Hindawi
Mathematical Problems in Engineering
Volume 2018, Article ID 1737685, 9 pages
https://doi.org/10.1155/2018/1737685

Research Article

On Robust Synchronization of Drive-Response Boolean Control Networks with Disturbances

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Received 6 April 2018; Revised 7 June 2018; Accepted 5 July 2018; Published 6 August 2018

Academic Editor: Jean Jacques Loiseau

This paper investigates the robust synchronization of drive-response Boolean control networks (BCNs) with disturbances via semi-tensor product of matrices. Firstly, the definition of robust synchronization is presented for the drive-response BCNs with disturbances. Then, based on the algebraic state space representation of drive-response BCNs, the robustly reachable states/sets are presented to investigate robust synchronization of disturbed BCNs. According to the set of robustly reachable states, some necessary and sufficient criteria are obtained for robust synchronization of drive-response BCNs with disturbances under a given state feedback controller. Finally, an illustrative example is presented to demonstrate the obtained theoretical results.

1. Introduction

As an efficient mathematical model of biological systems and genetic regulatory networks (GRNs), Boolean network (BN) has attracted much attention as a qualitative tool for analyzing GRNs. BN theory was firstly proposed by Kauffman in 1969 [1] to model a gene as a binary device for studying the behavior of large, randomly constructed networks of these genes. Thanks to Kauffman's pioneering work [1], investigations on Boolean networks (BNs) have attracted great attention from biologists and systems scientists. Consequently, many fundamental and excellent results have been established for BNs [2]. In a BN, each node is characterized by two possible values, 1 and 0, and its value (1 or 0) indicates its measured abundance (expressed or unexpressed, active or inactive). Meanwhile, the state evolution of each node on network is determined by a series of Boolean rules. In the past few years, some basic problems of BNs have been investigated in [3], including the fixed points, cycles, the basin of attractors, and the transient time. BNs with additional inputs called Boolean control networks (BCNs) were firstly proposed in [2] and could be used to design and analyze therapeutic intervention strategies. For example, in [4], Huang and Inber established a simple Boolean control network (BCN) to simulate the dynamics of signaling system within capillary endothelia cells, where two external inputs represent growth factors and cell shape (spreading). Recently, the controllability of BCNs has been widely investigated by numerous researchers [5–9].

Recently, a new matrix product called semi-tensor product (STP) of matrices proposed by Cheng and his colleagues [10, 11] has been successfully used to study Boolean (control) networks and game theory on static games [12, 13], and many excellent results have been obtained [14–23]. Its main idea is to convert a logical function into an algebraic function, and, thus, the dynamic of BN can be uniquely converted into a standard discrete algebraic dynamic [11]. However, it should be noted that one main drawback of the algebraic state expression of BNs is its computational complexity. The algebraic state representation converts a BN with n state variables and m control inputs into a state-control-space of size 2^{n+m}. Thus, any algorithm based on this approach has an exponential time-complexity. Moreover, many problems like determining fixed points and observability of BCNs have already been proved to be NP-hard. Hence, the computational complexity is intrinsic and also independent of the models adopted to describe BNs.

Since BN is an efficient model to provide general features of living organism and to well illustrate GRNs, the synchronization problem for BNs (or BCNs) has received considerable research interests in the past few years. The researches on synchronization of BNs (or BCNs) can capture lots of useful information on the evolution of biological systems whose corresponding subsystems influence each other. For example, investigations on synchronized BNs are beneficial to better understand synchronization between two coupled lasers [24]. Hence, there are both theoretical and practical importance for studying the synchronization problem of BNs (or BCNs). Recently, some important results corresponding to complete synchronization for two deterministic BNs have been obtained [25, 26].

In the past few years, much attention has been paid on the synchronization phenomenon of collective behavior by large amount of researches [27–34]. Aside from synchronization, the ability of disturbances resisting for synchronized networks has also attracted much attention [35–38]. In real world, the external disturbances are ubiquitous and may also lead the coupled networks to some unexpected behaviors or even break the phenomenon of synchronization. Hence, it is desirable to investigate the ability of coupled networks to resist external disturbances. In order to deal with this situation, it is practically significant to investigate robust synchronization of dynamical networks, which has gained much research attention [39–41]. Gene regulation is an intrinsically noisy process, and it is always subject to intracellular and extracellular disturbances and environment fluctuations [42–44]. For example, a cell apoptosis Boolean model is given to study the antiapoptotic pathway under external disturbance inputs, one may be interested in whether there exists a response system that can achieve synchronization with the apoptosis network without disturbances. Thus, when modeling GRNs, disturbances should be considered, as the states of genes may be subject to instantaneous perturbations and experience abrupt disturbances [45, 46]. These disturbance inputs may prohibit the effectiveness of control strategies in keeping the cellular states of biological systems and GRNs in a desirable set. Thus, it is necessary for us to design some control mechanisms under which some coupled GRNs are robustly synchronized to the external disturbances. Unfortunately, concerning BCNs, to the best of our knowledge, there is no result available corresponding to robust synchronization.

In this paper, using STP technique, the robust synchronization problem of drive-response BCNs with disturbances is investigated, and some necessary and sufficient criteria are obtained to check whether a given drive-response BCNs with disturbances can be robustly synchronized. Moreover, a set of robustly reachable states from a given initial state and a union set of robustly reachable states from the initial states set $\Delta_{2^{2n}}$ are proposed to check robust synchronization. Finally, one numerical example is presented to illustrate the main results.

The rest of this paper is structured as follows. In Section 2, we present some necessary notations and preliminaries on STP technique. In Section 3, we present our problem formulation and our main results on robust synchronization of drive-response BCNs with disturbances, by resorting to the matrix expression of logical functions and the STP technique. Illustrative example is given to show the validness of our main results in Section 4.

2. Some Preliminaries

In this section, we present some preliminaries on semi-tensor product (STP) and matrix expression of logical functions, which will be used in later analysis. The main tool used in this paper is STP of matrices. Using STP of matrices, a Boolean (control) network can be expressed in its equivalent algebraic representation.

For statement ease, we firstly present some basic notations: $\mathbf{1}_k = [\underbrace{1,\ldots,1}_{k}]^T$, where superscript T presents the transpose of a matrix or a vector; $\mathscr{D} = \{0, 1\}$ with the usual operations (sum +, product $\cdot$, and negation $\neg$); let δ_n^i and $\Delta_n = \{\delta_n^1, \ldots, \delta_n^n\}$ be the i-th column and the columns' set of identity matrixes I_n. When $n = 2$, we simply use $\Delta = \Delta_2$; let $\mathrm{Col}_i(A)$ ($\mathrm{Row}_i(A)$) and $\mathrm{Col}(A)$ ($\mathrm{Row}(A)$) be the i-th column (row) and the set of columns (rows) of matrix A; given an $m \times mn$ matrix A, denote by $\mathrm{Blk}_i(A)$, $i = 1, \ldots, n$, the i-th block of an $m \times mn$ matrix A; i.e., $A = (\mathrm{Blk}_1(A), \ldots, \mathrm{Blk}_n(A))$; $L = [\delta_n^{i_1}, \ldots, \delta_n^{i_s}]$ is called a logical matrix if $\mathrm{Col}(L) \subseteq \Delta_n$, which can be simply denoted as $L = \delta_n[i_1, \ldots, i_s]$. Moreover, here we call $i_1, \ldots, i_s$ the row indexes of columns for matrix L. The set of $m \times n$ logical matrices is denoted by $\mathscr{L}_{m\times n}$. Assume that $A \in \mathbb{R}^{m\times n}$ is a real matrix ($\mathbb{R}^{m\times n}$ denotes the set of real matrices with order $m \times n$), if all the entries of A are positive (nonnegative), that is, $A_{ij} > 0$ ($A_{ij} \geq 0$) for any $1 \leq i \leq m, 1 \leq j \leq n$, then simply denote $A > 0$ ($A \geq 0$).

Now, we present the definitions of Boolean addition and Boolean product of Boolean matrices. An $m \times n$ matrix $A = (a_{ij})_{m\times n}$ is called a Boolean matrix if $a_{ij} \in \mathscr{D}$, where $\mathscr{B}_{m\times n}$ denotes the set of all $m \times n$ Boolean matrices. Then, we define the Boolean addition below: $x +_{\mathscr{B}} y := x \vee y$, $x, y \in \mathscr{D}$. Therefore, for two Boolean matrices $X, Y \in \mathscr{B}_{m\times n}$, we can obtain the Boolean addition for X and Y: $X +_{\mathscr{B}} Y = (x_{ij} +_{\mathscr{B}} y_{ij})_{m\times n}$, $1 \leq i \leq m, 1 \leq j \leq n$. Moreover, we can also define the Boolean product of two Boolean matrices $A \in \mathscr{B}_{m\times n}$ and $B \in \mathscr{B}_{n\times p}$, denoted by $A \times_{\mathscr{B}} B = (c_{ij})_{m\times p} \in \mathscr{B}_{m\times p}$ with $c_{ij} = \sum_{\mathscr{B}}{}_{k=1}^{p} a_{ik} \wedge b_{kj}$, $1 \leq i \leq m, 1 \leq j \leq p$. Particularly, if $A \in \mathscr{B}_{m\times m}$, then $A^{(k)} = A^{(k-1)} \times_{\mathscr{B}} A$.

Then, we introduce the STP "$\ltimes$" between matrices (and in particular, vectors) as follows [10].

Definition 1 (see [10]). Consider $n \times m$ matrix A and $p \times q$ matrix B. The STP of A and B is defined as follows: $A \ltimes B = (A \otimes I_{l/m})(B \otimes I_{l/p})$, where $l = \mathrm{l.c.m.}(m, p)$, denotes the least common multiple of m and p. Here, $\otimes$ denotes the Kronecker product of matrices.

Remark 2. If $m = p$, then $A \ltimes B = AB$, which is the standard matrix product. Thus, the STP of matrices is a generalization of the standard matrix product providing a new way to multiply two matrices with arbitrary dimensions.

Definition 3 (see [10]). If $\sigma \in \Delta_{2^n}$, then $\sigma \ltimes \sigma = \Phi_{2^n}\sigma$, where Φ_{2^n} is called power-reducing matrix for 2^n-valued logical vectors and $\Phi_{2^n} = \delta_{2^{2n}}[1, 2^n + 2, \ldots, (2^n - 2)\cdot 2^n + 2^n - 1, 2^{2n}]$.

Definition 4 (see [10]). An $mn \times mn$ matrix $W_{[m,n]}$ is called a swap matrix, if it is constructed by the way: label its columns by $(11, 12, \ldots, 1n, \ldots, m1, \ldots, mn)$ and similarly label its rows by $(11, 21, \ldots, m1, \ldots, 1n, 2n, \ldots, mn)$. Then its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(I,J),(i,j)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

When $m = n$, we denote $W_{[m,n]}$ by $W_{[n]}$ or $W_{[m]}$.

Identify Boolean variables 1 and 0 with vectors δ_2^1 and δ_2^2. That is to say, we consider a Boolean variable $X \in \mathscr{D}$ as a vector $x \in \Delta$; thus a Boolean function with n variables $f : \mathscr{D}^n \longrightarrow \mathscr{D}$ is equivalent with a map $f : (\Delta_2)^n \longrightarrow \Delta_2$.

Proposition 5 (see [10]). *Let $f : (\mathscr{D})^n \longrightarrow \mathscr{D}$ be a Boolean function. Then there exists a unique matrix $F \in \mathscr{L}_{2\times 2^n}$ such that $f(\sigma_1, \ldots, \sigma_n) = F \ltimes \sigma_1 \ltimes \cdots \ltimes \sigma_n$, for every $(\sigma_1, \ldots, \sigma_n) \in (\Delta_2)^n$, where F is called structure matrix of the logical function f.*

3. Problem Formulation and Main Results

3.1. Problem Formulation. Since disturbance is an important factor when modeling GRNs, here we consider the following drive-response BCNs with disturbances:

$$\begin{aligned} x_1(t+1) &= f_1\left(\xi_1(t), \ldots, \xi_p(t), x_1(t), \ldots, x_n(t)\right), \\ &\vdots \\ x_n(t+1) &= f_n\left(\xi_1(t), \ldots, \xi_p(t), x_1(t), \ldots, x_n(t)\right), \\ y_1(t+1) &= g_1\left(u_1(t), \ldots, u_m(t), y_1(t), \ldots, y_n(t)\right), \\ &\vdots \\ y_n(t+1) &= g_n\left(u_1(t), \ldots, u_m(t), y_1(t), \ldots, y_n(t)\right). \end{aligned} \tag{2}$$

Here, x_i and y_j represent the i-th node and j-th node of drive and response BN, respectively, $f_i : \{1,0\}^{n+p} \longrightarrow \{1,0\}$, $g_j : \{1,0\}^{n+m} \longrightarrow \{1,0\}$ are Boolean functions, $i = 1, \ldots, n$, $j = 1, \ldots, n$. In addition, u_α and ξ_β are controls and disturbance inputs, $\alpha = 1, \ldots, m$, $\beta = 1, \ldots, p$. We simply denote by $x(t) = (x_1(t), \ldots, x_n(t))^T$ and $y(t) = (y_1(t), \ldots, y_n(t))^T$ the states of drive and response BN, respectively. We can observe that the state evolution of drive-response BCNs depends on the following initial states $x(0) = (x_1(0), \ldots, x_n(0))^T$ and $y(0) = (y_1(0), \ldots, y_n(0))^T$. Correspondingly, the dynamic of the state feedback control is in the form of

$$\begin{aligned} u_1(t) &= \psi_1\left(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t)\right), \\ &\vdots \\ u_m(t) &= \psi_n\left(x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t)\right). \end{aligned} \tag{3}$$

Here, $\psi_i : \{1,0\}^{2n} \longrightarrow \{1,0\}$ $(i = 1, \ldots, m)$ are Boolean functions.

Now, we present the definition of robust synchronization for drive-response BCNs (2) with disturbances.

Definition 6. Consider the drive-response BCNs (2) with disturbances and the state feedback control (3). Then, system (2) is said to be robustly synchronized if, for any initial states and any disturbance inputs, there exists an integer k such that $x(t) = y(t)$, $\forall t \geq k$.

Remark 7. As we can see from system (2), the disturbances are imposed on the states of drive BN, which implies that the states of drive BN will be affected by the values of disturbances $\xi_i(t)$ $(i = 1, \ldots, p)$. Thus, if there exists a state feedback control in form of (3) such that the response BN can always synchronize with the drive BN regardless of the disturbances, it will be welcome. The main objective of this paper is to establish robust synchronization criteria for the drive-response BCNs (2) with disturbances.

3.2. Algebraic form of Drive-Response BCNs with Disturbances. In this subsection, we will convert the drive-response BCNs (2) with disturbances and the feedback control (3) into equivalent algebraic forms by applying STP. Using the vector form of Boolean variables and denoting $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{2^n}$, $y(t) = \ltimes_{i=1}^n y_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{i=1}^m u_i(t) \in \Delta_{2^m}$, and $\xi(t) = \ltimes_{i=1}^p \xi_i(t) \in \Delta_{2^p}$, by Proposition 5, we can convert systems (2) and (3) into the following algebraic forms:

$$\begin{aligned} x(t+1) &= F\xi(t)\,x(t), \\ y(t+1) &= Gu(t)\,y(t), \end{aligned} \tag{4}$$

and

$$u(t) = Kx(t)\,y(t), \tag{5}$$

where $F \in \mathscr{L}_{2^n\times 2^{n+p}}$, $G \in \mathscr{L}_{2^n\times 2^{m+n}}$, and $K \in \mathscr{L}_{2^m\times 2^{2n}}$ are called the state transition matrices of drive BN and response BN and the state feedback gain matrix, respectively.

Further, denote $z(t) = x(t) \ltimes y(t) \in \Delta_{2^{2n}}$, and we can combine (4) into the following equation:

$$\begin{aligned} z(t+1) &= x(t+1)\,y(t+1) = F\xi(t)\,x(t)\,Gu(t)\,y(t) \\ &= F\left(I_{2^{n+p}} \otimes G\right)\xi(t)\,x(t)\,u(t)\,y(t) \\ &= F\left(I_{2^{n+p}} \otimes G\right)\left(I_{2^p} \otimes W_{[2^m,2^n]}\right)\xi(t)\,u(t)\,z(t) \\ &\triangleq L\xi(t)\,u(t)\,z(t), \end{aligned} \tag{6}$$

where $L = F(I_{2^{n+p}} \otimes G)(I_{2^p} \otimes W_{[2^m,2^n]}) \in \mathscr{L}_{2^{2n}\times 2^{p+m+2n}}$.

Thus, for the drive-response BCNs (4) with disturbances and the feedback control (5), we obtain another equivalent algebraic form as follows:

$$\begin{aligned} z(t+1) &= L\xi(t)u(t)z(t), \\ u(t) &= Kz(t), \end{aligned} \tag{7}$$

where $L \in \mathscr{L}_{2^{2n}\times 2^{p+m+2n}}$ and $K \in \mathscr{L}_{2^m\times 2^{2n}}$ are the state transition matrices of system (6) and state feedback gain matrix of system (5).

Remark 8. Defining $z(t) = x(t)y(t)$, we transfer the disturbed BCNs (4) and control system (5) into system (7). It has been proved in [10] that, by defining $z(t) = x(t)y(t)$, we can get a bijective mapping $(\Delta)_{2^{2n}} \longrightarrow \Delta_{2^n}\times\Delta_{2^n}$. For example, if $z(t) = \delta_4^3$, we can derive that $x(t) = \delta_2^2$ and $y(t) = \delta_2^1$.

3.3. Main Results on Robust Synchronization. In the following subsection, we will investigate the robust synchronization of the drive-response BCNs (2) with disturbances based on the equivalent algebraic form (7).

Consider the drive-response BCNs (2) with disturbances. For a given state feedback control $u(t) = Kz(t)$ with $K \in \mathscr{L}_{2^m\times 2^{2n}}$, one can see from Proposition 5 that

$$\begin{aligned} z(t+1) &= L\xi(t)u(t)z(t) = L\xi(t)K\Phi_{2n}z(t) \\ &= L\left(I_{2^p}\otimes K\Phi_{2n}\right)\xi(t)z(t) \triangleq \mathbb{L}\xi(t)z(t), \end{aligned} \tag{8}$$

where $\mathbb{L} = L(I_{2^p}\otimes K\Phi_{2n}) = [\mathrm{Blk}_1(\mathbb{L}),\dots,\mathrm{Blk}_{2^p}(\mathbb{L})] \in \mathscr{L}_{2^{2n}\times 2^{2n+p}}$.

Then, in the following sequel, we introduce a set of robustly reachable states from a given initial state at given step.

Definition 9. A state ϖ is said to be a reachable state from the initial state $z(0)$ at the k-th step if there exists a disturbance sequence $\xi(0),\dots,\xi(k-1)$, such that $\varpi = z(k)$. Then, for all possible disturbance sequences $\xi(0),\dots,\xi(k-1)$, we can obtain the set of all possible reachable states, which is called the set of robustly reachable states from the initial state $z(0)$ at the k-th step, denoted by $R_{\mathbb{S}}^k(z(0))$.

According to (8), suppose that $\xi(0) = \delta_{2^p}^{\mu}$ and $z(0) = \delta_{2^{2n}}^{j}$, and we can obtain the set of robustly reachable states from $z(0)$ at the first step, which is $R_{\mathbb{S}}^1(z(0)) = \{\mathrm{Col}_j(\mathrm{Blk}_\mu(\mathbb{L})), \mu = 1,\dots,2^p\}$. Then, we can further define the union set of robustly reachable states from the initial states set $\Delta_{2^{2n}}$ at the k-th step as follows:

$$R_{\mathbb{S}}^k(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k\left(\delta_{2^{2n}}^1\right)\cup R_{\mathbb{S}}^k\left(\delta_{2^{2n}}^2\right)\cup\cdots\cup R_{\mathbb{S}}^k\left(\delta_{2^{2n}}^{2^{2n}}\right). \tag{9}$$

Particularly, when $k=0$, we denote $R_{\mathbb{S}}^0(\Delta_{2^{2n}}) = \Delta_{2^{2n}}$.

The following proposition gives some important properties for the union set $R_{\mathbb{S}}^k(\Delta_{2^{2n}})$.

Proposition 10. *(1) Let $R_{\mathbb{S}}^k(\Delta_{2^{2n}})$ be the union set of robustly reachable states from the initial states set $\Delta_{2^{2n}}$ at the k-th step, and then we have the following relationship:*

$$\Delta_{2^{2n}} \coloneqq R_{\mathbb{S}}^0(\Delta_{2^{2n}}) \supseteq R_{\mathbb{S}}^1(\Delta_{2^{2n}})\cdots\supseteq R_{\mathbb{S}}^k(\Delta_{2^{2n}})\cdots. \tag{10}$$

(2) If there exists an integer $k>0$ such that $R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k(\Delta_{2^{2n}})$, then we have $R_{\mathbb{S}}^t(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k(\Delta_{2^{2n}})$, $t\ge k$.

Proof. (1) Since $R_{\mathbb{S}}^1(\Delta_{2^{2n}})\subseteq\Delta_{2^{2n}}$, we have

$$R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k\left(R_{\mathbb{S}}^1(\Delta_{2^{2n}})\right)\subseteq R_{\mathbb{S}}^k(\Delta_{2^{2n}}), \quad k=1,2,\dots. \tag{11}$$

(2) Suppose that there exists an integer k such that $R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k(\Delta_{2^{2n}})$; i.e., $R_{\mathbb{S}}^k(\Delta_{2^{2n}}) = R_{\mathbb{S}}^1(R_{\mathbb{S}}^k(\Delta_{2^{2n}}))$. Then we have

$$\begin{aligned} R_{\mathbb{S}}^k(\Delta_{2^{2n}}) &= R_{\mathbb{S}}^1\left(R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}})\right) = R_{\mathbb{S}}^{k+2}(\Delta_{2^{2n}}) \\ &= R_{\mathbb{S}}^2\left(R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}})\right) = R_{\mathbb{S}}^{k+3}(\Delta_{2^{2n}}) \\ &\vdots \\ &= R_{\mathbb{S}}^{k+\tau}(\Delta_{2^{2n}}), \quad \tau = 1,2,\dots. \end{aligned} \tag{12}$$

This completes the proof. □

By resorting to the above important properties of the union set $R_{\mathbb{S}}^k(\Delta_{2^{2n}})$, one can derive the following result, which will be an auxiliary result for the robust synchronization.

Proposition 11. *Assume that π is the smallest positive integer such that $R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k(\Delta_{2^{2n}})$, and then we have $0\le\pi\le 2^{2n}-1$.*

Proof. If $R_{\mathbb{S}}^1(\Delta_{2^{2n}}) = \Delta_{2^{2n}}$, then we have $R_{\mathbb{S}}^0(\Delta_{2^{2n}}) = R_{\mathbb{S}}^1(\Delta_{2^{2n}}) = \Delta_{2^{2n}}$, which implies that $\pi = 0$. Now we claim that $1\le\pi\le 2^{2n}-1$. To prove this claim, it is sufficient to prove that $|R_{\mathbb{S}}^k(\Delta_{2^{2n}})|\le 2^{2n}-k$ holds for every $1\le k\le 2^{2n}-1$. We draw the conclusion by induction on k. Consider the case of $k=1$. Suppose on the contrary that $R_{\mathbb{S}}^1(\Delta_{2^{2n}}) > 2^{2n}-1$, which implies that $|R_{\mathbb{S}}^1(\Delta_{2^{2n}})| = 2^{2n}$. This implies the fact of $R_{\mathbb{S}}^1(\Delta_{2^{2n}}) = \Delta_{2^{2n}}$, and then we have $R_{\mathbb{S}}^t(\Delta_{2^{2n}}) = \Delta_{2^{2n}}, t = 1,\dots$. This is a contradiction with the fact that $R_{\mathbb{S}}^0(\Delta_{2^{2n}})\supsetneq R_{\mathbb{S}}^1(\Delta_{2^{2n}})$; i.e., $\pi\ge 1$. Thus, we prove that, for the case of $k=1$, $R_{\mathbb{S}}^1(\Delta_{2^{2n}})\le 2^{2n}-1$.

Now, let $1<k\le\pi$ and suppose that $|R_{\mathbb{S}}^k(\Delta_{2^{2n}})|\le 2^{2n}-k$ by induction. According to Proposition 10, we have $R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}})\subseteq R_{\mathbb{S}}^k(\Delta_{2^{2n}})$. This together with the induction hypothesis yields that $|R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}})|\le|R_{\mathbb{S}}^k(\Delta_{2^{2n}})|\le 2^{2n}-k$. If $|R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}})| > 2^{2n}-k-1$, then we have $|R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}})| = |R_{\mathbb{S}}^k(\Delta_{2^{2n}})| = 2^{2n}-k$, which implies that $R_{\mathbb{S}}^{k+1}(\Delta_{2^{2n}}) = R_{\mathbb{S}}^k(\Delta_{2^{2n}})$. This contradicts with the fact of the minimality of π. Thus, one can derive that $|R_{\mathbb{S}}^k(\Delta_{2^{2n}})|\le 2^{2n}-k$. This completes the proof. □

Based on Proposition 10, we obtain the following necessary and sufficient condition for robust synchronization of the drive-response BCNs (2) with disturbances.

Theorem 12. *Consider the drive-response BCNs (2) with disturbances. Let $R_{\mathbb{S}}^{k}(\Delta_{2^{2n}})$ be the union set of robustly reachable states from the initial states set $\Delta_{2^{2n}}$ at the k-th step. Then system (2) can be robustly synchronized if and only if there exists an integer k satisfying $1 \le k \le 2^{2n}-1$, such that*

$$R_{\mathbb{S}}^{k}\left(\Delta_{2^{2n}}\right) \subseteq \Xi, \tag{13}$$

where $\Xi = \{\delta_{2^{2n}}^{(i-1)2^n+i} : i = 1,\dots,2^n\}$ denotes the synchronized states set.

Proof. Based on Proposition 10, if there exists an integer k satisfying $1 \le k \le 2^{2n}-1$, such that $R_{\mathbb{S}}^{k}(\Delta_{2^{2n}}) \subseteq \Xi$, then, given any integer $t \ge k$, $R_{\mathbb{S}}^{t}(\Delta_{2^{2n}}) \subseteq \Xi$. Then, according to the definition of set Ξ, the states of $x(t)$ and $y(t)$ reach synchronization after k steps. The proof of sufficient part can be directly obtained, which is omitted here. This completes the proof. □

Remark 13. If $x(t) = y(t) = \delta_{2^n}^{j}$, then the equality $z(t) = x(t) \ltimes y(t)$ yields that $z(t) = \delta_{2^{2n}}^{(j-1)2^n+j}$. Thus, we use the set $\Xi = \{\delta_{2^{2n}}^{(i-1)2^n+i} : i = 1,\dots,2^n\}$ to denote the synchronized states set. According to Proposition 10, for the initial states set $\Delta_{2^{2n}}$, we can always calculate the union set of robustly reachable states at given step by recurrence method. Moreover, according to Proposition 11, we can conclude that one only needs to calculate the union set of robustly reachable states for finite steps in order to check robust synchronization.

Consider a set $\Pi = \{\delta_{2^{2n}}^{\lambda_1}, \delta_{2^{2n}}^{\lambda_2}, \dots, \delta_{2^{2n}}^{\lambda_\varepsilon}\}$. Note that each state $\delta_{2^{2n}}^{\lambda_j}$ $(j = 1,\dots,\varepsilon)$ is a column of the identity matrix $I_{2^{2n}}$, we define the column vector form of the set Π by Boolean summing up of all the states as follows:

$$\Pi_c = \Big(\dots,0,\overset{(\lambda_1)}{1},0,\dots,0,\overset{(\lambda_\varepsilon)}{1},\dots\Big)^T \in \mathscr{B}_{2^n\times 1}. \tag{14}$$

Here, the $\lambda_1,\dots,\lambda_\varepsilon$-th elements of Π_c are 1, and the rest are 0. Particularly, when $\Pi = \Delta_{2^{2n}}$, one can derive its corresponding column vector form of $\Delta_{2^{2n}}$, denoted by $(1,\dots,1)^T \in \mathscr{B}_{2^n\times 1}$. Thus, for the union set of robustly reachable states from the initial states $\Delta_{2^{2n}}$, i.e., $R_{\mathbb{S}}^{k}(\Delta_{2^{2n}})$, we use $\mathscr{R}_{\mathbb{S}}^{k}(\Delta_{2^{2n}})$ to denote the corresponding column vector form. Then, we have

$$\begin{aligned} R_{\mathbb{S}}^{k}\left(\Delta_{2^{2n}}\right) &= \bigcup_{j=1}^{2^{2n}} R_{\mathbb{S}}^{k}\left(\delta_{2^{2n}}^{j}\right) \sim \mathscr{R}_{\mathbb{S}}^{k}\left(\Delta_{2^{2n}}\right) \\ &= \sum_{j=1}^{2^{2n}}{}_{\mathscr{B}}\, \mathscr{R}_{\mathbb{S}}^{k}\left(\delta_{2^{2n}}^{j}\right). \end{aligned} \tag{15}$$

Since $\mathbb{L}$ in (8) is a $2^{2n} \times 2^{2n+p}$ matrix, we split it into 2^p equal blocks as $\mathbb{L} = [\mathbb{L}_1,\dots,\mathbb{L}_{2^p}]$, where $\mathbb{L}_j \in \mathscr{L}_{2^{2n}}$ $(j = 1,\dots,2^p)$. Then, the following result will provide an algebraic representation for $\mathscr{R}_{\mathbb{S}}^{k}(\Delta_{2^{2n}})$.

Theorem 14. *Consider the drive-response BCNs (2) with disturbances. Let $R_{\mathbb{S}}^{k}(\Delta_{2^{2n}})$ be the union set of robustly reachable states from the initial states set $\Delta_{2^{2n}}$ at the k-th step. Then, we have the following equation for the column vector form $\mathscr{R}_{\mathbb{S}}^{k}(\Delta_{2^{2n}})$:*

$$\mathscr{R}_{\mathbb{S}}^{k}\left(\Delta_{2^{2n}}\right) = \sum_{j=1}^{2^{2n}}{}_{\mathscr{B}}\, \mathrm{Col}_j\left(\check{\mathbb{L}}^{(k)}\right), \quad \check{\mathbb{L}} = \sum_{j=1}^{2^p}{}_{\mathscr{B}}\, \mathbb{L}_j. \tag{16}$$

Proof. We prove the result by induction. Consider the case of $k = 1$. Given an initial state $z(0) = \delta_{2^{2n}}^{j}$, one can obtain the column vector form of $R_{\mathbb{S}}^{1}(\delta_{2^{2n}}^{j})$ as follows:

$$\begin{aligned} \mathscr{R}_{\mathbb{S}}^{1}\left(\delta_{2^{2n}}^{j}\right) &= \left(\mathbb{L} \ltimes \delta_{2^p}^{1}\right)\delta_{2^{2n}}^{j} +_{\mathscr{B}} \cdots +_{\mathscr{B}} \left(\mathbb{L} \ltimes \delta_{2^p}^{2^p}\right)\delta_{2^{2n}}^{j} \\ &= \sum_{i=1}^{2^p}{}_{\mathscr{B}}\, \mathbb{L}_i \delta_{2^{2n}}^{j} = \mathrm{Col}_j\left(\check{\mathbb{L}}\right). \end{aligned} \tag{17}$$

Thus, the union set of robustly reachable states from the initial states set $\Delta_{2^{2n}}$ at the first step is $\mathscr{R}_{\mathbb{S}}^{1}(\Delta_{2^{2n}}) = \sum_{\mathscr{B}}{}_{j=1}^{2^{2n}} \mathscr{R}_{\mathbb{S}}^{1}(\delta_{2^{2n}}^{j}) = \sum_{\mathscr{B}}{}_{j=1}^{2^{2n}} \mathrm{Col}_j(\check{\mathbb{L}})$, which proves the case of $k = 1$.

Then, we assume that (16) holds for the case of $k = n$ and consider the case of $k = n+1$. Suppose that the reachable states in $R_{\mathbb{S}}^{n}(\Delta_{2^{2n}})$ are $\delta_{2^{2n}}^{w_1},\dots,\delta_{2^{2n}}^{w_r}$, where $r = |R_{\mathbb{S}}^{n}(\Delta_{2^{2n}})|$. Note that the trajectory starting from $z(0) \in \Delta_{2^{2n}}$ to $z(n+1) \in R_{\mathbb{S}}^{n+1}(\Delta_{2^{2n}})$ can be decomposed into trajectory starting from $z(0)$ to some $z(n) = \delta_{2^{2n}}^{w_i}$ $(i \in \{1,\dots,r\})$ at the n-th step and the trajectory from the possible states $z(n) = \delta_{2^{2n}}^{w_i}$ to $z(n+1) \in R_{\mathbb{S}}^{n+1}(\Delta_{2^{2n}})$ at one step. Thus, one can derive that

$$\begin{aligned} \mathscr{R}_{\mathbb{S}}^{n+1}\left(\Delta_{2^{2n}}\right) &= \mathscr{R}_{\mathbb{S}}^{1}\left(\mathscr{R}_{\mathbb{S}}^{n}\left(\Delta_{2^{2n}}\right)\right) \\ &= \mathscr{R}_{\mathbb{S}}^{1}\left(\delta_{2^{2n}}^{w_1}\right) +_{\mathscr{B}} \cdots +_{\mathscr{B}} \mathscr{R}_{\mathbb{S}}^{1}\left(\delta_{2^{2n}}^{w_r}\right). \end{aligned} \tag{18}$$

This together with (17) yields

$$\begin{aligned} \mathscr{R}_{\mathbb{S}}^{n+1}\left(\Delta_{2^{2n}}\right) &= \sum_{l=1}^{r}{}_{\mathscr{B}} \sum_{j=1}^{2^p}{}_{\mathscr{B}}\, \mathbb{L}_j \delta_{2^{2n}}^{w_l} \\ &= \sum_{j=1}^{2^p}{}_{\mathscr{B}}\, \mathbb{L}_j \times_{\mathscr{B}} \sum_{l=1}^{r}{}_{\mathscr{B}}\, \delta_{2^{2n}}^{w_l} \\ &= \check{\mathbb{L}} \times_{\mathscr{B}} \mathscr{R}_{\mathbb{S}}^{n}\left(\Delta_{2^{2n}}\right). \end{aligned} \tag{19}$$

On the other hand, one can obtain that

$$\sum_{i=1}^{2^{2n}}{}_{\mathscr{B}}\, \mathrm{Col}_i\left(\check{\mathbb{L}}^{(k+1)}\right) = \sum_{i=1}^{2^{2n}}{}_{\mathscr{B}}\, \mathrm{Col}_i\left(\check{\mathbb{L}} \times_{\mathscr{B}} \check{\mathbb{L}}^{(k)}\right)$$

$$= \sum_{\substack{i=1\\ \mathscr{B}}}^{2^{2n}} \check{\mathbb{L}} \times_{\mathscr{B}} \mathrm{Col}_i\left(\check{\mathbb{L}}^{(k)}\right)$$
$$= \check{\mathbb{L}} \times_{\mathscr{B}} \sum_{\substack{i=1\\ \mathscr{B}}}^{2^{2n}} \mathrm{Col}_i\left(\check{\mathbb{L}}^{(k)}\right). \tag{20}$$

Applying the induction hypothesis, by (19), we have $\sum_{\mathscr{B}\, i=1}^{2^{2n}} \mathrm{Col}_i(\check{\mathbb{L}}^{(n+1)}) = \check{\mathbb{L}} \times_{\mathscr{B}} \mathscr{R}_{\mathbb{S}}^{n}(\Delta_{2^{2n}}) = \mathscr{R}_{\mathbb{S}}^{n+1}(\Delta_{2^{2n}})$, which proves the case of $k = n+1$. Thus, according to the mathematical induction, we can conclude that (16) holds for any positive integer k, which completes the proof. □

Let $\overrightarrow{\Xi} = \sum_{j=1}^{2^n} \delta_{2^{2n}}^{(j-1)2^n+j}$ be the column vector form of the synchronized states set Ξ. Thus, based on Theorem 14, we can obtain the following result for robust synchronization.

Corollary 15. *Consider the drive-response BCNs (2) with disturbances. Let $\overrightarrow{\Xi}$ be the column vector form of the synchronized states set Ξ. Then system (2) can be robustly synchronized if and only if there exists an integer $1 \le k \le 2^{2n} - 1$, such that*

$$\overrightarrow{\Xi} - \sum_{\substack{j=1\\ \mathscr{B}}}^{2^{2n}} \mathrm{Col}_i\left(\check{\mathbb{L}}^{(k)}\right) \ge 0. \tag{21}$$

4. Illustrative Example

In this section, we present an illustrative example to demonstrate the effectiveness of our obtained results.

Consider the following drive-response BCNs with one disturbance input, and each BN has two nodes:

$$\begin{aligned} x_1(t+1) &= \xi(t) \wedge x_1(t) \wedge x_2(t) \\ x_2(t+1) &= \xi(t) \wedge x_1(t) \wedge \overline{x}_2(t), \end{aligned} \tag{22a}$$

and

$$\begin{aligned} y_1(t+1) &= [u_1(t) \wedge y_1(t)] \vee \{\overline{u}_1(t) \\ &\quad \wedge \{[u_2(t) \wedge y_1(t)] \vee [\overline{u}_2(t) \\ &\quad \wedge ((y_1(t) \wedge y_2(t)) \vee (\overline{y}_1(t) \wedge \overline{y}_2(t)))\}\} \\ y_2(t+1) &= [u_1(t) \wedge y_2(t)] \vee \{\overline{u}_1(t) \\ &\quad \wedge [(\overline{u}_2(t) \wedge y_2(t)) \vee (\overline{u}_2(t) \wedge \overline{y}_2(t))]\}, \end{aligned} \tag{22b}$$

where $x_1, x_2 \in \mathscr{D}$ and $y_1, y_2 \in \mathscr{D}$ are two state variables of drive BN and response BN respectively; $u_1, u_2 \in \mathscr{D}$ are two control inputs and $\xi \in \mathscr{D}$ is an external disturbance. Here, we use $\overline{x}$ to substitute the symbol $\neg x$, in order to simplify the logical functions. For example, if $x = 1$ ($x = 0$), we have $\overline{x} = 0$ ($\overline{x} = 1$).

Suppose that the state feedback control is given in the following form:

$$\begin{aligned} u_1(t) &\\ &= x_1(t) \\ &\quad \vee \{\overline{x}_1(t) \wedge [x_2(t) \vee (\overline{x}_2(t) \wedge \overline{y}_1(t) \wedge \overline{y}_2(t))]\} \\ u_2(t) &\\ &= x_1(t) \vee [\overline{x}_1(t) \wedge \overline{x}_2(t) \wedge \overline{y}_1(t) \wedge \overline{y}_2(t)] \\ &\quad \vee [\overline{x}_1(t) \wedge x_2(t)]. \end{aligned} \tag{23}$$

Then, using the vector forms of Boolean variables and denoting $x(t) = \ltimes_{i=1}^{2} x_i(t) \in \Delta_4$, $y(t) = \ltimes_{i=1}^{2} y_i(t) \in \Delta_4$, and $u(t) = \ltimes_{i=1}^{2} u_i(t) \in \Delta_4$, we can obtain the following algebraic forms of the drive-response BCNs (22a) and (22b) with disturbance and the state feedback control (23):

$$\begin{aligned} x(t+1) &= F\xi(t)\,x(t), \\ y(t+1) &= Gu(t)\,y(t), \\ u(t) &= Kx(t)\,y(t), \end{aligned} \tag{24}$$

where $F = \delta_4[2,3,4,4,3,3,4,4]$, $G = \delta_4[1,2,3,4,1,2,3,4,1,2,3,4,2,3,4,1]$, and $K = \delta_4[1,1,1,1,1,1,1,1,1,1,1,1,4,4,4,1]$.

Then, denote $z(t) = x(t) \ltimes y(t) \in \Delta_{16}$, and we can obtain the following equivalent algebraic form of the drive-response BCNs (24) with disturbance:

$$z(t+1) = L\xi(t)\,u(t)\,z(t), \tag{25}$$

where $L = F(I_{2^{n+p}} \otimes G)(I_{2^p} \otimes W_{[2^m,2^n]}) \in \mathscr{L}_{16\times 2^7}$. We present the row indexes of columns in matrix L as follows:

$$\begin{aligned} L = \delta_{16}\,[&5,6,7,8, \\ &9,10,11,12,13,14,15,16,13,14,15,16,5,6,7,8,9, \\ &11,12,13\ ,14,15,16,13,14,15,16,5,6,7,8,9, \\ &10,11,12,13,14,15,16,13,14,15,16,5,6,7,8,9, \\ &10,11,12,13,14,15,16,13,14,15,16,6,7,8, \\ &5,10,11,12,9,14,15,16,13,14,15,16,13,9, \\ &10,11,12,9, \\ &10,11,12,13,14,15,16,13,14,15,16,9, \\ &10,11,12,9, \\ &10,11,12,13,14,15,16,13,14,15,16,9, \\ &10,11,12,9, \\ &10,11,12,13,14,15,16,13,14,15,16,10,11,12,9, \\ &10,11,12,9,14,15,16,13,14,15,16,13]. \end{aligned} \tag{26}$$

Figure 1 shows the row indexes of columns in matrix L.

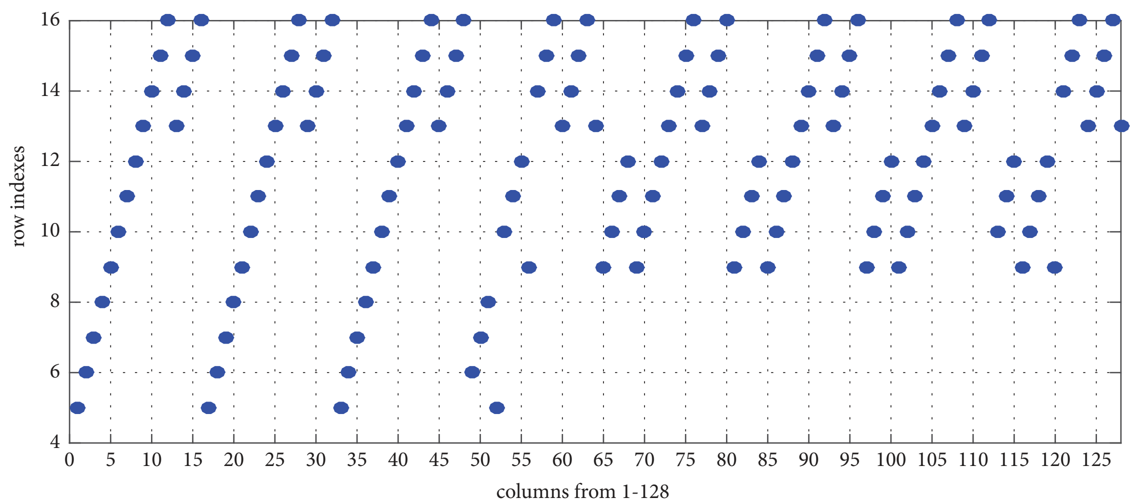

Figure 1: The row indexes of each column of matrix L obtained in system (25). Each point corresponds to the row index of each column, which implies the position of element 1.

Thus, under the state feedback control $u(t) = Kx(t)y(t)$, we obtain the following system: $z(t+1) = \mathbb{L}\xi(t)z(t)$, where $\mathbb{L} = L(I_2 \otimes K\Phi_4) \in \mathscr{L}_{16\times 32}$. A calculation yields

$$\begin{aligned} \mathbb{L} = \delta_{16}\,[&5,6,7,8,\\ &9,10,11,12,13,14,15,16,14,15,16,16,9,\\ &10,11,12,9,10,11,12,13,14,15,16,14,15,16,16]\,. \end{aligned} \tag{27}$$

Our objective is to check whether the drive-response BCNs (22a) and (22b) with disturbances can be robustly synchronized. As discussed in Remark 13, if $x(t) = y(t) = \delta_4^j$ $(1 \le j \le 4)$ then the equality $z(t) = x(t) \ltimes y(t)$ yields that $z(t) = \delta_{16}^{(j-1)4+j}$. Thus, we can obtain the synchronized states set $\Xi = \{\delta_{16}^i : i = 1,6,11,16\}$ and the unsynchronized states set $\overline{\Xi} = \{\delta_{16}^i : i = 2,3,4,5,7,8,9,10,12,13,14,15\}$. For example, if $z(t) = \delta_{16}^2$, one can obtain that $x(t) \ne y(t)$, since $x(t) = \delta_4^1$ and $y(t) = \delta_4^2$. A calculation yields

$$\begin{aligned} \check{\mathbb{L}} = \big[&\delta_{16}^5 + \delta_{16}^9, \delta_{16}^6 + \delta_{16}^{10}, \delta_{16}^7 + \delta_{16}^{11}, \delta_{16}^8 + \delta_{16}^{12}, \delta_{16}^9, \delta_{16}^{10},\\ &\delta_{16}^{11}, \delta_{16}^{12}, \delta_{16}^{13}, \delta_{16}^{14}, \delta_{16}^{15}, \delta_{16}^{16}, \delta_{16}^{14}, \delta_{16}^{15}, \delta_{16}^{16}, \delta_{16}^{16}\big]\,. \end{aligned} \tag{28}$$

Thus, according to Corollary 15, we can obtain that

$$\sum_{j=1}^{16}{}_{\mathscr{B}} \operatorname{Col}_j\left(\check{\mathbb{L}}^{(6)}\right) = \delta_{16}^{16} \le \vec{\Xi} = \sum_{j=1,6,11,16}{}_{\mathscr{B}} \delta_{16}^j, \tag{29}$$

which implies that the drive-response BCNs (22a) and (22b) with disturbance can be robustly synchronized. The set of trajectories starting from the initial states set Δ_{16} converging to synchronized states set $\Xi = \{\delta_{16}^i : i = 1,6,11,16\}$ from time $t = 0$ to time $t = 6$ can be given as follows:

$$\begin{gathered} \left\{\begin{matrix} z^1(0) = \delta_{16}^1 \\ z^2(0) = \delta_{16}^2 \\ \vdots \\ z^{16}(0) = \delta_{16}^{16} \end{matrix}\right\} \xrightarrow{\xi(0)} \cdots \xrightarrow{\xi(4)} \\ \left\{\begin{matrix} z^{15}(5) = \delta_{16}^{15} \\ z^{16}(5) = \delta_{16}^{16} \end{matrix}\right\} \xrightarrow{\xi(5)} \left\{z^{16}(6) = \delta_{16}^{16}\right\} \subseteq \Xi. \end{gathered} \tag{30}$$

As we can see from the above trajectories, at time $t = 4$, there exist two states δ_{16}^{14} and δ_{16}^{15} in $R_{\mathbb{S}}^4(\Delta_{16})$ that belong to the unsynchronized states set $\overline{\Xi}$. At time $t = 5$, there still exists one state δ_{16}^{15} in $R_{\mathbb{S}}^5(\Delta_{16})$ belonging to $\overline{\Xi}$. Thus, the drive-response BCNs (22a) and (22b) with disturbance can not be robustly synchronized at time $t = 5$. However, at time $t = 6$, there is only one state δ_{16}^{16} in $R_{\mathbb{S}}^6(\Delta_{16})$, which belongs to the synchronized states set Ξ. Thus, the drive-response BCNs (22a) and (22b) with disturbance is robustly synchronized at time $t = 6$. Moreover, the row indexes of the column vector $R_{\mathbb{S}}^k(\Delta_{16})$ versus time $k = 1,2,\ldots,6$ are plotted in Figure 2. Thus, the validness of Corollary 15 has been well illustrated by this example.

5. Conclusion

In this paper, we have investigated robust synchronization of drive-response BCNs with disturbances. Based on the fundamental properties of STP, we firstly have obtained the algebraic representations of drive-response BCNs with disturbances. By defining a set of robustly reachable states from a given initial state at a given time step, we have obtained the set of all possible reachable states for any given disturbance sequences from the global initial states set after

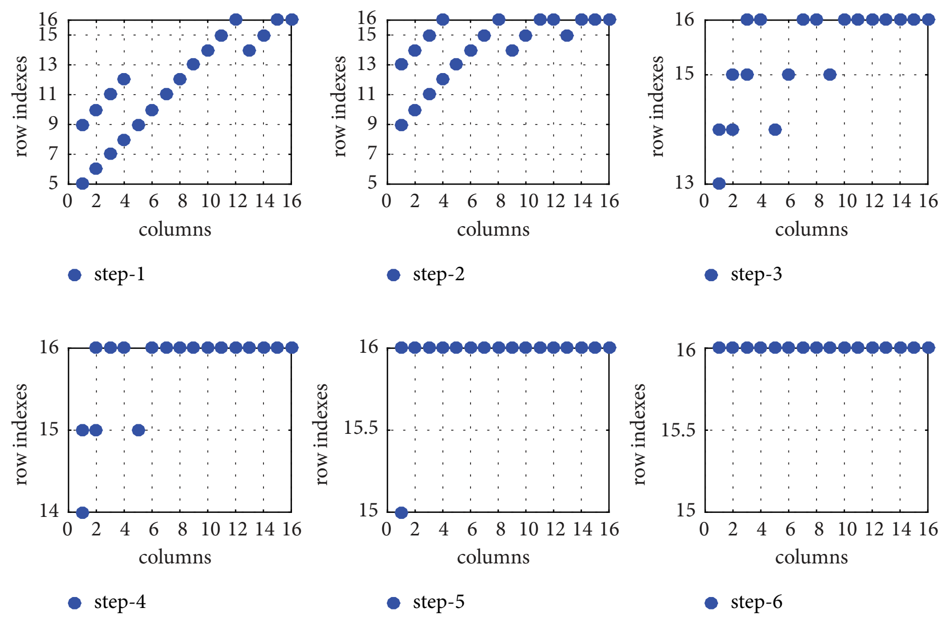

FIGURE 2: The whole row indexes of the column vectors obtained in $R_{\mathbb{S}}^{k}(\Delta_{16})$ $(k = 1, 2, \ldots, 6)$. Each point corresponds to the row index of each column, which implies the position of element 1.

given steps, which is called the union set of robustly reachable states from the global initial states set. Then, based on the defined union sets, several necessary and sufficient criteria have been obtained for robust synchronization of disturbed BCNs. Moreover, the obtained results also have been verified that one only needs to calculate finite steps to check whether the drive-response BCNs with disturbances can be robustly synchronized.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China [Grant nos. 61573102, 11671361], the Natural Science Foundation of Jiangsu Province of China [Grant BK20170019], China Postdoctoral Science Foundation [Grant nos. 2014M560377, 2015T80483], Jiangsu Province Six Talent Peaks Project [Grant no. 2015-ZNDW-002], and the Fundamental Research Funds for the Central Universities.

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