Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 6354979, 14 pages
https://doi.org/10.1155/2018/6354979

*Research Article*

# On Robust Stability Analysis of Uncertain Discrete-Time Switched Nonlinear Systems with Time Varying Delays

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Received 15 January 2018; Revised 29 April 2018; Accepted 3 June 2018; Published 16 August 2018

Academic Editor: Sabri Arik



This paper provides new sufficient conditions on robust asymptotic stability for a class of uncertain discrete-time switched nonlinear systems with time varying delays. The main focus will be dedicated to development of new algebraic criteria to break with classical criteria in terms of linear matrix inequalities (LMIs). Firstly, by contracting a new common Lyapunov-Krasovskii functional as well as resorting to the M-matrix proprieties, a novel robust stability criterion under arbitrary switching signals is derived. Secondly, the obtained result is extended for a class of switched nonlinear systems modeled by a set of differences equations by applying the aggregation techniques, the norm vector notion, and the Borne-Gentina criterion. Furthermore, a generalization for switched nonlinear systems with multiple delays is proposed. The main contribution of this work is that the obtained stability conditions are algebraic and simple. In addition, they provide a solution of the most difficult problem in switched systems, which is stability under arbitrary switching, and enable avoiding searching a common Lyapunov function considered as a very difficult task even for some low-order linear switched systems. Finally, two examples are given, with numerical simulations, to show the merit and effectiveness of the proposed approach.

## 1. Introduction

As a special class of hybrid dynamical systems, switched systems [1] are interestingly used amongst a variety of engineering domains particularly chemical processes, automotive engine control and aircraft control, power systems, power electronics, traffic control, network communications, and many other fields [1–3].

From a theoretical point of view, stability represents one of the most significant problems for switched systems. Indeed, it has attracted a growing attention in literature [1, 4–26]. Therefore, stability of switched systems is mainly divided into two aspects: one is how to contract switching laws under which switched systems are stable. In this context, dwell time [4] and average dwell time [5] switching signals methods still play a king role. The second aspect is how to make switched systems stable under arbitrary switching. In this framework, the individual stability of all the subsystems is mandatory. Moreover, the existence of a common Lyapunov function [6, 7] for all subsystems is the unique sufficient condition to ensure stability under arbitrary switching. Unfortunately, getting such a function is a very hard task even for discrete-time switched linear systems. Therefore, this problem becomes more complicated when switched nonlinear systems are involved, and relatively available results in this context are limited [6, 21]. It is worth noting that, in a real frame, switching laws can be unknown, even imposed under a random way. For this, stability under arbitrary switching which will be considered in this paper remains undoubtedly the most interesting issue.

On the other hand, time delays especially time varying delays are frequently imposed in diverse real-world engineering systems, which would lead to performance deterioration and, in some cases, it may lead to system malfunction and instability. Consequently, wide efforts have been devoted to address the challenge of switched systems with time varying delays [10, 11, 13, 16, 17, 19, 22–26]. In addition, from the practical standpoint, it is important to tackle uncertain switched systems [18, 19, 23, 25, 26]. Thus, in this investigation, uncertain switched nonlinear systems with time varying delays with polytopic uncertainties type are considered to give a strong practical aspect for this work.

Up to now, there are few results concerning stability analysis of uncertain switched nonlinear systems with time varying delays under arbitrary switching [23]. Thus, almost all existing works deal with the linear case or the linearization of the original nonlinear systems by using the TS fuzzy models [19, 20]. It should be noted that in [20] sufficient conditions are derived to ensure the robust stability for discrete-time randomly switched fuzzy systems with known sojourn probabilities, presented in the terms of linear matrix inequalities (LMIs). However, to the best of our knowledge, randomly uncertain switched nonlinear systems with time varying delays have not been largely considered yet.

This paper seeks new algebraic practical stability criteria. Firstly, a novel robust asymptotic stability criterion for a class of discrete-time switched nonlinear systems with time varying delays and subject to polytopic uncertainties is established via constructing a new common Lyapunov functional [9], according to the vector norm notion [9–13, 27–31] and $M$ - matrix properties [32]. Secondly, the derived results are extended for a class of switched systems given by a set of difference equations. In fact, new stability conditions are obtained by transforming the considered systems representation under the arrow form matrix [29] and employing the discrete-time Borne and Gentina practical stability criterion [30, 31]. Finally, these proposed results are generalized for a class of switched systems with multiple time varying delays.

In contrast with some existing results on underlined filed, the contributions of this paper are twofold. In fact, the obtained results guarantee asymptotic stability of these considered systems under arbitrary switching and may overcome the conservatism of searching a common Lyapunov function and the LMIs constraints. In addition, these stability criteria are expressed in terms of simple algebraic conditions, explicitly, and simple.

The rest of this paper is organized as follows. Section 2 presents the problem statement and some necessary preliminaries. The main results are proposed in Section 3. New delay-dependent sufficient robust stability conditions for a class of uncertain switched nonlinear systems with time varying delays described by a set of difference equations are given in Section 4. Section 5 generalizes the obtained result for switched systems with multiple delays. Finally, two case studies are presented in Section 6 to show the effectiveness of the provided results. Conclusions are given in Section 7.

*Notation*. The notation used here is fairly standard except where otherwise stated. For a matrix $A$, we denote the transpose by $A^T$. Let $\mathfrak{R}$ denote the field of real numbers and $\mathfrak{R}^n$ denote an $n$ dimensional linear vector space over the reals with the norm $\|.\|$. For any $u = (u_i)_{1\le i\le n}$, $v = (v_i)_{1\le i\le n} \in \mathfrak{R}^n$, we define the scalar product of the vectors $u$ and $v$ as $\langle u, v\rangle = \sum_{i=1}^{n} u_i v_i$, $\mathfrak{R}^{n\times n}$ the space of $n \times n$ matrices with real entries. $\mathfrak{R}_+$ is the set of positive real numbers. $I[k_1\ \ k_2]$ denotes the set of integers $\{k_1, k_1 + 1, k_1 + 2, ..., k_2\}$ and $I_n$ is the identity matrix with appropriate dimension.

## 2. Problem Statement and Preliminaries

*2.1. Problem Statement.* Consider the following uncertain discrete-time switched nonlinear system with time varying delays described by

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}(.)\,x(k) + D_{\sigma(k)}(.)\,x\left(k-\tau_{\sigma(k)}(k)\right) \\ x(\theta) &= \phi(\theta) \\ s &= -r_2, -r_2+1, \dots, -1 \end{aligned} \tag{1}$$

where $x(k) \in \mathfrak{R}^n$ denotes the state vector, $\sigma(k) : [0, \infty) \longrightarrow I[1\ \ L]$ is the switching signal which is a piecewise constant function, $L$ is the number of subsystems, and $\sigma(k) = i$ implies that the $i - th$ subsystem is activated. $\tau_i(k)$, $i = 1, 2, ..., L$ are the time varying delays satisfying $0 \le r_1 \le \tau_i(k) \le r_2$, where $r_1$ and $r_2$ are constant scalars.

Therefore, the switched system is composed of $L$ subsystems which are expressed as

$$x(k+1) = A_i(.)\,x(t) + D_i(.)\,x(k-\tau(k)), \quad i \in I[1\ \ L] \tag{2}$$

where $A_i(.)$ $i \in I[1\ \ L]$ and $D_i(.)$ $i \in I[1\ \ L]$ are matrices with appropriate dimensions having nonlinear elements.

Assuming that all subsystems are uncertain of polytopic type which can be described as

$$A_i(.) = \sum_{p=1}^{L_p} \mu_{ip}(k)\,A_{ip}(.), \quad i \in I[1\ \ L] \tag{3}$$

$$D_i(.) = \sum_{q=1}^{L_q} \lambda_{iq}(k)\,D_{iq}(.), \quad i \in I[1\ \ L] \tag{4}$$

where $A_{ip}(.)$, $p \in I[1\ \ L_p]$ and $D_{iq}(.)$ $q \in I[1\ \ L_q]$ are, respectively, the vertex matrices denoting the extreme points of the polytope $A_i(.)$, $i \in I[1\ \ L]$ and $D_i(.)$, $i \in I[1\ \ L]$, $L_p$ is the number of the vertex matrices $A_i(.)$, $L_q$ is the number of the vertex matrices $D_i(.)$, and the weighting factors $\mu_{ip}(k)$ $p \in I[1\ \ L_p]$, $\lambda_{iq}(k)$ $q \in I[1\ \ L_q]$ are unknown polytopic uncertainties parameters for each $i \in I[1\ \ L]$ belonging to $\mu_i(k) : \sum_{p=1}^{L_p} \mu_{ip}(k) = 1$, $\mu_{ip}(k) \ge 0$ and $\lambda_{iq}(k)$: $\sum_{q=1}^{N_q} \lambda_{iq}(k) = 1$, $\lambda_{iq}(k) \ge 0$.

*2.2. Preliminaries.* Now, the following lemmas, criterion, remark, and definitions are preliminarily presented for later development.

**Lemma 1** (see [32]). *The matrix $A(.)$ is said to be an $M -$* matrix *if the following properties are verified:*

(i) *All the eigenvalues of $A(.)$ have a positive real part.*

(ii) *The real eigenvalues are positives.*

(iii) *The principal minors of $A(.)$ are positive:*

$$(A(.))\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} > 0, \quad \forall j \in I[1\ \ n]. \tag{5}$$

(iv) *For any positive vector $x = (x_1, ..., x_n)^T$ the algebraic equation $A(.)x$ admits a positive solution $w = (w_1, ..., w_n)^T$.*

In Kotelyanski lemma [33], the real parts of the eigenvalues of the matrix $A(.)$, with nonnegative off-diagonal elements, are less than a real number $\mu$ if and only if all those of matrix $M(.)$, where $M(.) = \mu I_n - A(.)$, are positive, with $I_n$ the $n$ identity matrix.

In this case, all the principal minors of matrix $(-A(.))$ are positive. Then, the Kotelyanski lemma permits deducing on stability properties of the system given by $A(.)$.

Now, we will introduce the discrete-time Borne and Gentina practical stability criterion and the pseudo-overvaluing matrix.

In discrete-time Borne and Gentina practical stability criterion [30, 31], let us consider the discrete-time nonlinear system $x(k+1) = A(.)x(k)$ and the overvaluing matrix $T_M(.) = \{|a_{j,k}|\}\ \forall j,k = 1,...,n$. If the nonlinearities are isolated in either one row or one column of $T_M(.)$, the verification of the Kotelyanski condition would enable us to achieve stability conclusion of the original system characterized by $A(.)$. The Kotelyanski lemma applied to the overvaluing matrix obtained by the use of the regular vector norm $p(w) = [|w_1|, |w_2|, ...|w_n|]$ with $w = [w_1, w_2, ...w_n]$ leads to the following sufficient conditions of asymptotic stability of original system: $(I_n - T_M(.))\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} > 0$ $(j = 1, 2, ..., n)$.

*Definition 2* (see [28]). The matrix $T_M(.)$ is the comparison matrix of the system given by a matrix $A(.)$ with respect to the vector norm $p$ if the inequality

$$p(x(k+1)) \le T_M(.)\, p(x(k)) \tag{6}$$

$\forall x \in E$ and $k > 0$ is verified for each corresponding component. Then, the stability of the comparison system $z(k+1) = T_M(.)z(k)$ with the initial conditions such as $z_0 = p(x_0)$.

In this case, the following properties are satisfying:

(i) If all the elements of $T_M(.)$ are nonnegative, it is assumed that the eigenvalue of $T_M(.)$, the biggest in module, is both real and positive and is called main eigenvalue of $T_M(.)$.

(ii) If all the elements of matrix $T_M(.)$ are nonnegative, it is assumed that the principal minors of $(I_n - T_M(.))$ are all positive, the spectral radius of $T_M(.)$ is inferior to the unit, and all the elements of $(I_n - T_M(.))^{-1}$ are nonnegative.

(iii) When $T_M(.)$ is an irreducible matrix, the main eigenvector of $T_M(.)$ is the same as of $(T_M(.))^T$ and all their elements are nonnegative.

*Remark 3.* A discrete-time system given by a matrix $A(.)$ is stable if matrix $(I_n - T_M(.))$ verified the Kotelyanski conditions; in this case $(I_n - T_M(.))$ is an $M-$matrix.

## 3. Main Results

In this subsection, we present our first result on robust stability analysis for system (1).

**Theorem 4.** *System (1) is robustly asymptotically stable under arbitrary switching rule $\sigma(k)$ and all admissible uncertainties (3) and (4) if $(I_n - T_M(.))$ is an $M-$matrix, where*

$$\begin{aligned} T_M(.) = {} & \max_{\substack{i\in I[1\ \ L] \\ p\in I[1\ \ L_p]}} \left(\left|A_{ip}(.)\right|\right) \\ & + (r_2 - r_1 + 1) \max_{\substack{i\in I[1\ \ L] \\ q\in I[1\ \ L_q]}} \left(\left|\sup_{[.]}\left(D_{iq}(.)\right)\right|\right) \end{aligned} \tag{7}$$

*Proof.* Let us consider system (1) for all admissible uncertainties (3) and (4); let $w \in \Re^n$ with components $(w_p > 0, \forall p = 1, ..., n)$ and $x(k) \in \Re^n$ be the state vector.

Choose the common Lyapunov-Krasovskii functional:

$$\begin{aligned} V(x(k), k) = {} & V_1(x(k), k) + V_2(x(k), k) \\ & + V_3(x(k), k) \end{aligned} \tag{8}$$

where

$$V_1(x(k), k) = \langle |x(k)|, w\rangle \tag{9}$$

$$V_2(x(k), k) = \sum_{j=k-\tau_{\sigma(k)}(k)}^{k-1} \langle D_M |x(j)|, w\rangle \tag{10}$$

$$V_3(x(k), k) = \sum_{m=-r_2+2}^{-r_1+1} \sum_{d=k+m+1}^{k-1} \langle D_M |x(l)|, w\rangle \tag{11}$$

with

$$D_M = \max_{\substack{i\in I[1\ \ L] \\ q\in I[1\ \ L_q]}} \left(\left|\sup_{[.]}\left(D_{iq}(.)\right)\right|\right). \tag{12}$$

We can easily verify that $V(x(k), k) \ge 0$.

Hence, we get that the difference of the Lyapunov functional $V_1(x(k), k)$ alongside the trajectories of system (1) has the following form:

$$\begin{aligned} \Delta V_1(x(k), k) &= \langle |x(k+1)|, w\rangle - \langle |x(k)|, w\rangle \\ &= \left\langle \left|A_{\sigma(k)}(.)\,x(k) + D_{\sigma(k)}(.)\,x(k - \tau_{\sigma(k)}(k))\right|, w\right\rangle \\ &\quad - \langle |x(k)|, w\rangle \le \left\langle \left|A_{\sigma(k)}(.)\right| |x(k)| \right. \\ &\quad \left. + \left|D_{\sigma(k)}(.)\right| \left|x(k - \tau_{\sigma(k)}(k))\right|, w\right\rangle - \langle |x(k)|, w\rangle \\ &= \left\langle \left|A_{\sigma(k)}(.)\right| |x(k)|, w\right\rangle \\ &\quad + \left\langle \left|D_{\sigma(k)}(.)\right| \left|x(k - \tau_{\sigma(k)}(k))\right|, w\right\rangle - \langle |x(k)|, w\rangle \\ &\le \left\langle A_M(.)\,|x(k)|, w\right\rangle \\ &\quad + \left\langle D_M \left|x(k - \tau_{\sigma(k)}(k))\right|, w\right\rangle - \langle |x(k)|, w\rangle \end{aligned} \tag{13}$$

with

$$A_M(.) = \max_{\substack{i\in I[1\quad L]\\ p\in I[1\quad L_p]}} \left(\left|A_{ip}(.)\right|\right). \tag{14}$$

Moreover, $\Delta V_2(x(k),k)$ is given as follows:

$$\begin{aligned}
\Delta V_2(x(k),k) &= \sum_{j=k+1-\tau_{\sigma(k+1)}(k+1)}^{k} \left\langle D_M\left|x(j)\right|,w\right\rangle \\
&\quad - \sum_{j=k-\tau_{\sigma(k)}(k)}^{k-1} \left\langle D_M\left|x(j)\right|,w\right\rangle \\
&= \sum_{j=k+1-\tau_{\sigma(k+1)}(k+1)}^{k-r_1} \left\langle D_M\left|x(j)\right|,w\right\rangle \\
&\quad + \left\langle D_M\left|x(k)\right|,w\right\rangle \\
&\quad - \left\langle D_M\left|x\left(k-\tau_{\sigma(k)}(k)\right)\right|,w\right\rangle \\
&\quad + \sum_{j=k+1-r_1}^{k-1} \left\langle D_M\left|x(j)\right|,w\right\rangle \\
&\quad - \sum_{j=k+1-\tau_{\sigma(k)}(k)}^{k-1} \left\langle D_M\left|x(j)\right|,w\right\rangle .
\end{aligned} \tag{15}$$

Since $\tau_{\sigma(k)}(k) \geq r_1$ one can have

$$\sum_{j=k+1-r_1}^{k-1} \left\langle D_M\left|x(j)\right|,w\right\rangle - \sum_{j=k+1-\tau_{\sigma(k)}(k)}^{k-1} \left\langle D_M\left|x(j)\right|,w\right\rangle \leq 0. \tag{16}$$

From (15), we have

$$\begin{aligned}
\Delta V_2(x(k),k) &\leq \sum_{j=k+1-\tau_{\sigma(k+1)}(k+1)}^{k-r_1} \left\langle D_M\left|x(j)\right|,w\right\rangle \\
&\quad + \left\langle D_M\left|x(k)\right|,w\right\rangle \\
&\quad - \left\langle D_M\left|x\left(k-\tau_{\sigma(k)}(k)\right)\right|,w\right\rangle .
\end{aligned} \tag{17}$$

Finally, $\Delta V_3(x(k),k)$ is estimated as

$$\begin{aligned}
&\Delta V_3(x(k),k) \\
&= \sum_{m=-r_2+2}^{-r_1+1} \sum_{l=k+m}^{k} \left\langle D_M\left|x(l)\right|,w\right\rangle \\
&\quad - \sum_{m=-r_2+2}^{-r_1+1} \sum_{l=k+m+1}^{k-1} \left\langle D_M\left|x(l)\right|,w\right\rangle \\
&= \sum_{m=-r_2+2}^{-r_1+1} \sum_{l=k+m}^{k-1} \left\langle D_M\left|x(l)\right|,w\right\rangle + \left\langle D_M\left|x(k)\right|,w\right\rangle \\
&\quad - \sum_{m=-r_2+2}^{-r_1+1} \sum_{l=k+m}^{k-1} \left\langle D_M\left|x(l)\right|,w\right\rangle \\
&\quad - \sum_{m=-r_2+2}^{-r_m+1} \sum_{l=k+m}^{k-1} \left\langle D_M\left|x(k+m-1)\right|,w\right\rangle \\
&= \sum_{m=-r_2+2}^{-r_1+1} \left\langle D_M\left|x(k)\right|,w\right\rangle \\
&\quad - \sum_{m=-r_2+2}^{-r_1+1} \left\langle D_M\left|x(k+m-1)\right|,w\right\rangle \\
&= \left(r_2-r_1\right)\left\langle D_M\left|x(k)\right|,w\right\rangle \\
&\quad - \sum_{m=k+1+r_2}^{k-r_1} \left\langle D_M\left|x(m)\right|,w\right\rangle .
\end{aligned} \tag{18}$$

Since $\tau_{\sigma(k)}(k) \leq r_2$, we have

$$\sum_{j=k+1-\tau_{\sigma(k+1)}(k+1)}^{k-r_1} \left\langle D_M\left|x(j)\right|,w\right\rangle - \sum_{j=k+1-r_2}^{k-r_1} \left\langle D_M\left|x(j)\right|,w\right\rangle \leq 0. \tag{19}$$

Thus, it follows from (17) and (18) that

$$\begin{aligned}
&\Delta V_3(x(k),k) + \Delta V_2(x(k),k) \\
&\quad \leq \left(r_2-r_1\right)\left\langle D_M\left|x(k)\right|,w\right\rangle + \left\langle D_M\left|x(k)\right|,w\right\rangle \\
&\qquad - \left\langle D_M\left|x\left(k-\tau_{\sigma(k)}(k)\right)\right|,w\right\rangle .
\end{aligned} \tag{20}$$

Combining (13) and (20) gives

$$\begin{aligned}
&\Delta V(x(k),k) \\
&\quad \leq \left\langle A_M(.)\left|x(k)\right|,w\right\rangle \\
&\qquad + \left\langle D_M\left(r_2-r_1+1\right)\left|x(k)\right|,w\right\rangle - \left\langle\left|x(k)\right|,w\right\rangle \\
&\quad = \left\langle\left(A_M(.) + \left(r_2-r_1+1\right)D_M - I_n\right)\left|x(k)\right|,w\right\rangle \\
&\quad = \left\langle\left(T_M(.) - I_n\right)\left|x(k)\right|,w\right\rangle
\end{aligned} \tag{21}$$

where $T_M(.)$ is given in (4).

On the other hand, we assume that $(I_n - T_M(.))$ is an $M-$matrix. Indeed, we can give a vector $\rho \in \mathfrak{R}_+^{*n}$ ($\rho_p \in \mathfrak{R}_+^{*}$ $p = 1,\ldots,n$) such that $(I_n - T_M(.))^T w = \rho$, $\forall w \in \mathfrak{R}_+^{*n}$.

Thus, we get

$$\begin{aligned}
&\left\langle\left(I_n - T_M(.)\right)\left|x(k)\right|,w\right\rangle \\
&\quad = \left\langle\left(I_n - T_M(.)\right)^T w,\left|x(k)\right|\right\rangle = \left\langle\rho,\left|x(k)\right|\right\rangle .
\end{aligned} \tag{22}$$

Then we have

$$\langle (T_M(.) - I_n)|x(k)|, w\rangle = \langle -\rho, |x(k)|\rangle . \quad (23)$$

Finally, we obtain

$$\Delta v(k) \le \langle (T_M(.) - I_n)|x(k)|, w\rangle \le -\sum_{p=1}^{n}\rho_p |x_p(k)| < 0. \quad (24)$$

The proof is completed. □

According to the Lyapunov theory, robust stability condition of system (1), under switching law $\sigma(k) = i \in I[1 \;\; L]$ by Theorem 4, is immediately derived.

## 4. Application to Discrete-Time Uncertain Switched Nonlinear Systems with Time Varying Delays Defined by Difference Equations

In the sequel, to illustrate the effectiveness of the obtained results, as application of discrete-time uncertain switched nonlinear systems with time varying delays modeled by difference equations will be proposed.

Let us consider the following functional difference equation for all the subsystems:

$$S_i : y(k+n) + \left(\sum_{p=1}^{L_p}\mu_{ip}(k)\sum_{m=0}^{n-1}a_{ip}^{n-m}(.)\,y(k+m) + \cdot \sum_{q=1}^{L_q}\lambda_{iq}(k)\sum_{m=0}^{n-1}d_{iq}^{n-m}(.)\,y(k+m-\tau_i(k))\right) = 0 \quad (25)$$

where $\mu_{ip}(k)$ $p \in I[1 \;\; L_p]$ and $\lambda_{iq}(k)$ $q \in I[1 \;\; L_q]$ denote the polytopic uncertain parameters as defined in (3) and (4) for each $i \in I[1 \;\; L]$. $y(k) \in \Re$ is the output and $n$ is the subsystem order. $a_{ip}^{n-m}(.)$ and $d_{iq}^{n-m}(.)$ are nonlinear coefficients.

Consider the following change of variable:

$$x_m(k+1) = x_{m+1}(k), \quad m = 1, \ldots, n-1. \quad (26)$$

Combining (25) and (26) leads to

$$x_n(k+1) = -\sum_{p=1}^{L_p}\mu_{ip}(k)\sum_{m=0}^{n-1}a_{ip}^{n-m}(.)\,x_{m+1}(k) - \sum_{q=1}^{L_q}\lambda_{iq}(k)\sum_{m=0}^{n-1}d_{ip}^{n-m}(.)\,x_{m+1}(k-\tau_i(k)). \quad (27)$$

From (27), subsystems $S_i$, $i \in I[1 \;\; L]$ will be given under matrix representation as follows:

$$x(k+1) = \sum_{p=1}^{L_l}\mu_{ip}(k)A_{ip}(.)\,x(k) + \sum_{q=1}^{L_q}\lambda_{iq}(k)D_{iq}(.)\,x(k-\tau_i(k))$$
$$x(\theta) = \phi(\theta)$$
$$s = -r_2, -r_2+1, \ldots, -1 \quad (28)$$

$$A_{ip}(.) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ -a_{ip}^{n}(.) & -a_{ip}^{n-1}(.) & \cdots & -a_{ip}^{1}(.) \end{bmatrix}, \quad \forall i \in I[1 \;\; L],\ \forall p \in I[1 \;\; L_p] \quad (29)$$

$$D_{iq}(.) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -d_{iq}^{n}(.) & -d_{iq}^{n-1}(.) & \cdots & -d_{iq}^{1}(.) \end{bmatrix} \quad \forall i \in I[1 \;\; L],\ \forall q \in I[1 \;\; L_q]. \quad (30)$$

According to the switching signal $\sigma(k)$, the resultant switched system will be given by

$$x(k+1) = A_{\sigma(k)}(.)\,x(k) + D_{\sigma(k)}(.)\,x(k-\tau_{\sigma(k)}(k))$$
$$x(\theta) = \phi(\theta)$$
$$s = -r_2, -r_2+1, \ldots, -1. \quad (31)$$

The regular basis change $P$ permits characterizing the dynamics of subsystems $S_i$ by the change of coordinate defined by

$$z(k) = Px(k) \quad (32)$$

with

$$P = \begin{bmatrix} 1 & 1 & \ldots & 1 & 0 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{n-1} & 0 \\ (\alpha_1)^2 & (\alpha_2)^2 & \ldots & (\alpha_{n-1})^2 & \vdots \\ \vdots & \vdots & \ldots & \vdots & 0 \\ (\alpha_1)^{n-1} & (\alpha_2)^{n-1} & \ldots & (\alpha_{n-1})^{n-1} & 1 \end{bmatrix} \quad \alpha_j \ne \alpha_q \;\; \forall j, q. \quad (33)$$

Combining (30), (31), and (32) leads to the following state representation:

$$\begin{aligned} z(k+1) &= \sum_{p=1}^{L_p} \mu_{ip}(k) E_{ip}(.) z(k) \\ &\quad + \sum_{q=1}^{L_q} \lambda_{iq}(k) F_{iq}(.) z(k-\tau_i(k)) \end{aligned} \tag{34}$$
$$z(\theta) = P\phi(\theta)$$
$$s = -r_2, \ldots, -1, 0$$

where

$$E_{ip}(.) = P^{-1} A_{ip}(.) P = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 & \beta_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \\ \gamma_{ip}^1(.) & \cdots & \cdots & \gamma_{ip}^{n-1}(.) & \gamma_{ip}^n(.) \end{bmatrix}. \tag{35}$$

The elements of the vertex matrix $E_{ip}(.)$ are defined by

$$\begin{aligned} \gamma_{ip}^j(.) &= -G_{ip}(\alpha_j), \quad \forall j = 1, \ldots, n-1 \\ \gamma_{ip}^n(.) &= -a_{ip}^1(.) - \sum_{j=1}^{n-1} \alpha_j \end{aligned} \tag{36}$$

where

$$G_{A_{ip}(.)}(s,.) = s^n + \sum_{m=0}^{n-1} a_{ip}^{n-m}(.) s^m \tag{37}$$

and

$$\beta_j = \prod_{\substack{q=1 \\ q\neq j}}^{n-1} (\alpha_j - \alpha_q)^{-1}, \quad \forall j = 1, \ldots, n-1. \tag{38}$$

The vertex matrix $F_{iq}(.)$is given by

$$F_{iq}(.) = P^{-1} D_{iq}(.) P = \begin{bmatrix} 0_{n-1,n-1} \cdots & 0_{n-1,1} \\ \delta_{iq}^1(.) \cdots \delta_{iq}^{n-1}(.) & \delta_{iq}^n(.) \end{bmatrix} \tag{39}$$

with

$$\begin{aligned} \delta_{iq}^j(.) &= -N_{D_{iq}(.)}(\alpha_j), \quad \forall j = 1, \ldots, n-1 \\ \delta_{iq}^n(.) &= -d_{iq}^1(.) \end{aligned} \tag{40}$$

and

$$N_{D_{iq}(.)}(s,.) = \sum_{m=0}^{n-1} d_{iq}^{n-m}(.) s^m. \tag{41}$$

Thus, the matrix $T_i(.)$ is given as follows:

$$T_i(.) = \begin{bmatrix} |\alpha_1| & 0 & \cdots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & |\alpha_{n-1}| & |\beta_{n-1}| \\ t_i^1(.) & \cdots & \cdots & t_i^{n-1}(.) & t_i^n(.) \end{bmatrix}, \quad \forall i \in I[1 \quad L] \tag{42}$$

with

$$t_{i,p,q}^j(.) = |\gamma_{ip}^j(.)| + (r_2 - r_1 + 1) \left| \sup_{[.]} (\delta_{iq}^j(.)) \right|, \quad j = 1, \ldots, n-1 \tag{43}$$
$$t_{i,p,q}^n(.) = |\gamma_{ip}^n(.)| + (r_2 - r_1 + 1) \left| \sup_{[.]} (\delta_{iq}^n(.)) \right|.$$

Finally, the comparison matrix $T_M(.)$ of system (31) will be given as follows:

$$T_M(.) = \begin{bmatrix} |\alpha_1| & 0 & \cdots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & |\alpha_{n-1}| & |\beta_{n-1}| \\ \bar{t}^n(.) & \cdots & \cdots & \bar{t}^n(.) & \bar{t}^n(.) \end{bmatrix} \tag{44}$$

with

$$\begin{aligned} \bar{t}^n(.) &= \max_{\substack{i\in I[1 \quad L] \\ p\in I[1 \quad L_p] \\ q\in I[1 \quad L_q]}} (t_{i,p,q}^n(.)) \\ \bar{t}^j(.) &= \max_{\substack{i\in I[1 \quad L] \\ p\in I[1 \quad L_p] \\ q\in I[1 \quad L_q]}} (t_{i,p,q}^j(.)), \quad j = 1, \ldots, n-1. \end{aligned} \tag{45}$$

Now, based on the Borne and Gentina practical stability criterion, we are in a position to give sufficient robust stability conditions of system (31) that are illustrated in the following theorem.

**Theorem 5.** *System (31) is robustly asymptotically stable, under arbitrary switching signal $\sigma(k)$ and all admissible uncertainties (3) and (4), if there exist $\alpha_j$ ($j = 1, \ldots, n-1$), $\alpha_j \neq \alpha_q, \forall j \neq q$, satisfying the following conditions:*

$$(i) \quad 1 - |\alpha_j| > 0, \quad j = 1, \ldots n-1 \tag{46}$$

$$(ii) \quad 1 - (\bar{t}^n(.)) - \sum_{j=1}^{n-1} (\bar{t}^j(.)) |\beta_j| (1 - |\alpha_j|)^{-1} > 0 \tag{47}$$

*Proof.* For an arbitrary choice $|\alpha_j| < 1$ $(j = 1, \ldots, n-1)$, $\alpha_j \neq \alpha_q, \forall j \neq q$ and from (44), it is clear that all the elements of $T_M(.)$ are isolated in one row and positive. The verification of the Borne and Gentina practical stability criterion enables concluding to the stability of the original system (31).

Therefore, it comes the following sufficient robust asymptotic stability conditions:

$$\left(I_n - T_M(.)\right)\begin{pmatrix} 1 & 2 & \cdots & j \\ 1 & 2 & \cdots & j \end{pmatrix} > 0 \quad (j = 1, \ldots, n). \tag{48}$$

It is clear that, for $j = 1, \ldots n-1$, condition (46) is verified as follows: $0 < |\alpha_j| < 1$. The last condition $j = n$ gives us

$$\det\left(I_n - T_M(.)\right) = \begin{vmatrix} 1-|\alpha_1| & 0 & \cdots & 0 & -|\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1-|\alpha_{n-1}| & -|\beta_{n-1}| \\ -\bar{t}^1(.) & \cdots & \cdots & -\bar{t}^{n-1}(.) & 1-\bar{t}^n(.) \end{vmatrix}. \tag{49}$$

This implies $1 - (\bar{t}^n(.)) - \sum_{j=1}^{n-1}(\bar{t}^j(.))|\beta_j|(1-|\alpha_j|)^{-1} > 0$.

This completes this proof. □

To simplify the use of the obtained stability conditions, Theorem 5 can be reduced to Corollary 6.

**Corollary 6.** *If system (31) is robustly asymptotically stable under arbitrary switching $\sigma(k)$ and all admissible uncertainties (3) and (4), the following conditions are satisfied $\forall\alpha_j \in ]0 \;\; 1[$ $(j = 1, \ldots, n-1)$ $\alpha_j \neq \alpha_q$ $\forall j \neq q$, for each $i \in I[1 \;\; L]$, $p \in I[1 \;\; L_p]$, and $q \in I[1 \;\; L_q]$:*

(i) $$\beta_j\left(G_{A_{ip}(.)}(\alpha_j,.) + (r_2 - r_1 + 1)\sup_{[.]}\left(N_{D_{iq}(.)}(\alpha_j,.)\right)\right) < 0 \tag{50}$$

(ii) $$\left(G_{A_{ip}(.)}(s = 1,.) + (r_2 - r_1 + 1)\sup_{[.]}\left(N_{D_{iq}(.)}(s = 1,.)\right)\right) > 0 \tag{51}$$

(iii) $$\left(\gamma_{ip}^n(.) + (r_2 - r_1 + 1)\sup_{[.]}\left(\delta_{iq}^n(.)\right)\right) > 0 \tag{52}$$

*Proof (see [9]).* By (47), we can obtain the following result:

$$\max_{\substack{i\in I[1\;\;L]\\ p\in I[1\;\;L_p]\\ q\in I[1\;\;L_q]}}\left(\left|\gamma_{ip}^n(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\delta_{iq}^n(.)\right)\right|\right) + \sum_{j=1}^{n-1}\left(\max_{\substack{i\in I[1\;\;L]\\ p\in I[1\;\;L_p]\\ q\in I[1\;\;L_q]}}\left(\left|\gamma_{ip}^j(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\delta_{iq}^j(.)\right)\right|\right)\right)\left|\beta_j\right|\left(1-\left|\alpha_j\right|\right)^{-1} < 1. \tag{53}$$

This implies

$$1 - \max_{\substack{i\in I[1\;\;L]\\ p\in I[1\;\;L_p]\\ q\in I[1\;\;L_q]}}\left(\left|\gamma_{ip}^n(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\delta_{iq}^n(.)\right)\right|\right) - \sum_{j=1}^{n-1}\left(\max_{\substack{i\in I[1\;\;L]\\ p\in I[1\;\;L_p]\\ q\in I[1\;\;L_q]}}\left(\left|\gamma_{ip}^j(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\delta_{iq}^j(.)\right)\right|\right)\right)\left|\beta_j\right|\left(1-\left|\alpha_j\right|\right)^{-1} > 0. \tag{54}$$

It is easy to deduce that the relation below is more restrictive than (54)

$$1 - \left(\left|\gamma_{ip}^n(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\delta_{iq}^n(.)\right)\right|\right) - \sum_{j=1}^{n-1}\left(\left(\left|\gamma_{ip}^j(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\delta_{iq}^j(.)\right)\right|\right)\right) \cdot \left|\beta_j\right|\left(1-\left|\alpha_j\right|\right)^{-1} > 0. \tag{55}$$

Therefore, In order to deduce the stability conditions of system (31), we consider (50) and (52) to find (51).

Thus, we deduce from (50), (52), and (55) that

$$1 + \left(a_{ip}^1(.) + (r_2 - r_1 + 1)\sup_{[.]}\left(d_{iq}^1(.)\right)\right) + \sum_{j=1}^{n-1}\alpha_j$$

$$+\sum_{j=1}^{n-1}\left(\frac{1}{\left(1-\alpha_j\right)}\left(\frac{\left(s-\alpha_j\right)\left(G_{A_{ip}(.)}(s,.)+\left(r_2-r_1+1\right)\sup_{[.]}\left(N_{D_{iq}(.)}(s,.)\right)\right)}{H(s)}\right)\right)_{s=\alpha_j}>0 \tag{56}$$

with

$$H(s)=\prod_{j=1}^{n-1}\left(s-\alpha_j\right). \tag{57}$$

To complete this proof, let us first observe that

$$\begin{aligned}&\frac{\left(G_{A_{ip}(.)}(s,.)+\left(r_2-r_1+1\right)\sup_{[.]}\left(N_{D_{iq}(.)}(s,.)\right)\right)}{H(s)}\\&=s+\left(a_{ip}^1(.)+\left(r_2-r_1+1\right)\sup_{[.]}\left(d_{iq}^1(.)\right)\right)+\sum_{j=1}^{n-1}\alpha_j\\&+\sum_{j=1}^{n-1}\left(\frac{\left(s-\alpha_j\right)\left(G_{A_{ip}(.)}(s,.)+\left(r_2-r_1+1\right)\sup_{[.]}\left(N_{D_{iq}(.)}(s,.)\right)\right)}{\left(1-\alpha_j\right)H(s)}\right)_{s=\alpha_j}.\end{aligned} \tag{58}$$

From (56), (57), and (58), we have

$$\left(\frac{G_{A_{ip}(.)}(s,.)+\left(r_2-r_1+1\right)\sup_{[.]}\left(N_{D_{iq}(.)}(s,.)\right)}{H(s)}\right)_{s=1}>0. \tag{59}$$

By a simple verification, we obtain

$$H(s=1)=\prod_{j=1}^{n-1}\left(1-\alpha_j\right)>0 \quad \forall\alpha_j\in\,]0\ \ 1[ \tag{60}$$

which yields

$$\left(G_{A_{ip}(.)}(s=1,.)+\left(r_2-r_1+1\right)\sup_{[.]}\left(N_{D_{iq}(.)}(s=1,.)\right)\right)>0. \tag{61}$$

This ends the proof. □

## 5. Extension Results for Uncertain Switched Nonlinear Systems with Multiple Varying Delays

This subsection is aimed at generalizing the previously reached results to uncertain switched nonlinear systems with multiple varying delays systems.

Considering a class of uncertain switched nonlinear systems with multiple varying delays formed by $L$ subsystems given by

$$\begin{aligned}x(k+1)&=A_{\sigma(k)}(.)\,x(k)\\&+\sum_{l=1}^{h}D_{\sigma(k),l}(.)\,x\left(k-\tau_{l,\sigma(k)}(k)\right)\\x(\theta)&=\phi(\theta)\\&\theta=-\max_{1\le l\le m}\left(r_{2,l}\right),-\max_{1\le l\le m}\left(r_{2,l}\right)+1,\ldots,-1\end{aligned} \tag{62}$$

where $\sigma(k)$ is the switching signal given in (1), $A_i(.)\in\mathfrak{R}^{n\times n}$ and $D_{l,i}(.)\in\mathfrak{R}^{n\times n}(l=1,\ldots h)$ are matrices of appropriate dimensions with nonlinear elements of appropriate dimensions. $\tau_{i,l}(k)$, $i\in I\left[1\ \ L\right]$ and $l\in I\left[1\ \ h\right]$ are the time varying delays satisfying $0\le r_{1,l}\le\tau_{i,l}(k)\le r_{2,l}$, where $r_{1,l}$ and $r_{2,l}$ $l\in I\left[1\ \ h\right]$ are constant scalars.

We assume that $A_i(.)$ and $D_{l,i}(.)$ are uncertain of polytopic type described by

$$A_i(.)=\sum_{p=1}^{L_p}\mu_{ip}(k)\,A_{ip}(.),\quad i\in I\left[1\ \ L\right] \tag{63}$$

$$D_{i,l}(.)=\sum_{q=1}^{L_{q,l}}\lambda_{iq,l}(k)\,D_{iq,l}(.),\quad i\in I\left[1\ \ L\right]. \tag{64}$$

Now, it became easier to establish the following sufficient stability conditions for system (62) by generalizing the common Lyapunov-Krasovskii functional (8).

**Theorem 7.** *System (62) is globally robustly asymptotically stable under arbitrary switching* $\sigma(k)$ *and all admissible uncertainties (63) and (64) if* $(I_n - T_{l,M}(.))$ *is an* $M-$ matrix, *where*

$$T_{M,l}(.) = \max_{\substack{i\in I[1\ \ L]\\ p\in I[1\ \ L_p]}} \left(\left|A_{ip}(.)\right|\right) + (r_2 - r_1 + 1) \max_{\substack{i\in I[1\ \ L]\\ q\in I[1\ \ L_q]}} \left(\left|\sup_{[.]}\left(\sum_{l=1}^{h}\left(D_{iq,l}(.)\right)\right)\right|\right) \tag{65}$$

*with* $r_2 = \max_{l\in I[1\ h]}(\tau_{2,l})$ *and* $r_1 = \min_{l\in I[1\ h]}(\tau_{1,l})$.

*Proof.* It suffices to choose the following Lyapunov function $V(x(k),k)$ and follow the same steps as given in the proof of Theorem 5.

$$V(x(k),k) = V_1(x(k),k) + V_2(x(k),k) + V_3(x(k),k) \tag{66}$$

where

$$V_1(x(k),k) = \langle |x(k)|, w\rangle \tag{67}$$

$$V_2(x(k),k) = \sum_{l=1}^{h}\sum_{j=k-\tau_{l,\sigma(k)}}^{k-1} \langle D_{M,l}\left|x(j)\right|, w\rangle \tag{68}$$

$$V_3(x(k),k) = \sum_{l=1}^{h}\sum_{j=k-\tau_{l,\sigma(k)}}^{k-1}\sum_{m=-r_{l,2}+2}^{-r_{1,l}+1}\sum_{d=k+m+1}^{k-1} \langle D_{M,l}\left|x(l)\right|, w\rangle \tag{69}$$

with

$$D_{M,l} = \max_{\substack{i\in I[1\ \ L]\\ q\in I[1\ \ L_{q,l}]}} \left(\left|\sup_{[.]}\left(D_{iq,l}(.)\right)\right|\right). \tag{70}$$

Hence, we will apply this result to determine delay-dependent stability criteria for systems described by a set of difference equations with multiple time varying delays:

$$S_i : y(k+n) + \left(\sum_{p=1}^{L_p}\mu_{ip}(k)\sum_{m=0}^{n-1}a_{ip}^{n-m}(.)\,y(k+m) + \cdot \sum_{l=1}^{h}\sum_{q=1}^{L_{q,l}}\lambda_{iq,l}(k)\sum_{m=0}^{n-1}d_{iq,l}^{n-m}(.)\,y\left(k+m-\tau_{i,l}(k)\right)\right) = 0 \tag{71}$$

where $y(k) \in \mathfrak{R}$ is the output and $a_{ip}^{n-m}(.)$ and $d_{iq,l}^{n-m}(.)$are nonlinear coefficients.

From (26) and taking into consideration the switched rule signal $\sigma(k)$, the resulting studied switched nonlinear system can be described by the following state space representation:

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}(.)\,x(k) + \sum_{l=1}^{h} D_{\sigma(k),l}(.)\,x\left(k-\tau_{l,\sigma(k)}(k)\right) \\ x(\theta) &= \phi(\theta) \\ \theta &= -\max_{l\in I[1\ \ h]}\left(r_{2,l}\right), -\max_{l\in I[1\ \ h]}\left(r_{2,l}\right)+1,\ldots,-1. \end{aligned} \tag{72}$$

The matrix $D_{iq,l}(.)$ is represented by

$$D_{iq,l}(.) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ -d_{iq,l}^{n}(.) & -d_{iq,l}^{n-1}(.) & \cdots & -d_{iq,l}^{1}(.) \end{bmatrix} \tag{73}$$

and $A_{ip}(.)$ is given in (29).

Therefore, the new polynomial $N_{D_{l,iq}(.)}(s,.)$ for $i \in I\left[1\ \ l\right]$, $l \in I\left[1\ \ h\right]$ and $q \in I\left[1\ \ L_q\right]$ is defined by

$$N_{D_{l,iq}(.)}(s,.) = \sum_{m=0}^{n-1} d_{l,iq}^{m}(.)\,s^m. \tag{74}$$

Then, according to (30), system (72) will be represented in the arrow form, $\forall i \in I\left[1\ \ l\right]$, $\forall p \in I\left[1\ \ L_p\right]$, $\forall l \in I\left[1\ \ h\right]$, and $\forall q \in I\left[1\ \ L_q\right]$ as follows:

$$\begin{aligned} z(k+1) &= \sum_{p=1}^{L_p}\mu_{ip}(k)\,E_{ip}(.)\,z(k) + \sum_{l=1}^{h}\sum_{q=1}^{L_q}\lambda_{iq}(k)\,F_{iq,l}(.)\,z\left(k-\tau_{i,l}(k)\right) \\ z(\theta) &= P\phi(\theta) \\ \theta &= -\max_{l\in I[1\ \ h]}\left(r_{2,l}\right), -\max_{l\in I[1\ \ h]}\left(r_{2,l}\right)+1,\ldots,-1 \end{aligned} \tag{75}$$

where $F_{ip}(.)$ is given in (35) and $E_{iq,l}(.)$ is represented by

$$E_{iq,l}(.) = P^{-1}D_{l,i}(.)\,P = \begin{bmatrix} 0_{n-1,n-1}\cdots & 0_{n-1,1} \\ \delta_{iq,l}^{1}(.)\cdots\delta_{iq,l}^{n-1}(.) & \delta_{iq,l}^{n}(.) \end{bmatrix} \tag{76}$$

with

$$\begin{aligned} \delta_{iq,l}^{j}(.) &= -N_{D_{iq}(.)}(s,.), \\ &\qquad \forall j = 1,\ldots,n-1,\ \ l = 1,\ldots,h \\ \delta_{iq,l}^{n}(.) &= -d_{iq,l}^{1}(.). \end{aligned} \tag{77}$$

Consequently, matrix $T_{ipq,l}(.)$, $\forall i \in I[1\ \ L]$, $\forall p \in I[1\ \ L_p]$, $\forall q \in I[1\ \ L_q]$, and $\forall l \in I[1\ \ h]$, is given as follows:

$$T_{ipq,l}(.) = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \alpha_{n-1} & |\beta_{n-1}| \\ t^1_{ipq,l}(.) & \cdots & \cdots & t^{n-1}_{ipq,l}(.) & t^n_{ipq,l}(.) \end{bmatrix} \quad (78)$$

$$t^j_{ipq,l}(.) = \left|\gamma^j_{ip}(.)\right| + (r_2 - r_1 + 1)\left|\sup_{[.]}\left(\sum_{l=1}^{h}\delta^j_{iq,l}(.)\right)\right|, \quad j = 1,\ldots,n-1 \quad (79)$$

$$t^n_{ipq,l}(.) = \left|\gamma^n_{ip}(.)\right| + (r_2 - r_1 + 1)\sup_{[.]}\left(\sum_{l=1}^{h}\delta^n_{iq,l}(.)\right).$$

Finally, the comparison matrix $T_M(.)$ of system (72) is given by

$$T_M(.) = \begin{bmatrix} |\alpha_1| & 0 & \cdots & 0 & |\beta_1| \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & |\alpha_{n-1}| & |\beta_{n-1}| \\ \bar{t}^n(.) & \cdots & \cdots & \bar{t}^n(.) & \bar{t}^n(.) \end{bmatrix} \quad (80)$$

where

$$\bar{t}^n(.) = \max_{\substack{i\in I[1\ \ L]\\ p\in I[1\ \ L_p]\\ q\in I[1\ \ L_q]}}\left(t^n_{ipq,l}(.)\right)$$
$$\bar{t}^j(.) = \max_{\substack{i\in I[1\ \ L]\\ p\in I[1\ \ L_p]\\ q\in I[1\ \ L_q]}}\left(t^j_{ipq,l}(.)\right), \quad j = 1,\ldots,n-1. \quad (81)$$

Thus, using the special form of system (72) we can announce the following Theorem. □

**Theorem 8.** *System (72) is globally robustly asymptotically stable, for any arbitrary switching $\sigma(k)$ and all admissible uncertainties (63) and (64), if there exist $\alpha_j$ $(j = 1, \ldots, n-1)$, $\alpha_j \neq \alpha_q, \forall j \neq q$, such as*

$$(i)\ 1 - \left|\alpha_j\right| > 0, \quad j = 1,\ldots n-1 \quad (82)$$

$$(ii)\ 1 - \left(\bar{t}^n(.)\right) - \sum_{j=1}^{n-1}\left(\bar{t}^j(.)\right)\left|\beta_j\right|\left(1-\left|\alpha_j\right|\right)^{-1} > 0. \quad (83)$$

Next, Theorem 8 can be simplified to the following corollary.

**Corollary 9.** *System (72) is globally robustly asymptotically stable under arbitrary switching rule $\sigma(k)$ and all admissible uncertainties (63) and (64), if there exist $\alpha_j \in \mathfrak{R}_-$ $(j = 1, \ldots, n-1)$, $\alpha_j \neq \alpha_q$, $\forall j \neq q$ and $\forall i \in I[1\ \ L]$, $\forall p \in I[1\ \ L_p]$, $\forall q \in I[1\ \ L_q]$, $\forall l \in I[1\ \ h]$, such as*

$$(i)\ \beta_j\left(G_{A_{ip}(.)}\left(\alpha_j,.\right) + (r_2 - r_1 + 1)\sup_{[.]}\left(\sum_{l=1}^{h}N_{D_{iq,l}(.)}\left(\alpha_j,.\right)\right)\right) < 0 \quad (84)$$

$$(ii)\ \left(G_{A_{ip}(.)}(s=1,.) + (r_2 - r_1 + 1)\sup_{[.]}\left(\sum_{l=1}^{h}N_{D_{iq}(.)}(s=1,.)\right)\right) > 0 \quad (85)$$

$$(iii)\ \left(\gamma^n_{ip}(.) + (r_2 - r_1 + 1)\sup_{[.]}\left(\sum_{l=1}^{h}\delta^n_{iq,l}(.)\right)\right) > 0 \quad (86)$$

## 6. Illustrative Examples

*Example 1.* Let us consider system (72) with three subsystems, where the randomly switched model is given as

$$A_{11}(.) = \begin{bmatrix} 0 & 1 \\ -0.45 + 0.08f(.) & 0.95 - 0.43\Phi(.) \end{bmatrix},$$

$$A_{12}(.) = \begin{bmatrix} 0 & 1 \\ -0.6 + 0.08f(.) & 0.9 - 0.4\Phi(.) \end{bmatrix},$$

$$A_{21}(.) = \begin{bmatrix} 0 & 1 \\ -0.4 + 0.085f(.) & 1 - 0.5\Phi(.) \end{bmatrix},$$

$$A_{22}(.) = \begin{bmatrix} 0 & 1 \\ -0.5 + 0.07f(.) & 1.2 - 0.8\Phi(.) \end{bmatrix},$$

$$A_{31}(.) = \begin{bmatrix} 0 & 1 \\ -0.55 + 0.085f(.) & 0.9 - 0.55\Phi(.) \end{bmatrix},$$

$$A_{32}(.) = \begin{bmatrix} 0 & 1 \\ -0.6 + 0.075f(.) & 1 - 0.6\Phi(.) \end{bmatrix}$$

$$D_{11,1}(.) = \begin{bmatrix} 0 & 0 \\ -0.012 + 0.02\psi(.) & 0.04 - 0.03\psi(.) \end{bmatrix},$$

$$D_{11,2}(.) = \begin{bmatrix} 0 & 0 \\ -0.018 + 0.015\psi(.) & 0.037 - 0.035\psi(.) \end{bmatrix}$$

$$D_{12,1}(.) = \begin{bmatrix} 0 & 0 \\ -0.01 + 0.02\psi(.) & 0.04 - 0.023\psi(.) \end{bmatrix},$$

$$D_{12,2}(.) = \begin{bmatrix} 0 & 0 \\ -0.01 + 0.025\psi(.) & 0.037 - 0.04\psi(.) \end{bmatrix},$$

$$D_{21,1}(.) = \begin{bmatrix} 0 & 0 \\ -0.015 + 0.02\psi(.) & 0.05 - 0.03\psi(.) \end{bmatrix},$$

$$D_{21,2}(.) = \begin{bmatrix} 0 & 0 \\ -0.01 + 0.017\psi(.) & 0.04 - 0.035\psi(.) \end{bmatrix},$$

$$D_{22,1}(.) = \begin{bmatrix} 0 & 0 \\ -0.013 + 0.01\psi(.) & 0.03 - 0.01\psi(.) \end{bmatrix},$$

$$D_{22,2}(.) = \begin{bmatrix} 0 & 0 \\ -0.02 + 0.0184\psi(.) & 0.04 - 0.0357\psi(.) \end{bmatrix}$$

$$D_{31,1}(.) = \begin{bmatrix} 0 & 0 \\ -0.015 + 0.025\psi(.) & 0.045 - 0.024\psi(.) \end{bmatrix},$$

$$D_{31,2}(.) = \begin{bmatrix} 0 & 0 \\ -0.02 + 0.03\psi(.) & 0.03 - 0.045\psi(.) \end{bmatrix},$$

$$D_{32,1}(.) = \begin{bmatrix} 0 & 0 \\ -0.007 + 0.017\psi(.) & 0.021 - 0.043\psi(.) \end{bmatrix}$$

and $D_{32,2}(.)$

$$= \begin{bmatrix} 0 & 0 \\ -0.013 + 0.04\psi(.) & 0.03 - 0.02\psi(.) \end{bmatrix} \quad (87)$$

with $f(.)$ $\Phi(.)$ and $\psi(.)$ being general nonlinear functions.

Hence, we suppose that $\psi(.) \in E([0.8, 1, 1.2])$ and the corresponding delay functions are listed as follows: $r_{1,1}(k) = 4 + \sin^2(k\pi/2)$, $r_{1,2}(k) = 3 + \sin^2(k\pi/2)$, $r_{2,1}(k) = 4 + \sin^2(k\pi/2)$, $r_{2,2}(k) = 2 + \sin^2(k\pi/2)$ $r_{3,1}(k) = 3 + \sin^2(k\pi/2)$, $r_{3,2}(k) = 2 + \sin^2(k\pi/2)$ $k = 0, 1, 2.....$

Thus, by Corollary 9, with $\alpha = 0.2$, we obtain the following stability conditions:

$$\begin{aligned} &\Phi(.) < -2.04 + 0.5f(.) \\ &\Phi(.) > -0.54 + 0.17f(.) \\ &\Phi(.) < 1.254. \end{aligned} \quad (88)$$

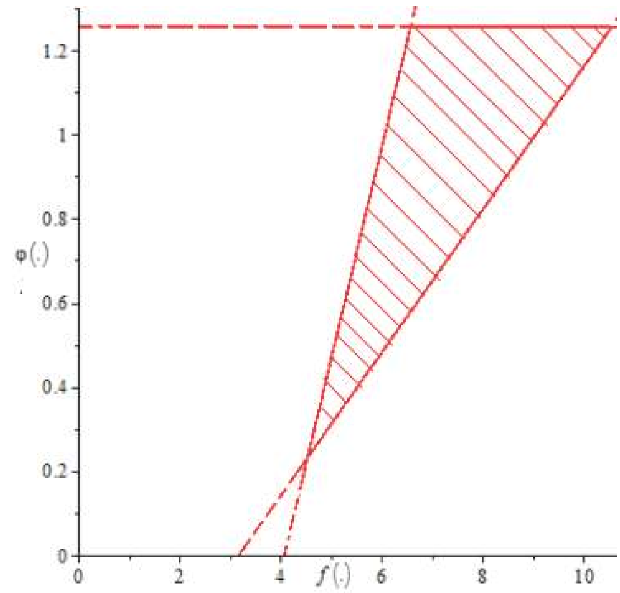


FIGURE 1: Stability domain for Example 1 obtained from Corollary 9.

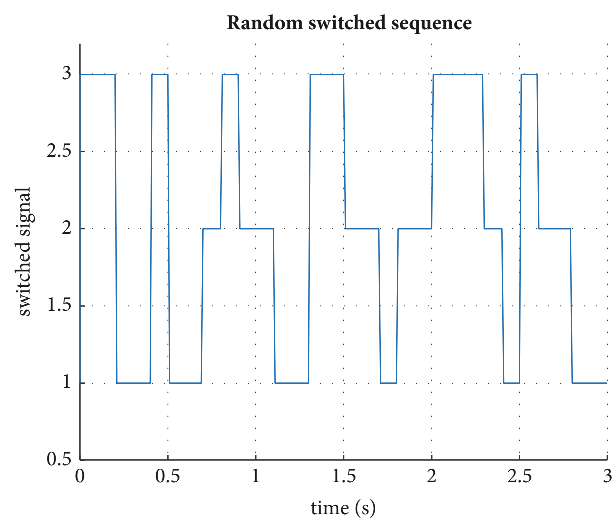


FIGURE 2: Random switching signal for system given in Example 1.

Due to those inequalities, the robust stability domain given by the nonlinear $\Phi(.)$ relative to the nonlinear $f(.)$ is illustrated in Figure 1.

By fixing the sampling time at $T_e = 0.1s$, consider the random switching sequence depicted in Figure 2; the initial conditions are $\phi(k) = [-0.8 \ \ 0.7]^T$, the nonlinearities $f(.) = 4.5$, $\varphi(.) = 0.5$, and $\psi(.) = 1$, and the incertitude parameters are chosen as $\mu_{11}(k) = \mu_{21}(k) = \mu_{31}(k) = 0.6$ and $\lambda_{11}(k) = \lambda_{21}(k) = \lambda_{31}(k) = 0.7$. The simulation results, respectively, given in Figures 3 and 4 hint about the state responses of the system and state norm where the switching signal is randomly generated.

Noting that, due to the complexity of this randomly switched system alongside the important number of the subsystems, it is hard to find a common Lyapunov function, then,

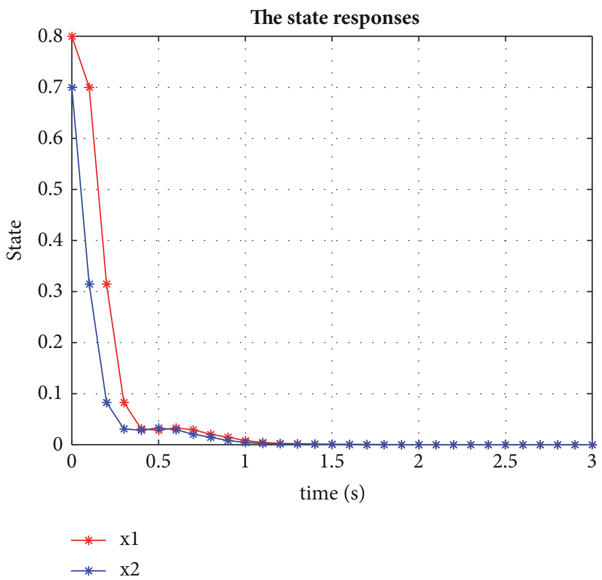


FIGURE 3: The state responses of the system given in Example 1.

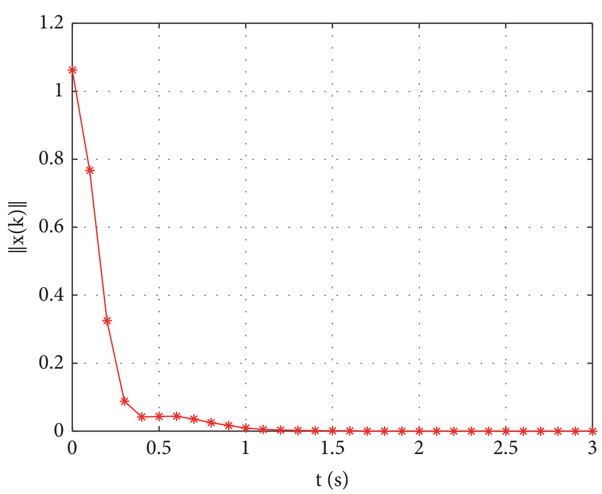


FIGURE 4: The state norm of the system given in Example 1.

we cannot guarantee stability of this switched system under randomly switching that values our proposed approach.

*Example 2* (see [27, 34]). Consider a randomly switched system represented by a set of differential equations given as follows:

$$\ddot{x}(t)+\sum_{p=1}^{2}\mu_{ip}(t)\,a_{ip}\dot{x}(t)+\sum_{p=1}^{2}\mu_{ip}(t)\,\frac{\varphi_{ip}(x)}{x}x(t)+\sum_{q=1}^{2}\lambda_{iq}(t)\,b_{iq}\dot{x}(t-\tau_i(t))+\sum_{q=1}^{2}\lambda_{iq}(t)\,c_{iq}x(t-\tau_i(t))=0, \tag{89}$$

where $a_{ip}$, $b_{iq}$, and $c_{iq}$ are parameters and $i\in\{1,2\}$.

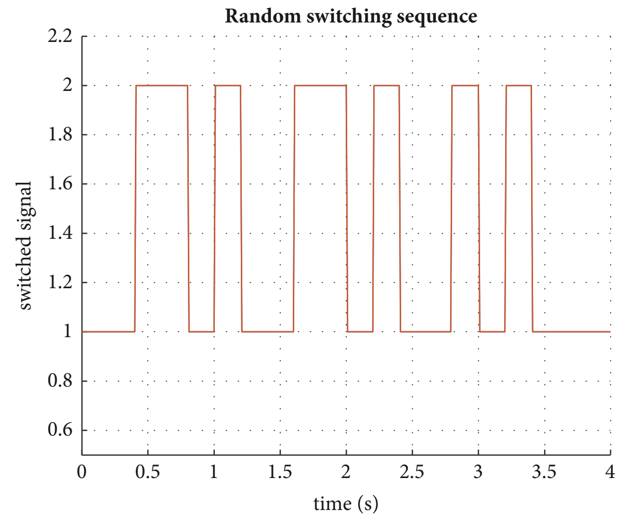


FIGURE 5: Random switching signal for system given in Example 2.

By the Taylor-Young formula, $\ddot{x}(k)=(x(k+2)-2x(k+1)+x(k))/(T_e)^2$ and $\dot{x}(k)=(x(k+1)-x(k))/T_e$, where $T_e$ is the sampling time.

According to (25), all subsystems can be represented under the following matrix representation: $x(k+1)=A_i(.)x(k)+D_ix(k-\tau_i(k))$, where $x(k)$ is the state vector and the matrices $A_i(.)=\sum_{p=1}^{2}\mu_{ip}(k)A_{ip}(.)$ and $D_i=\sum_{q=1}^{2}\lambda_{iq}(k)D_{iq}$ are given by

$$A_{ip}(.)=\begin{pmatrix}0 & 1\\ -1+a_{ip}T_e\dfrac{\varphi_{ip}(x)}{x} & 2-a_{ip}T_e\end{pmatrix}$$
$$\text{and } D_{iq}=\begin{pmatrix}0 & 0\\ b_{iq}T_e-c_{iq}(T_e)^2 & -b_{iq}T_e\end{pmatrix}. \tag{90}$$

Let us consider the following particular values: $\left\{\begin{smallmatrix}a_{11}=8\\ b_{11}=0.01\\ c_{11}=0.2\end{smallmatrix}\right.$, $\left\{\begin{smallmatrix}a_{12}=7.5\\ b_{12}=0.017\\ c_{12}=0.35\end{smallmatrix}\right.$, $\left\{\begin{smallmatrix}a_{21}=7.2\\ b_{21}=0.013\\ c_{21}=0.25\end{smallmatrix}\right.$, and $\left\{\begin{smallmatrix}a_{21}=8.5\\ b_{21}=0.015\\ c_{21}=0.3\end{smallmatrix}\right.$. The sampling time $T_e$ is fixed at $0.2s$ and the time varying delays functions are $\tau_1(k)=1+3\cos^2(k\pi/2)$, $k=0,1,2....$ and $\tau_2(k)=1+2\cos^2(k\pi/2)$, $k=0,1,2.....$

By Corollary 6, with $\alpha=0.2$, the following stability conditions can be deduced:

(i) $\varphi^*_{11\min}(.)=0.616<\varphi_{11}(x)/x<\varphi^*_{11\max}(.)=1.02$.
(ii) $\varphi^*_{12\min}(.)=0.65<\varphi_{12}(x)/x<\varphi^*_{12\max}(.)=1.03$.
(iii) $\varphi^*_{21\min}(.)=0.66<\varphi_{21}(x)/x<\varphi^*_{21\max}(.)=1.027$.
(iv) $\varphi^*_{22\min}(.)=0.59<\varphi_{22}(x)/x<\varphi^*_{22\max}(.)=1.0282$.

According to the previous stability conditions and with a particular choice of functions $\varphi_{11}(x)/x=0.6$, $\varphi_{12}(x)/x=0.8$, $\varphi_{21}(x)/x=0.9$, and $\varphi_{22}(x)/x=1$ and by considering the random switching sequence given in Figure 5, the vector-valued initial function $\phi(s)=[1\ \ 0.6]^T$ and the incertitude

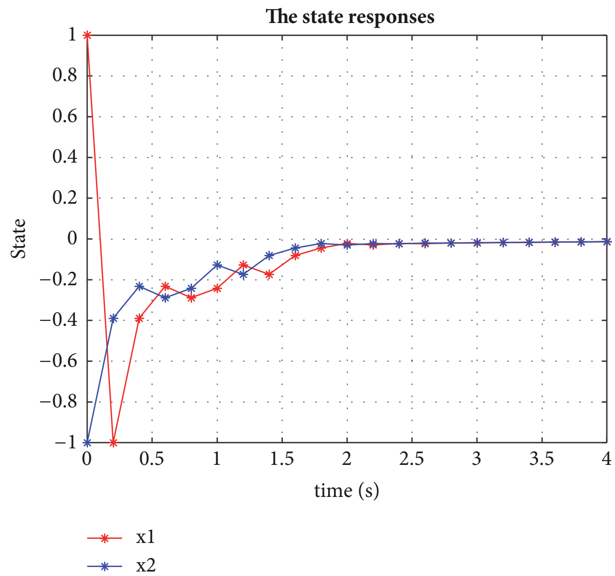


Figure 6: The state responses of the system given in Example 2.

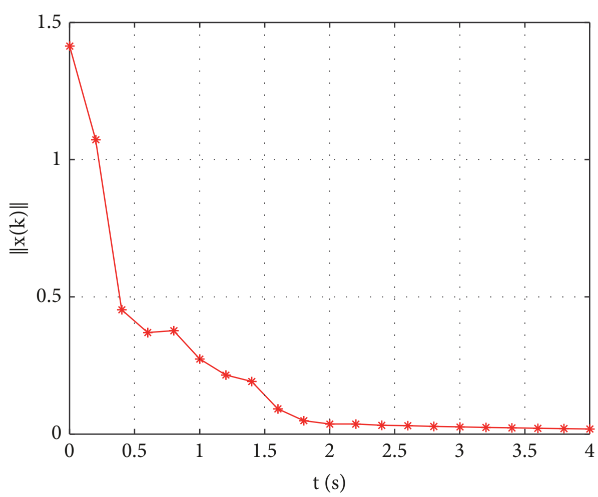


Figure 7: The state norm of the system given in Example 2.

parameters $\mu_{11}(k) = \mu_{21}(k) = 0.6$ and$\lambda_{11}(k) = \lambda_{21}(k) = 0.7$. Then, the simulation results are obtained as follows: a typical result plotted in Figures 6 and 7 shows that the system state and the norm state converge to zero, where the switching signal is randomly generated. Thus, the simulation affirms the theoretical results.

## 7. Conclusion

In this paper, we have interestingly addressed the stability issue for a class of discrete-time switched nonlinear systems with time varying delays and with polytopic uncertainties. New sufficient stability conditions have been yielded by constructing a new common Lyapunov-Krasovskii functional alongside resorting to the $M$ − matrix properties, Borne-Gentina practical stability criterion, the aggregation techniques, and the vector norms notion. Numerical simulations are given to illustrate the effectiveness of our results. It is expected that the idea and the technique in this paper will be worth of use for the future research works on that filed.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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