$.

Take any distinct two points a, b in E^c and the balls U_a, U_b centered at a, b with disjoint closures in E^c. We may assume that Ω_n contains $\overline{U_a \cup U_b}$ for all n. For a function g and a set U, we denote by $g|_U$ the restriction of g to U. There exist the principal functions $P_{i,n}$ $(i=0, 1)$ with respect to Ω_n with the following properties ([6]):

(1) $P_{i,n}$ is harmonic in $\Omega_n-(\{a\}\cup\{b\})$,

(2) $$P_{i,n}|_{U_a}=\frac{1}{\sigma_N|x-a|^{N-2}}+h_{i,n},$$

$$P_{i,n}|_{U_b}=\frac{-1}{\sigma_N|x-b|^{N-2}}+f_{i,n},$$

where $h_{i,n}$ and $f_{i,n}$ are harmonic in U_a and U_b respectively and $f_{i,n}(b)=0$,

(3) $$\frac{\partial P_{0,n}}{\partial n}=0 \qquad \text{on } \partial\Omega_n^j,$$

$$P_{1,n}|_{\partial\Omega_n^j}=C_n^j \text{ (constant) and } \int_{\partial\Omega_n^j}\frac{\partial P_{1,n}}{\partial n}dS=0, \qquad \text{for } j=1,\cdots,j(n).$$

On letting $n\to\infty$, we can see that the following limits exist:

$$P_i=\lim_{n\to\infty} P_{i,n}, \qquad h_i=\lim_{n\to\infty} h_{i,n}, \qquad f_i=\lim_{n\to\infty} f_{i,n} \quad (i=0, 1).$$

Here the convergence is uniform on every compact subset of E^c and these limit functions do not depend on the choice of exhaustion; see [6].

Let $\{\tilde{\Omega}_n\}_{n=1}^{\infty}$ be an approximation of E^c towards E such that

(1) $\tilde{\Omega}_n$ is an unbounded subdomain of E^c,

(2) $\partial\tilde{\Omega}_n$ consists of a finite number of compact C^1-surfaces such that the interior of each surface of $\partial\tilde{\Omega}_n$ contains at least one point of E,

(3) $\bar{\tilde{\Omega}}_n \subset \tilde{\Omega}_{n+1} (n=1, 2, \cdots)$ and $\bigcup_{n=1}^{\infty} \tilde{\Omega}_n = E^c$.

Let g and u be harmonic functions which are defined in $U_E - E$ and have finite Dirichlet integrals on $U_E - E$, where U_E is an open neighborhood of E. We may assume that U_E contains $\partial\tilde{\Omega}_n$ for all n. Then the limit of $\int_{\partial\tilde{\Omega}_n} g \frac{\partial u}{\partial n} dS$ exists and does not depend on the choice of an approximation $\{\tilde{\Omega}_n\}$. Therefore we use the symbolic expression

$$\int_{\partial E} g \frac{\partial u}{\partial n} dS = \lim_{n\to\infty} \int_{\partial\tilde{\Omega}_n} g \frac{\partial u}{\partial n} dS.$$

For the purpose of observing a relation between KD-null sets and the principal functions we shall give the following lemma and introduce the notion of span.

LEMMA 1. *The following properties hold regarding* g *and* $P_i (i=0, 1)$:

(1) $\int_{\partial E} g \frac{\partial P_0}{\partial n} dS = 0$,

(2) *if* $\int_{\partial\tilde{\Omega}_n^j} \frac{\partial g}{\partial n} dS = 0$ *is satisfied for every component* $\partial\tilde{\Omega}_n^j$ *of any* $\partial\tilde{\Omega}_n$, *then* $\int_{\partial E} P_1 \frac{\partial g}{\partial n} dS = 0$.

For the proof, see [6]. From this lemma we can derive

$$\int_{\partial E} P_0 \frac{\partial P_0}{\partial n} dS = \int_{\partial E} P_1 \frac{\partial P_1}{\partial n} dS = 0.$$

Let u be a harmonic function defined in E^c such that

(1) $D_{E^c}(u) < \infty$, (2) $u(b) = 0$,

(3) there exists a constant C_u such that $u + C_u$ is regular at infinity,

(4) $\int_{\tau} \frac{\partial u}{\partial n} dS = 0$ for all τ in $\{\tau\}_{E^c}$.

Using Green's formula and Lemma 1, we have the equality

(3.1) $$D_{E^c}(u - P_0 + P_1) = D_{E^c}(u) - 2u(a) + h_0(a) - h_1(a).$$

We set $S(a, b)=h_0(a)-h_1(a)$ and call it the span of E^c with respect to (a, b) (cf. [6]). If we set $u=0$ in (3.1), then we obtain $S(a, b)=D_{E^c}(P_0-P_1)$. From this we have $0\leqq S(a, b)<\infty$. Accordingly the property $S(a, b)=0$ means that P_0-P_1 is a constant.

THEOREM 3. *A compact set E belongs to the class N_{KD} if and only if the span $S(a, b)$ of E^c is equal to zero for all couples (a, b) of different points in E^c.*

PROOF. Assume that there exist two different points a, b such that $S(a, b)\neq 0$. Then P_0-P_1 is a non-constant harmonic function in E^c with finite Dirichlet integral. By using the properties of P_0, P_1 and the maximum principle, we can conclude that P_0-P_1 is a bounded harmonic function outside a sufficently large sphere. Since any bounded harmonic function defined outside a compact set is expressed as the sum of a constant and a harmonic function which is regular at infinity, there exists a constant C such that P_0-P_1-C is regular at infinity.

Using Green's formula and the boundary properties of $P_i (i=0, 1)$, we have that for all τ in $\{\tau\}_{E^c}$

$$\int_\tau \frac{\partial(P_0-P_1)}{\partial n} dS=0.$$

Accordingly P_0-P_1 belongs to the class $KD(E^c)$. This shows that $E\notin N_{KD}$.

Conversely assume that $S(a, b)=0$ for all points a, b in E^c. Let u be a harmonic function in $KD(E^c)$. By making use of Green's formula and Lemma 1, we have

$$D_{E^c}(u,\ P_0-P_1)=-\int_{\partial E} u\frac{\partial(P_0-P_1)}{\partial n} dS=u(a)-u(b).$$

Since $S(a, b)=0$ implies $P_0-P_1=$const., it follows that $u(a)=u(b)$. Letting a vary in $E^c-\{b\}$, we have $u=$const. in E^c. Since u is regular at infinity, we have $u=0$. By Theorem 2 we conclude that $E\in N_{KD}$.

REMARK. From the latter half of the above proof we can derive $E\in N_{KD}$ under the condition that $S(a, b)=0$ for any point a in an open set G in E^c and some b in E^c.

Let us observe a relation between $V(E)$, the N-dimensional Lebesgue measure of E, and $E\in N_{KD}$.

LEMMA 2. *If $S(a, b)=0$ for two distinct points a, b, then $V(E)=0$.*

PROOF. Set

$$P(x)=\frac{1}{\sigma_N}\left(\frac{1}{|x-a|^{N-2}}-\frac{1}{|x-b|^{N-2}}\right)$$

Using Lemma 1 we have

$$D_{E^c}\left(P-\frac{P_0+P_1}{2}\right)=-\int_{\partial E}P\frac{\partial P}{\partial n}dS+\frac{1}{4}S(a, b).$$

Since P is harmonic on E, from the definition of Dirichlet integral that $D_E(P)=\inf_G D_G(P)$, where G runs over all open sets containing E, it follows that $\int_{\partial E}P\frac{\partial P}{\partial n}dS=D_E(P)$. By the assumption that $S(a, b)=0$ we have that

$$0\leqq D_{E^c}\left(P-\frac{P_0+P_1}{2}\right)=-D_E(P)\leqq 0,$$

so that $D_E(P)=0$. On the other hand, since the N-dimensional Lebesgue measure of the set $\left\{x \,\middle|\, \frac{\partial P}{\partial x_i}=0, i=1,\cdots, N\right\}$ equals zero, we conclude $V(E)=0$.

By Lemma 2 and Theorem 3, we have the following corollary.

COROLLARY 2. *If* $E\in N_{KD}$, *then* $V(E)=0$.

The converse of Corollary 2 is not always true. In fact, let E be a compact part of a hyperplane and Ω be a bounded domain containing E. We set

$$U^\mu(x)=\int_{e_0}\frac{d\mu(y)}{|x-y|^{N-2}}-\int_{e_1}\frac{d\mu(y)}{|x-y|^{N-2}},$$

where e_0 and e_1 are disjoint compact $(N-1)$-dimensional balls with the same radius on E and μ is the measure which consists of the equilibrium mass-distributions on e_i $(i=0, 1)$. In the same way as the proof of Theorem 1, we see that U^μ belongs to $KD(\Omega-E)$ but does not belong to $KD(\Omega)$. Therefore $E\notin N_{KD}$. In this example $V(E)=0$.

Now we shall consider another class of harmonic functions and compare this class with the KD-class.

Let $HD(\Omega)$ be the class of harmonic functions defined in a bounded domain Ω with finite Dirichlet integral. The expression $E\in N_{HD}$ is defined in the same way as N_{KD}. It is well known that $E\in N_{HD}$ if and only if the Newtonian capacity $C(E)$ of E is equal to zero; see [2]. We have obviously the inclusion $HD(\Omega-E)\supset KD(\Omega-E)$, which implies $N_{HD}\subset N_{KD}$.

We take a compact set E in Ω such that $V(E)=0$ and $C(E)>0$. Let μ be the equilibrium mass-distribution of unit mass on E and consider the potential

$$\int_E\frac{d\mu(y)}{|x-y|^{N-2}}$$

It is easy to show that this function belongs to $HD(\Omega-E)$ but does not belong to $KD(\Omega-\mathrm{E})$. Accordingly the inclusion $HD(\Omega-E)\supset KD(\Omega-E)$ is proper. Sario [9] showed a relation between N_{HD} and the span for the identity partition of E. Thus our Theorem 3 gives a result corresponding to Sario's.

§4. Extremal length

Let γ denote a locally rectifiable curve in R^N and Γ be a family of such curves. A non-negative Borel measurable function ρ is called admissible in association with Γ if $\int_\gamma \rho ds \geqq 1$ for each $\gamma \in \Gamma$. The module $M(\Gamma)$ is defined by $\inf_\rho \int \rho^2 dV$, where ρ is admissible in association with Γ, and the extremal length $\lambda(\Gamma)$ is defined by $\frac{1}{M(\Gamma)}$. The following properties are known:

(4.1) if $\Gamma' \subset \Gamma$, then $M(\Gamma') \leqq M(\Gamma)$,

(4.2) if $\Gamma=\Gamma_1 \cup \Gamma_2$ and $M(\Gamma_2)=0$, then $M(\Gamma)=M(\Gamma_1)$.

A property will be said to hold almost everywhere (=a.e.) on Γ if the extremal length of the subfamily of exceptional curves is infinite.

Let Ω be a bounded domain in R^N which contains a compact set E and $\tilde{\Gamma}$ be the family of locally rectifiable curves γ in Ω each of which starts from a point x_γ of Ω and tends to $\partial\Omega$. We shall denote by $BLD(\Omega)$ the class of Borel measurable functions f defined in Ω which are absolutely continuous along a.e. $\gamma \in \tilde{\Gamma}$ and which have finite Dirichlet integrals. We shall write $f(\gamma)$ for the limit, in case it exists, as the variable starts from x_γ and proceeds towards $\partial\Omega$ along γ. We know that for $f \in BLD(\Omega)$, $f(\gamma)$ exists and is finite for a.e. $\gamma \in \tilde{\Gamma}$; see [4].

Let α_0, α_1 be non-empty compact subsets of $\partial\Omega$ such that $\alpha_0 \cap \alpha_1 = \emptyset$ and $\tilde{\Gamma}_i$ be the subfamily of $\tilde{\Gamma}$ such that each curve of $\tilde{\Gamma}_i$ tends to α_i $(i=0, 1)$. We shall denote by $\mathcal{D}(\Omega)$ the class of functions belonging to $BLD(\Omega)$ such that $f(\gamma)=0$ for a.e. $\gamma \in \tilde{\Gamma}_0$ and $f(\gamma)=1$ for a.e. $\gamma \in \tilde{\Gamma}_1$. Let Γ(resp. Γ') be the family of locally rectifiable curves in Ω (resp. $\Omega-E$) connecting α_0 and α_1.

The following lemma is important.

LEMMA 3. (Ohtsuka) *Set*

$$C(\alpha_0, \alpha_1)=\inf_f D_{\Omega-E}(f),$$

where f runs over all elements of $\mathcal{D}$ $(\Omega-E)$. Then there exists a unique har-

monic function $f_0 \in \mathcal{D}$ $(\Omega-E)$ *such that* $C(\alpha_0, \alpha_1)=D_{\Omega-E}(f_0)$. *Moreover we have the equality* $C(\alpha_0, \alpha_1)=M(\Gamma')$.

PROOF. The proof of the first half is the same as that in [4] when we replace a Riemann surface by $\Omega-E$. Regarding the latter half, we sketch the proof given by Ohtsuka in his lectures: Extremal length in 3-space. First, note that

$$\int_\gamma |\text{grad } f_0|\, ds \geqq \left|\int_\gamma df_0\right| = 1 \qquad \text{for a.e. } \gamma \in \Gamma'.$$

Accordingly we have $M(\Gamma') \leqq C(\alpha_0, \alpha_1)$ by property (4.2). On the other hand, for any ε, $0<\varepsilon<\frac{1}{2}$, we can take a C^∞-function $\beta(x)$ in $\Omega-E$ such that $0<\beta(x)<\text{dist }(x, \partial(\Omega-E))$ and $|\text{grad } \beta|<\varepsilon$ hold. We denote by $U(x, r)$ the closed ball with center x and radius r. Take any ρ admissible in association with Γ'. Let

$$f(x)=\frac{1}{\sigma_N \beta(x)^N}\int_{U(x,\ \beta(x))} \rho\, dV$$

in $\Omega-E$ and extend it by 0 on the rest of R^N. This function is continuous in $\Omega-E$. We can see that $(1+\varepsilon)f$ is admissible in association with Γ' and obtain the inequality

$$\int_{\Omega-E} f^2 dV \leqq (1+\varepsilon)\int_{\Omega-E} \rho^2 dV.$$

For this reason we may restrict admissible ρ to be continuous in $\Omega-E$ in defining $M(\Gamma')$. Suppose $M(\Gamma')<\infty$. For a continuous function ρ admissible in association with Γ', we set

$$g(x)=\inf_\gamma \int_\gamma \rho\, ds \qquad \text{in } \Omega-E,$$

where γ is a curve in $\Omega-E$ starting from $x \in \Omega-E$ and terminating at a point of α_0. Then we can see that $g(\gamma)=0$ for a.e. $\gamma \in \tilde{\Gamma}_0$ and $g(\gamma) \geqq 1$ for a.e. $\gamma \in \tilde{\Gamma}_1$. If the segment $\overline{xx'}$ is included in $\Omega-E$, then

$$|g(x)-g(x')| \leqq \int_{\overline{xx'}} \rho\ ds.$$

From this inequality we infer that g is absolutely continuous along every curve in $\Omega-E$. Moreover, by Rademacher-Stepanov's theorem we see that grad g exists a.e. in $\Omega-E$, and that $|\text{grad } g| \leqq \rho$ a.e. in $\Omega-E$. Accordingly min $(g, 1) \in \mathcal{D}(\Omega-E)$, and hence

$$C(\alpha_0, \alpha_1) \leqq \int_{\Omega-E} |\text{grad min } (g, 1)|^2 dV \leqq \int_{\Omega-E} \rho^2 dV.$$

This implies that $C(\alpha_0, \alpha_1) \leqq M(\Gamma')$.

Next, we shall show a necessary condition for $E \in N_{KD}$.

THEOREM 4. *If $E \in N_{KD}$, then $M(\Gamma) = M(\Gamma')$ for every Ω, α_0 and α_1.*

PROOF. In view of $M(\Gamma') \leqq M(\Gamma)$, we may assume $M(\Gamma') < \infty$. Let f_0 be the extremal function in Lemma 3 such that $D_{\Omega-E}(f_0) = M(\Gamma')$. Take an open set G such that $E \subset G \subset \bar{G} \subset \Omega$ and ∂G consists of a finite number of compact C^1-surfaces. Since f_0 is the harmonic function with the smallest Dirichlet integral in the class of harmonic functions defined in $G-E$ and having boundary values f_0 on ∂G, we have $\int_\tau \frac{\partial f_0}{\partial n} dS = 0$ for all τ in $\{\tau\}_{G-E}$; cf. [3], Satz 15.1. Since any τ in $\{\tau\}_{\Omega-E}$ is homologous to a finite number of surfaces of $\{\tau\}_{G-E}$, we have $\int_\tau \frac{\partial f_0}{\partial n} dS = 0$ for any τ in $\{\tau\}_{\Omega-E}$. Therefore $f_0 \in KD(\Omega-E)$. Hence there exists a harmonic function $\hat{f}_0$ belonging to $KD(\Omega)$ such that $f_0 = \hat{f}_0$ holds in $\Omega - E$. It follows that

$$\int_\gamma |\operatorname{grad} \hat{f}_0|\, ds \geqq \left| \int_\gamma d\hat{f}_0 \right| \geqq 1 \qquad \text{for a.e. } \gamma \in \Gamma.$$

By property (4.2) this shows that $M(\Gamma) \leqq D_\Omega(\hat{f}_0)$. Accordingly we have

$$D_{\Omega-E}(f_0) = M(\Gamma') \leqq M(\Gamma) \leqq D_\Omega(\hat{f}_0).$$

By Corollary 2 the equality $D_\Omega(\hat{f}_0) = D_{\Omega-E}(f_0)$ is true.
These imply $M(\Gamma) = M(\Gamma')$.

It is open whether the converse of Theorem 4 is true or not.

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