ON ROBUSTNESS AND RELATED PROPERTIES ON TORIC IDEALS

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Abstract. A toric ideal is called robust if its universal Gröbner basis is a minimal set of generators, and is called generalized robust if its universal Gröbner basis equals its universal Markov basis (the union of all its minimal sets of binomial generators). Robust and generalized robust toric ideals are both interesting from both a Commutative Algebra and an Algebraic Statistics perspective. However, only a few nontrivial examples of such ideals are known. In this work we study these properties for toric ideals of both graphs and numerical semigroups. For toric ideals of graphs, we characterize combinatorially the graphs giving rise to robust and to generalized robust toric ideals generated by quadratic binomials. As a byproduct, we obtain families of Koszul rings. For toric ideals of numerical semigroups, we determine that one of its initial ideals is a complete intersection if and only if the semigroup belongs to the so-called family of free numerical semigroups. Hence, we characterize all complete intersection numerical semigroups which are minimally generated by one of its Gröbner basis and, as a consequence, all the Betti numbers of the toric ideal and its corresponding initial ideal coincide. Moreover, also for numerical semigroups, we prove that the ideal is generalized robust if and only if the semigroup has a unique Betti element and that there are only trivial examples of robust ideals. We finish the paper with some open questions.

Contents

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1. Introduction

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subseteq \mathbb{N}^n$ be a finite set of nonzero vectors and $\mathbb{N}A := \{l_1\mathbf{a}_1 + \cdots + l_m\mathbf{a}_m \mid l_i \in \mathbb{N}\}$ the corresponding affine monoid. We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the *$A$-degree* of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be $\deg_A(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A$, while we denote the usual degree $u_1 + \cdots + u_m$ of $\mathbf{x}^{\mathbf{u}}$ by $\deg(\mathbf{x}^{\mathbf{u}})$. The *toric ideal* I_A associated to A is the ideal generated by all the binomials $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ such that $\deg_A(\mathbf{x}^{\mathbf{u}}) = \deg_A(\mathbf{x}^{\mathbf{v}})$. It is a prime ideal of height $m - r$, being r the rank of the subgroup of $\mathbb{Z}^m$ spanned by A (see, e.g., [39]). Toric ideals have applications in several areas such as: algebraic statistics, biology, computer algebra, computer aided geometric design, dynamical systems, hypergeometric differential equations, integer programming, toric geometry, graph theory, e.t.c. (see, e.g. [1, 3, 20, 21, 28, 31, 34, 35, 39, 45, 46]).

A binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ in I_A is called *primitive* if there is no other binomial $\mathbf{x}^{\mathbf{w}} - \mathbf{x}^{\mathbf{z}}$ in I_A, such that $\mathbf{x}^{\mathbf{w}}$ divides $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{z}}$ divides $\mathbf{x}^{\mathbf{v}}$. The set of primitive binomials, which is finite, is the *Graver basis* of I_A and is denoted by Gr_A. The *universal Gröbner basis* of an ideal I_A, is denoted by $\mathcal{U}_A$ and is defined as the union of all reduced Gröbner bases $G_{\prec}$ of I_A, as $\prec$ runs over all term orders. Since I_A is generated by binomials, then every reduced Gröbner basis of I_A consists of binomials (see, e.g. [21]). Thus, the universal Gröbner basis of I_A is a finite subset of binomials in I_A and it is a Gröbner basis for the ideal with respect to all term orders. By [39, Proposition 4.11], we have that $\mathcal{U}_A \subseteq Gr_A$. A *Markov basis* M_A is a minimal binomial generating set of the toric ideal I_A (its name Markov basis comes from its relation with Markov chains, see [20, Theorem 3.1]). The *universal Markov basis* of the ideal is denoted by $\mathcal{M}_A$ and is defined as the union of all the Markov bases of the ideal. The elements of $\mathcal{M}_A$ are called *minimal binomials.* Since $A \subseteq \mathbb{N}^n$, then $\mathbb{N}A$ is *pointed* (that is, $\mathbb{N}A \cap (-\mathbb{N}A) = \{0\}$). As a consequence $\mathcal{M}_A \subseteq Gr_A$ and, hence, $\mathcal{M}_A$ is also a finite set (see [14, Theorem 2.3] and [15]). The Graver basis, the universal Gröbner basis and the universal Markov basis are usually called *toric bases.*

An ideal I is called *robust* if its universal Gröbner basis is a minimal set of generators of the ideal. Even in the context of toric ideals, robustness is a property that has not been fully described. It is known that Lawrence ideals are robust and robustness has also been studied in [12] for toric ideals of graphs, and in [11] for toric ideals which are generated by quadratic binomials. Some of the interests in studying robustness stems from the fact that they are ideals which are minimally generated by a Gröbner basis, see [4, 16]. Whenever I is an ideal with a quadratic Gröbner basis, then $\mathbb{K}[x_1, \ldots, x_m]$ is a Koszul algebra. Hence, another interesting feature of robust ideals generated by quadrics, is that they provide examples of Koszul algebras. Nevertheless, robustness is a rare property and there are very few nontrivial examples of robust ideals. This makes it natural to consider a wider family of ideals that shares many of its good properties. In this paper we study the property of a toric ideal being *generalized robust.* An ideal is called generalized robust if its universal Gröbner basis is equal to its universal Markov basis. The notion of generalized robustness of a toric ideal was introduced in [41] and, as its name indicates, it is a family containing robust toric ideals (see [41, Corollary 3.5]). Since determining or computing the universal Gröbner basis of I is a very difficult and computationally demanding problem, it is still difficult to determine whether a toric ideal is generalized robust. The main goal of this paper is to provide several families of generalized robust toric ideals.

The present paper is divided into the following sections. In Section 2, we collect some basic facts related to the toric bases, that can be found in the literature or are easy consequences of known results. In particular, we give a description of the universal Markov basis (Proposition 2.1) and study how the toric bases behave with respect to the elimination of variables (Propositions 2.3 and 2.4). In Theorem 2.5 we prove that a homogeneous toric ideal is generalized robust and generated by quadrics if and only if its universal Gröbner basis only consists of quadrics.

The main results of this paper are divided in two parts. The first one is presented in Section 3, in which we completely characterize the graphs giving rise to robust and generalized robust toric ideals that both are generated by quadrics. More precisely, we provide the following structural theorem, which summarizes Theorem 3.7 and Theorem 3.13 (see also Definition 3.9 and Definition 3.11):

Theorem 1.1. *Let G be a finite, connected and simple graph. The toric ideal associated to G is generalized robust and generated by quadrics if and only if G has at most one non bipartite block which is either a K_4, or a $K_{2,\ell} \cup \{e\}$ or a double-$K_{2,(r,s)}$ or a necklace-$K_{2,\ell}$ graph. The bipartite blocks of G are either $K_{2,\ell}$ or cut edges.*

Interestingly, $\mathbb{K}[x_1,\ldots,x_m]/I_G$ is a Koszul ring for all the graphs G described in this theorem.

As a direct consequence of this, our second main result is Theorem 3.17, in which we characterize all graphs whose toric ideal is robust and is generated by quadratic binomials. It should be noticed that this result also completes [12, Corollary 5.2],

Theorem 1.2. *Let G be a non bipartite graph. The toric ideal associated to G is robust and is generated by quadrics if and only if all the blocks of G are bipartite except one which is either a $K_{2,\ell} \cup \{e\}$ or a double-$K_{2,(r,s)}$ or a necklace-$K_{2,\ell}$ graph. The bipartite blocks of G are of type $K_{2,\ell}$ or cut edges.*

The second part of the main results is presented in Section 4, where we work in the framework of toric ideals associated to submonoids of $\mathbb{N}$. More precisely, given a submonoid $\mathcal{S}$ of $(\mathbb{N},+)$, then it has a unique minimal set of generators $A = \{a_1,\ldots,a_m\}$, and the toric ideal of $\mathcal{S}$ is defined as $I_{\mathcal{S}} := I_A$. Taking $d := \gcd(a_1,\ldots,a_m)$ and $A' := \{a_1/d,\ldots,a_m/d\}$, then $I_A = I_{A'}$. Hence, one may assume without loss of generality that $A = \{a_1,\ldots,a_m\}$ consists of relatively prime positive integers and, in this case, $\mathcal{S}$ is called a numerical semigroup (for a detailed study of numerical semigroups we refer the reader to [6, 36]). Since $I_{\mathcal{S}}$ has height $m-1$, we have that $I_{\mathcal{S}}$ is a complete intersection if and only if one of its Markov basis (and, thus, all its Markov bases) consists of $m-1$ binomials. Complete intersection numerical semigroups have been widely studied in the literature, see, e.g., [5, 8, 9, 17, 18, 27].

Take $\prec$ a monomial order in $\mathbb{K}[x_1,\ldots,x_m]$ and denote by $\mathrm{in}_\prec(I_{\mathcal{S}})$ the corresponding initial ideal of $I_{\mathcal{S}}$. Since $\mathrm{ht}(\mathrm{in}_\prec(I_{\mathcal{S}})) = \mathrm{ht}(I_{\mathcal{S}}) = m-1$, then $\mathrm{in}_\prec(I_{\mathcal{S}})$ is a complete intersection if it can be generated by $m-1$ monomials. In other words, $\mathrm{in}_\prec(I_{\mathcal{S}})$ is a complete intersection if and only if the reduced Gröbner basis of $I_{\mathcal{S}}$ with respect to $\prec$ consists of $m-1$ binomials. Since Gröbner bases are generating sets of the ideal, whenever $\mathrm{in}_\prec(I_{\mathcal{S}})$ is a complete intersection for a monomial order $\prec$, then $I_{\mathcal{S}}$ so is.

Our main results in this section are summarized in the following diagram:

$$
\begin{array}{ccc}
\mathcal{S} = \langle a_1, a_2 \rangle & \Longleftrightarrow & \mathcal{S} \text{ is robust} \\
\Downarrow & & \\
\mathcal{S} \text{ has a unique Betti element} & \Longleftrightarrow & \mathcal{S} \text{ is generalized robust} \\
\Downarrow & & \\
\mathcal{S} \text{ is free} & \Longleftrightarrow & \mathrm{in}_{\prec}(I_{\mathcal{S}}) \text{ is a C.I. for some } \prec \\
\Downarrow & & \\
I_{\mathcal{S}} \text{ is a C.I.} & &
\end{array}
$$

Theorem 4.7 states that $I_{\mathcal{S}}$ has a complete intersection initial ideal if and only if $\mathcal{S}$ is a free numerical semigroup, a family of semingroups studied in [7, 27, 37]. Since in this case, both $I_{\mathcal{S}}$ and the corresponding initial ideal $\mathrm{in}_{\prec}(I_{\mathcal{S}})$ are complete intersections, it turns out that all the Betti numbers in the whole minimal graded free resolution of $I_{\mathcal{S}}$ and $\mathrm{in}_{\prec}(I_{\mathcal{S}})$ coincide, providing examples of *robustness of Betti numbers*. This is an interesting phenomenon which is known to occur for robust toric ideals generated by quadrics [11] and also to the ideal of maximal minors of a generic matrix and its Gröbner basis with respect to a certain monomial order [16]. In Theorem 4.12 we determine all toric ideals of numerical semigroup that are generalized robust. It turns out that this property is characterized by a known subfamily of numerical semigroups studied in [23, 24, 29], namely the semigroups with a unique Betti element. As an easy consequence, we get in Corollary 4.17 that there are no nontrivial examples of robust toric ideal of a numerical semigroup, that is, $I_{\mathcal{S}}$ is robust if and only if $I_{\mathcal{S}}$ a principal ideal or, in other words, if $\mathcal{S}$ is a 2-generated numerical semigroup.

Finally, we state some conclusions and formulate some conjectures and open problems concerning robustness, generalized robustness and related properties in toric ideals. These conjectures are supported by some experimental evidence with the computer softwares CoCoA [1] and Singular [19].

2. Remarks on toric bases

Let $A \subseteq \mathbb{N}^n$ be a finite set of nonzero vectors. In this section we will discuss some known basic facts about the bases associated to the toric ideal $I_A \subseteq \mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, \ldots, x_n]$, namely the universal Markov basis $\mathcal{M}_A$, the universal Gröbner basis $\mathcal{U}_A$, the Graver basis Gr_A and the set of the circuits $\mathcal{C}_A$. A binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_A$ with $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbf{v} = (v_1, \ldots, v_m) \in \mathbb{N}^m$ is called a *circuit* if it has minimal support (with respect to set containment), if $\gcd(\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}) = 1$ and the nonzero entries of $\mathbf{u} + \mathbf{v}$ are relatively prime.

First of all, it is worth pointing out that, since $\mathbb{N}A$ is pointed, then the graded version of Nakayama's lemma holds. As a consequence, if we consider a set $\{g_1, \ldots, g_r\}$ of A-homogeneous polynomials, one has that $I_A = \langle g_1, \ldots, g_r \rangle$ if and only if the cosets of $g_1, \ldots, g_r$ span the $\mathbb{K}$-vector space $I_A / \langle x_1, \ldots, x_m \rangle \cdot I_A$. As a consequence, any minimal set of A-homogeneous generators of I_A has the same cardinality, which is $\mu(I_A) := \dim_{\mathbb{K}}(I_A / \langle x_1, \ldots, x_m \rangle \cdot I_A)$. Moreover, Nakayama's lemma also guarantees that the A-degrees appearing in any minimal set of A-homogeneous generators are invariant, these values are usually called *Betti degrees* of I_A. Since every binomial in I_A is A-homogeneous, we have the following result.

Proposition 2.1. *The universal Markov basis $\mathcal{M}_A$ is the set of binomials in I_A that do not belong to $\langle x_1, \ldots, x_m \rangle \cdot I_A$.*

Moreover, the following inclusions of toric bases hold:

Theorem 2.2. [39, Proposition 4.11], [14, Theorem 2.3] *For any toric ideal it holds*

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

Moreover, since A defines a pointed semigroup, then $\mathcal{M}_A \subseteq Gr_A$.

In the forthcoming we use many times how the toric bases behave with respect to an elimination of variables, these are summarized in the following result (see, e.g., [39, Proposition 4.13]).

Proposition 2.3. *Let $A' \subseteq A$ and denote $\mathbb{K}[\mathbf{x}_{A'}] := \mathbb{K}[x_i \,|\, a_i \in A']$, then*
(a) $I_{A'} = I_A \cap \mathbb{K}[\mathbf{x}_{A'}]$.
(b) $\mathcal{C}_{A'} = \mathcal{C}_A \cap \mathbb{K}[\mathbf{x}_{A'}]$.
(c) $\mathcal{U}_{A'} = \mathcal{U}_A \cap \mathbb{K}[\mathbf{x}_{A'}]$.
(d) $Gr_{A'} = Gr_A \cap \mathbb{K}[\mathbf{x}_{A'}]$.
(e) $\mathcal{M}_{A'} \supseteq \mathcal{M}_A \cap \mathbb{K}[\mathbf{x}_{A'}]$.

In general we do not have equality in Proposition 2.3.(e). For example, for $A' = \{a_1, a_2\} \subseteq A = \{a_1, a_2, a_3\} \subseteq \mathbb{N}$ with $a_1 = 4, a_2 = 5$ and $a_3 = 6$, we have that $\mathcal{M}_A = \{x_1^3 - x_3^2, x_2^2 - x_1x_3\}$, and hence $\mathcal{M}_A \cap \mathbb{K}[x_1, x_2] = \emptyset$, whereas $\mathcal{M}_{A'} = \{x_1^5 - x_2^4\}$.

In [11, Proposition 2.5] the authors proved that robustness is preserved under an elimination of variables, that is, if an ideal I_A is robust and $A' \subseteq A$, then the ideal $I_{A'}$ is also robust. We do not know if generalized robustness is preserved under elimination of variables, see Question 5.1. Nevethelesss, the following results hold.

Proposition 2.4. *Let $A \subseteq \mathbb{N}^n$ be a finite set of nonzero vectors and $A' \subseteq A$. Then,*
(a) *If $\mathcal{U}_A \subseteq \mathcal{M}_A$, then $\mathcal{U}_{A'} \subseteq \mathcal{M}_{A'}$.*
(b) *If $\mathcal{C}_A \subseteq \mathcal{M}_A$, then $\mathcal{C}_{A'} \subseteq \mathcal{M}_{A'}$.*
(c) *If $Gr_A = \mathcal{M}_A$, then $Gr_{A'} = \mathcal{M}_{A'}$.*

Proof. (a) If $\mathcal{U}_A \subseteq \mathcal{M}_A$, by Proposition 2.3 we have that $\mathcal{U}_{A'} = \mathcal{U}_A \cap \mathbb{K}[\mathbf{x}_{A'}] \subseteq \mathcal{M}_A \cap \mathbb{K}[\mathbf{x}_{A'}] \subseteq \mathcal{M}_{A'}$. The proof of (b) is analogue to the one of (a).

(c) If $Gr_A = \mathcal{M}_A$, by Proposition 2.3 we have that $Gr_{A'} = Gr_A \cap \mathbb{K}[\mathbf{x}_{A'}] = \mathcal{M}_A \cap \mathbb{K}[\mathbf{x}_{A'}] \subseteq \mathcal{M}_{A'}$, the inclusion $\mathcal{M}_{A'} \subseteq Gr_{A'}$ follows from Proposition 2.2. □

The next theorem gives a nice property which motivates the study of generalized robust toric ideals which are generated by quadrics.

Theorem 2.5. *A homogeneous toric ideal $I_A \subseteq \mathbb{K}[x_1, \ldots, x_m]$ is generalized robust and generated by quadrics if and only if the universal Gröbner basis of I_A only consists of quadrics.*

Proof. ($\Longrightarrow$) Since I_A is generated by quadrics, then its universal Markov basis $\mathcal{M}_A$ consists of quadrics. The result follows from the definition of a generalized robust toric ideal.

($\Longleftarrow$) Since the universal Gröbner basis is a set of generators, then I_A is generated by quadrics. Let us see now that $\mathcal{M}_A = \mathcal{U}_A$.

Let $f \in \mathcal{U}_A$, then $f \in I_A$ is a quadric and $f \notin \langle x_1, \ldots, x_m \rangle \cdot I_A$ (since the elements of $\langle x_1, \ldots, x_m \rangle \cdot I_A$ have degree at least three). By the graded version of Nakayama's lemma it follows that $f \in \mathcal{M}_A$ and thus $\mathcal{U}_A \subseteq \mathcal{M}_A$. Conversely, take $f \in \mathcal{M}_A$. Since the ideal I_A is generated by quadrics and due to the fact that there exists a Markov basis of the ideal which consists of quadrics, by Nakayama's lemma it follows that every Markov basis of I_A consists of quadrics and thus f is a quadratic binomial. After reindexing the variables and considering $-f$ if necessary, we have that either $f = x_3x_4 - x_1x_2$, or $f = x_3^2 - x_1x_2$, otherwise the ideal is not prime. In both cases the binomial f is in the reduced Gröbner basis of I_A with respect to the lexicographic order with $x_m > \cdots > x_1$. Thus, $f \in \mathcal{U}_A$ and therefore $\mathcal{M}_A \subseteq \mathcal{U}_A$. □

3. Quadratic robust and generalized robust toric ideals of graphs

3.1. **Preliminaries.** In this section we study the robustness and generalized robustness property for toric ideals of graphs which are generated by quadratic binomials. In the rest of the present section, we consider finite, simple and connected graphs. Let G be a graph with vertices $V(G) = \{v_1, \ldots, v_n\}$ and edges $E(G) = \{e_1, \ldots, e_m\}$. Let $\mathbb{K}[e_1, \ldots, e_m]$ be the polynomial ring in the m variables $e_1, \ldots, e_m$ over a field $\mathbb{K}$. We associate each edge $e = (v_i, v_j) \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of the vertices of G, i.e. each vertex $v_j \in V(G)$ is associated with the vector $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$, where the nonzero component is in the j position. We denote by I_G the toric ideal I_{A_G} in $\mathbb{K}[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subseteq \mathbb{N}^n$. Toric ideals of graphs are homogeneous prime ideals and many of their algebraic properties can be described in terms of the underlying graph. For example, I_G has height

$$\mathrm{ht}(I_G) = |E(G)| - |V(G)| + b(G), \tag{1}$$

where $b(G)$ denotes the number of connected components of the graph G that are bipartite (see, e.g., [47]). In particular, if G is connected, then $b(G) = 1$ if G is bipartite and $b(G) = 0$ otherwise (see, e.g., [26]). Also, binomial generating sets of I_G can be described in terms of some walks in the graph. To present this result, we first recall some basic elements from graph theory (for unexplained terminology and basics on graphs we refer to [13]).

A *walk* connecting $u \in V(G)$ and $u' \in V(G)$ is a finite sequence of vertices of the graph $w = (u = u_0, u_1, \ldots, u_{\ell-1}, u_\ell = u')$, with each $e_{i_j} = (u_{j-1}, u_j) \in E(G)$, for $j = 1, \ldots, \ell$. The *length* of the walk w is the number ℓ of its edges. An *even* (respectively *odd*) *walk* is a walk of even (respectively odd) length. A walk $w = (u_0, u_1 \ldots, u_{\ell-1}, u_\ell)$ is called *closed* if $u_0 = u_\ell$. A *cycle* is a closed walk $(u_0, u_1, \ldots, u_{\ell-1}, u_\ell)$ with $u_k \neq u_j$, for every $1 \leq k < j \leq \ell$.

Consider an even closed walk $w = (u_0, u_1, u_2, \ldots, u_{2s-1}, u_{2s} = u_0)$ of length $2s$ with $e_{i_j} = (u_{j-1}, u_j) \in E(G)$, for $j = 1, \ldots, 2s$. The binomial

$$B_w = e_{i_1} e_{i_3} \cdots e_{i_{2s-1}} - e_{i_2} e_{i_4} \cdots e_{i_{2s}} \tag{2}$$

belongs to the toric ideal I_G. Actually, Villarreal proved in [46] that

$$I_G = \langle B_w \mid w \text{ is an even closed walk} \rangle, \tag{3}$$

that is, the toric ideal I_G is generated by the binomials corresponing to even closed walks of the graph G.

In [33], Ohsugi and Hibi gave the following combinatorial criterion for the toric ideal of a graph G to be generated by quadrics:

Theorem 3.1. [33, Theorem 1.2] *Let G be a finite connected simple graph. Then, the toric ideal I_G of G is generated by quadrics if and only if the following conditions are satisfied:*

i) *if c is an even cycle of G of length ≥ 6, then either c has an even chord or c has three odd chords e, e', e'' such that e and e' cross in c,*
ii) *if c_1 and c_2 are odd chordless cycles of G having exactly one common vertex, then there exists a bridge between them,*
iii) *if c_1 and c_2 are odd chordless cycles of G having no common vertex, then there exist at least two bridges between c_1 and c_2.*

Also, the only even closed walks of length four in a simple graph are cycles. Hence, whenever B_w is a quadric, then w is a cycle of length four. Thus, we have the following.

Corollary 3.2. *Let G be a graph. If the toric ideal I_G is generated by quadrics, then all its minimal binomials are of the form B_w, where w is a cycle of length four.*

The Graver basis and the universal Markov basis of the toric ideal of a graph G, which we denote by Gr_G and $\mathcal{M}_G$ correspondingly, were described in [38], Theorem 3.2 and Theorem 4.13 correspondingly, while the universal Gröbner basis, which we denote by $\mathcal{U}_G$, was described in [42, Theorem 3.4]. For the sake of brevity we refer the reader to the above articles. A necessary and sufficient characterization for generalized robust toric ideals of graphs was given in [41]:

Theorem 3.3. [41, Theorem 3.4] *Let G be a graph and let I_G be its corresponding toric ideal. The ideal I_G is generalized robust if and only if $\mathcal{M}_G = Gr_G$.*

For general toric ideals, it may happen that the universal Markov basis is not contained into the universal Gröbner basis (see Section 5 for an example). Nevertheless, in the context of toric ideals of graphs we have that $\mathcal{M}_G \subseteq \mathcal{U}_G$ for any graph G (see [41, Proposition 3.3]). This fact together with Proposition 2.4 yields that the generalized robustness property of a graph G is a hereditary property, in the sense that it holds also for any subgraph of G.

Corollary 3.4. *Let H be a subgraph of a graph G. If the ideal I_G is generalized robust, then I_H is generalized robust.*

3.2. **Quadratic generalized robust graphs**; **the bipartite case.** We state some properties of a generalized robust toric ideal of a graph, which stem from directly of the results in [38, 41, 42] and will be useful for us in the sequel. By *chordless graph* we mean a graph in which every cycle has no chords.

Corollary 3.5. *Let G be a bipartite graph.*

α) *The ideal I_G is generalized robust if and only if the graph G is chordless.*
β) *The ideal I_G is generalized robust and generated by quadrics if and only if all the cycles of the graph G have length four.*

Proof. (α) In [41, Proposition 4.3] the author proved that an even cycle in a graph of a generalized robust toric ideal has only odd chords (if it has). Since the graph G is bipartite it follows that it is chordless. Conversely, it is known that $\mathcal{M}_G \subseteq Gr_G$ and let $B_w \in Gr_G$. The graph is bipartite, thus the walk w is an even cycle, see [38, Theorem 3.2]. Since the graph is chordless, the cycle w is chordless and therefore $B_w \in \mathcal{M}_G$, see [38, Theorem 4.13]. The result follows from Theorem 3.3.

(β) It follows from the previous argument (α) and from Theorem 3.1.

□

We remark that in the case of non bipartite graphs, none of the two of the implications of Corollary 3.5 (α) are true. For example in Figure 1, we present a non bipartite graph G_1 which is not chordless and whose toric ideal is generalized robust since it is principal with $I_{G_1} = \langle ac - bd \rangle$, see [38, Theorem 4.13]. Also, in Figure 1, we present a chordless non bipartite graph G_2 whose corresponding toric ideal is not generalized robust. Indeed, one can verify that it not satisfies the condition in Theorem 3.3. More precisely, the binomial $B_w = acf^2h - be^2ig$ is in the Graver basis of the ideal and is not minimal since the corresponding walk $w = (a, b, c, e, f, i, h, g, f, e)$ has a bridge d, see [38, Theorems 3.2 and 4.13].

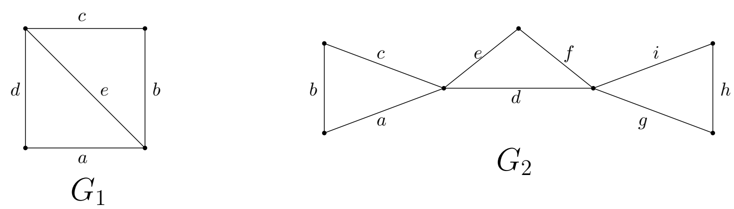

FIGURE 1. Corollary 3.5.(α) does not hold for non bipartite graphs.

By Corollary 3.5, the bipartite graphs yielding generalized robust toric ideals of graphs which are generated by quadrics are exactly the graphs whose cycles have all length 4. In Theorem 3.7 we describe precisely these graphs. Before proceeding with its statement and proof, we introduce some definitions and notation.

We denote by K_n the complete graph on n vertices and by $K_{r,s}$ the complete bipartite graph with partitions of sizes $r \in \mathbb{Z}^+$ and $s \in \mathbb{Z}^+$. In Figure 2 we see the complete graph on four vertices K_4 and the complete bipartite graph $K_{3,3}$.

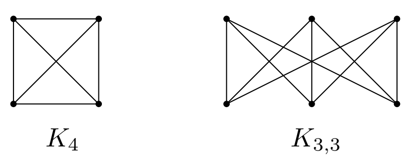

FIGURE 2. The complete graphs K_4 and $K_{3,3}$

A cut edge (respectively cut vertex) is an edge (respectively vertex) of the graph whose removal increases the number of connected components of the remaining subgraph. A graph is called biconnected if it is connected and does not contain a cut vertex. A block is a maximal biconnected subgraph of a graph. For a given graph G and a set $S \subseteq V(G)$, we denote by $[S]$ the corresponding induced subgraph of G, that is, the graph with vertex set S and whose edge set consists of the edges in $E(G)$ having both endpoints in S.

We have the following combinatorial lemma.

Lemma 3.6. *The only biconnected graphs whose cycles have all length* 4 *are cut edges and complete bipartite graphs* $K_{2,\ell}$ *with* $\ell \geq 2$.

Proof. If the graph has no cycles, then it is biconnected if and only if it is a cut edge. The graph $K_{2,\ell}$ has only cycles of length 4 for all $\ell \geq 2$. Consider now a non-acyclic biconnected graph B whose cycles all have length 4. We have that B is bipartite and we denote by U and V the bipartition of $V(B)$. Take $u \in U$ and $v \in V$, since B is biconnected, there are two disjoint paths from u to v. Moreover, B has only cycles of length four, so one of these paths has length 1 and, hence, $\{u, v\}$ is an edge of B. As a consequence B is a complete bipartite graph $B = K_{t,\ell}$ for some $2 \leq t \leq \ell$. If $t \geq 3$, then B has a 6-cycle, a contradiction. Thus, we conclude that $B = K_{2,\ell}$ for some $\ell \geq 2$. □

As a consequence of this lemma and Corollary 3.5 we get the following.

Theorem 3.7. *Let* G *be a bipartite graph. The ideal* I_G *is generalized robust and generated by quadrics if and only if all the blocks of* G *are* $K_{2,\ell}$ *or cut edges, for some* $\ell \geq 2$.

Proof. ($\Longrightarrow$) Let I_G be a generalized robust toric ideal which is generated by quadratic binomials and let B be one of the blocks of G. By Corollary 3.4 and Lemma 3.6 we have that B is a cut edge or a $K_{2,\ell}$, for some $\ell \geq 2$.

($\Longleftarrow$) The graph G is consists of blocks which are either $K_{2,\ell}$ or cut edges. Therefore all the cycles of G have length four. By Corollary 3.5 (β) it follows that the ideal is generalized robust and is generated by quadrics. □

In Figure 3 we present an example of a bipartite graph G whose corresponding toric ideal is generalized robust and is generated by quadrics.

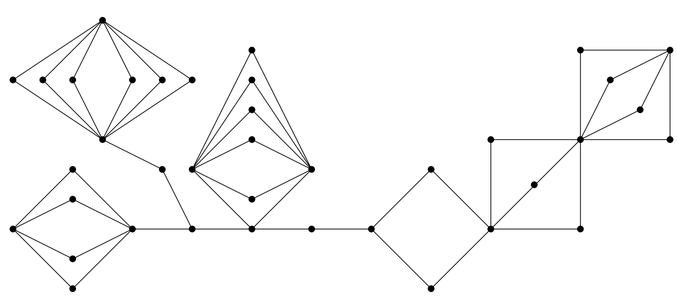

FIGURE 3. A bipartite graph G such that the ideal I_G is a quadratic generalized robust

3.3. **Quadratic generalized robust graphs; the general case.** We are moving on to the general case of non bipartite graphs. As we can see in the next lemma, when the toric ideal of a graph is generated by quadratic binomials, then it has at most one non bipartite block.

Lemma 3.8. *Let* G *be a simple connected graph such that the corresponding toric ideal* I_G *is generated by quadrics. Then the graph* G *has at most one non bipartite block.*

Proof. Let G be a connected graph with at least two non bipartite blocks and let them be B_1 and B_2. Let c_1 and c_2 be two chordless odd cycles of the blocks B_1 and B_2 correspondingly. We separate the proof in two cases: either the cycles have (exactly) one common vertex or they are vertex disjoint.

In the first case, by Theorem 3.1.(ii) there exists a bridge between the cycles c_1, c_2, but this contradicts the fact that the cycles belong to different blocks. In the second case, Theorem 3.1.(iii) guarantees that there exist $e_1 = (x_1, y_1)$ and $e_2 = (x_2, y_2)$ two bridges between c_1, c_2, where $x_1, x_2 \in V(c_1)$ and $y_1, y_2 \in V(c_2)$. There are two cases, either $x_1 \neq x_2$ and $y_1 \neq y_2$ or $x_1 = x_2$ (similarly if $y_1 = y_2$). The first case is not possible because c_1, c_2 are in different blocks. The second case yields two odd cycles in different blocks with one common vertex (the vertex $x_1 = x_2$), and we already discussed that this is not possible. $\square$

In order to give the structural characterization of a graph G such that I_G is generalized robust and generated by quadrics, we need to introduce the notions of the *double-$K_{2,(r,s)}$ graph* and the *necklace-$K_{2,\ell}$ graph.* We remind that a *2-clique sum* of the graphs G_1 and G_2 is obtained by identifying an edge e_1 of G_1 and an edge e_2 of G_2.

Definition 3.9. *We consider the non bipartite graph $G_1 = K_{2,r} \cup \{e\}$, where e is an edge connecting two vertices of $K_{2,r}$ and the bipartite graph $G_2 = K_{2,s}$, where $r, s \geq 2$. A graph G is called a double-$K_{2,(r,s)}$ if it is a 2-clique sum of the graphs G_1 and G_2 obtained by identifying the edge e of G_1 with any edge of G_2.*

For example, in Figure 4 we present the two non-isomorphic double-$K_{2,(3,4)}$ graphs.

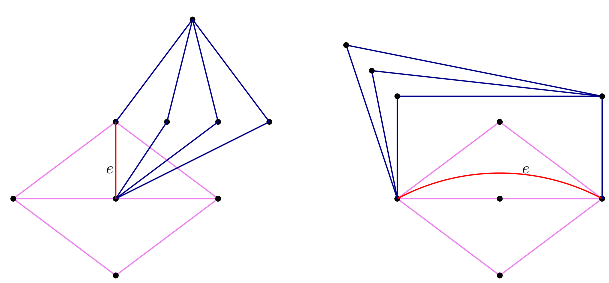

FIGURE 4. The two non-isomorphic double-$K_{2,(3,4)}$ graphs

Proposition 3.10. *If G is a double-$K_{2,(r,s)}$, then I_G is a generalized robust toric ideal generated by quadrics.*

Proof. Let G be a double-$K_{2,(r,s)}$ graph and we consider the graph G' which consists of two connected components, the $K_{2,r}$ and the $K_{2,s}$. We observe that $I_{G'} \subseteq I_G$ since every even closed walk in G' is in G. Since both connected components of G' are bipartite, we have that $b(G') = 2$ and, by (1):

$$
\begin{aligned}
\mathrm{ht}(I_{G'}) &= |E(G')| - |V(G')| + 2 = 2(r+s) - (r+s+4) + 2 \\
&= 2(r+s) - (r+s+2) = r+s-2 \\
&= |E(G)| - |V(G)| = \mathrm{ht}(I_G).
\end{aligned}
$$

So both I_G and $I_{G'}$ are prime ideals of the same height and $I_{G'} \subseteq I_G$, hence $I_G = I_{G'}$ and the proof follows from Theorem 3.7. $\square$

In [44] the authors define the necklace graph as a graph which comes from identifying two vertices at odd distance of a chain of bipartite blocks. Following the same structure, we define the necklace-$K_{2,\ell}$ graphs. Let T_G be the *block tree* of a graph G, that is, the bipartite graph with bi-partition $(\mathbb{B}, S)$ where $\mathbb{B}$ is the set of the blocks of G and S is the set of the cut vertices of G, such that (B, u) is an edge if and only if $u \in B$. A chain of bipartite blocks is a graph G such that its block tree T_G is a path.

Definition 3.11. *Let R be a bipartite graph consisting of a chain of (bipartite) blocks $B_1, \ldots, B_k$ where $k \geq 2$ and either $B_i = K_{2,n_i}$ for some $n_i \geq 2$, or B_i is a cut edge of R, for $i = 1, \ldots, k$. Let $x_1 \in V(B_1)$ and $x_2 \in V(B_k)$ be two non-adjacent vertices of R at odd distance which are not cut vertices of R. We define a necklace-$K_{2,\ell}$ graph as the graph G obtained after identifying the vertices x_1 and x_2. That is, the graph on the vertex set*

$$V(G) = (V(R) \setminus \{x_1, x_2\}) \cup \{x\}$$

and edges

$$E(G) = E(R \setminus \{x_1, x_2\}) \cup \{\{u, x\} \mid \text{ either } \{u, x_1\} \in E(R) \text{ or } \{u, x_2\} \in E(R)\}.$$

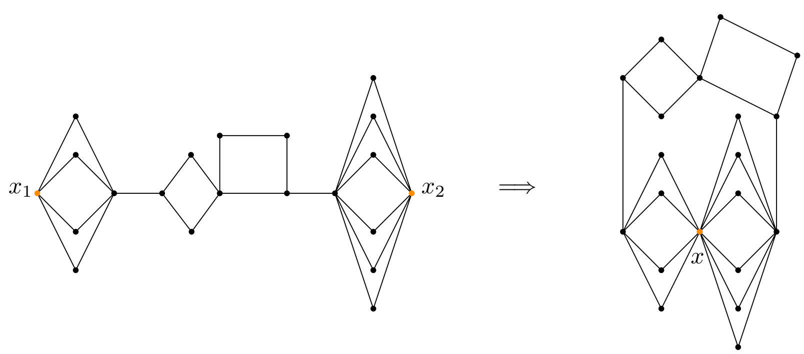

FIGURE 5. The construction of a necklace-$K_{2,\ell}$ graph

Proposition 3.12. *If G is a necklace-$K_{2,\ell}$, then I_G is a generalized robust toric ideal generated by quadrics.*

Proof. Take R as in the definition of necklace-$K_{2,\ell}$. Since every walk in R is a walk in G we have that $I_R \subseteq I_G$. Moreover, the graph R is bipartite and therefore $\mathrm{ht}(I_R) = |E(R)| - |V(R)| + 1$, while the graph G is not bipartite, that is $\mathrm{ht}(I_G) = |E(G)| - |V(G)|$. By (1) it follows that

$$\begin{aligned} \mathrm{ht}(I_G) &= |E(G)| - |V(G)| = |E(R)| - (|V(R)| - 1) \\ &= |E(R)| - |V(R)| + 1 = \mathrm{ht}(I_R). \end{aligned}$$

Hence $I_R = I_G$ and the proof follows from Theorem 3.7. $\square$

Next, we state the main result of this section in which we give a complete characterization of the graphs G such that I_G is a generalized robust toric ideal generated by quadratic binomials. By Theorem 2.5, this class coincides with the graphs G such that $\mathcal{U}_G$ consists of quadrics.

Theorem 3.13. *Let G be a non bipartite graph. The ideal is generalized robust and is generated by quadrics if and only if all the blocks of G are bipartite except one, which is either a K_4 or a $K_{2,\ell} \cup \{e\}$ or a double-$K_{2,(r,s)}$ or a necklace-$K_{2,\ell}$ graph. Every bipartite block of the graph G are of type $K_{2,\ell}$ for some ℓ or cut edges.*

In the poof of this result we repeatedly use the following lemma, which is a consequence of [38, Corollary 3.3].

Lemma 3.14. *Let G be a connected graph. If I_G is generalized robust and generated by quadrics, then:*

(a) *it has no even cycles of length ≥ 6,*
(b) *it has no two edge disjoint odd cycles.*

Proof. (a) Let w be an even cycle of G. By [38, Corollary 3.3] we have that $B_w \in Gr_G$. Since I_G is generalized robust, by Theorem 3.3, we have that $B_w \in \mathcal{M}_G$. But since I_G is generated by quadrics, then B_w is quadric and we conclude that w is a 4-cycle.

(b) Assume that there are two edge disjoint odd cycles. By [38, Corollary 3.3], there exists an even closed walk w (which consists of the above two odd cycles and a walk connecting them) such that $B_w \in Gr_G$. Since the ideal is generalized robust, by Theorem 3.3 we have that B_w is also minimal and has degree ≥ 3, a contradiction due to the fact that the ideal is generated by quadrics. $\square$

Proof of Theorem 3.13. ($\Longrightarrow$) Assume that I_G is generalized robust and generated by quadrics. By Lemma 3.8, the graph G has exactly one non bipartite block which we denote by B. By Corollary 3.4 and Theorem 3.7 the rest of the blocks are of the form $K_{2,\ell}$ or cut edges. We separate two cases: either (i) there exists an edge $e \in E(B)$ such that the graph $B \setminus \{e\}$ is bipartite, or (ii) for every edge $e \in E(B)$ the graph $B \setminus \{e\}$ is not bipartite; where $B \setminus \{e\}$ denotes the graph with vertex set $V(B)$ and edge set $E(B) - \{e\}$.

(i) Let $e \in E(B)$ such that $B \setminus \{e\}$ is bipartite. Combining Theorem 3.7 and Corollary 3.4, it follows that the blocks of $B \setminus \{e\}$ are $K_{2,\ell}$ or cut edges. If $B \setminus \{e\}$ is still biconnected, then B is a $K_{2,\ell} \cup \{e\}$. If $B \setminus \{e\}$ is not biconnected, then it is a chain of blocks, where each block is either a $K_{2,\ell}$ or a cut-edge. We get thus that B is a necklace-$K_{2,\ell}$ graph.

(ii) Assume that for every edge $e \in E(B)$ the graph $B \setminus \{e\}$ is not bipartite.

Claim 1: Every edge of B belongs to a cycle of length four of B.

Suppose not and let ϵ be an edge of B that does not belong to a cycle of length four of B. Since I_B is generated by quadrics and every 4-cycle of B is in $B - \{\epsilon\}$, by Corollary 3.2, it follows that $I_B \subseteq I_{B\setminus\{\epsilon\}}$. Obviously we have that $I_{B\setminus\{\epsilon\}} \subseteq I_B$ and therefore $I_{B\setminus\{\epsilon\}} = I_B$. But the graph $B \setminus \{\epsilon\}$ is not bipartite and connected, then

$$\begin{aligned} \mathrm{ht}(I_{B\setminus\{\epsilon\}}) &= |E(B \setminus \{\epsilon\})| - |V(B \setminus \{\epsilon\})| \\ &= |E(B)| - 1 - |V(B)| = ht(I_B) - 1, \end{aligned}$$

a contradiction.

Let $G_1, \dots, G_k$ be the maximal subgraphs (with respect to the inclusion) of type $K_{2,\ell}$ of B. By (Claim 1) we have that every edge of the block B belongs to a $K_{2,\ell}$, thus we have that $E(G_1) \cup \dots \cup E(G_k) = E(B)$.

Claim 2: $E(G_i) \cap E(G_j) = \emptyset$ for all $1 \leq i < j \leq k$.

Otherwise, suppose that there exist distinct $i, j \in \{1, \dots, k\}$ such that G_i and G_j have at least one edge in common. Let $G_i = K_{2,r}$ and $G_j = K_{2,s}$ with $r, s \geq 2$, and denote by $\{v_1, v_2\}, \{w_1, w_2, \dots, w_r\}$ and $\{u_1, u_2\}, \{x_1, x_2, \dots, x_s\}$ their corresponding bipartitions. Let e be the common edge of G_i, G_j and without loss of generality we suppose that $e = (v_1, w_1) = (u_1, x_1)$ (otherwise we rename the vertices). We note that the vertex $u_2 \notin V(G_i)$. Indeed, $u_2 = v_2$ contradicts the maximality of G_j, and $u_2 = w_2$ implies that the odd cycle (v_1, w_1, u_2) belongs to the bipartite graph G_i, a contradiction. Similarly, we note that the vertex $x_2 \notin V(G_i)$. We conclude that we have a length 6 cycle $(v_1 = u_1, x_2, u_2, w_1 = x_1, v_2, w_2, v_1)$ in B, a contradiction to Lemma 3.14, and (Claim 2) is proved.

We denote by $[G_1], \dots, [G_k]$ the induced subgraphs with vertices $V(G_1), \dots, V(G_k)$ correspondingly. We split the proof in two cases: either (ii_a) the graphs $[G_1], \dots, [G_k]$ are bipartite or (ii_b) there exists $i \in \{1, \dots, k\}$ such that the graph $[G_i]$ is non bipartite.

(ii_a) First of all, we remark that $|V(G_i) \cap V(G_j)| \leq 1$ for all $1 \leq i < j \leq k$.

Otherwise, suppose that there exist distinct $i, j \in \{1, \dots, k\}$ such that G_i and G_j have at least two vertices in common. Let $G_i = K_{2,r}$ and $G_j = K_{2,s}$ with $r, s \geq 2$, and denote by $\{v_1, v_2\}, \{w_1, w_2, \dots, w_r\}$ and $\{u_1, u_2\}, \{x_1, x_2, \dots, x_s\}$ their corresponding bipartitions. Since both G_i and G_j are complete bipartite graphs and they do not share edges by (Claim 2), then the two common vertices are not adjacent. By the maximality of G_i and G_j, the common vertices have to be $x_i = w_{i'}$ and $x_j = w_{j'}$, for some $i, j \in \{1, \dots, r\}$ and $i', j' \in \{1, \dots, s\}$ and $r, s > 2$. Then a cycle of length 6 arises; namely the cycle $(x_i = w_{i'}, u_1, x_j = w_{j'}, v_1, x_k, v_2, x_i = w_{i'})$ with $k \in \{1, \dots, s\} \setminus \{i, j\}$, a contradiction to Lemma 3.6.

Consider now G_1, G_2 two maximal subgraphs of type $K_{2,\ell}$ with one vertex in common (there are such subgraphs since B is biconnected) and let $\{x\}$ be the common vertex of G_1, G_2 (the vertex which we discussed above). Take G' the graph with vertex set

$$V(G') = V(B \setminus \{x\}) \cup \{x_1, x_2\}$$

and edge set

$$E(G') = E(B \setminus \{x\}) \cup \{\{x_1, u\} : \{u, x\} \in E(G_1)\} \cup \{\{x_2, v\} : \{v, x\} \in E(B \setminus G_1)\}.$$

By (Claim 1) and (Claim 2) we know that $E(B) = E(G_1) \sqcup \dots \sqcup E(G_k)$, then $|E(G')| = |E(B)|$. As a consequence, we have that

$$\text{ht}(I_{G'}) = \begin{cases} |E(B)| - (|V(B)| + 1) + 1, & \text{if } G' \text{ is bipartite} \\ |E(B)| - (|V(B)| + 1), & \text{if } G' \text{ is not bipartite} \end{cases} \tag{4}$$

while

$$\text{ht}(I_B) = |E(B)| - |V(B)| \tag{5}$$

Combining (4) and (5) we have that

$$\text{ht}(I_B) - \text{ht}(I_{G'}) = \begin{cases} 0, & \text{if } G' \text{ is bipartite} \\ 1, & \text{if } G' \text{ is not bipartite} \end{cases} \tag{6}$$

Since every walk in G' corresponds to a walk in B, we have that $I_{G'} \subseteq I_B$. Let us prove the converse statement. We know that I_B is generated by quadrics, i.e., binomials which correspond to cycles of length four (see Corollary 3.2) . Let $c = (v_1, v_2, v_3, v_4)$ be a cycle of B, we are going to build a cycle c' in G' such that $B_c = B_{c'}$. If c does not pass through x, then we take $c' = c$. In case that c passes through x, then it has the form (x, v_1, v_2, v_3, x). Since the vertices of c form a $K_{2,2}$, then there exists an $i \in \{1, \ldots, k\}$ such that $\{x, v_1, v_2, v_3, x\} \subseteq V(G_i)$. If $i = 1$ we choose $c' := (x_1, v_1, v_2, v_3, x_1)$ and, if $i \geq 2$ we choose $c' := (x_2, v_1, v_2, v_3, x_2)$. It follows then that $I_{G'} \subseteq I_B$ and, therefore,

$$\mathrm{ht}(I_B) = \mathrm{ht}(I_{G'}) \tag{7}$$

Combining (6) and (7) we conclude that G' is bipartite. Since the ideal I_G is generalized robust and generated by quadrics, by Corollary 3.4 it follows that $I_B = I_{G'}$ so is. Since G' is bipartite, by Theorem 3.7 all the blocks of G' are of type $K_{2,\ell}$ or cut edges. By construction of the graph G' we have that the block B is a necklace graph of bipartite blocks each of them is of type $K_{2,\ell}$. Note also that every path from x_1 to x_2 has odd length because B is not bipartite and G is.

(ii_b) We assume that there exists $i \in \{1, \ldots, k\}$ such that the graph $[G_i]$ is not bipartite. Then $[G_i]$ is a $K_{2,\ell}$ graph plus at least one more edge. We observe that we can only have one more edge or $[G_i] = K_4$, otherwise we would be in one of the three cases shown in Figure 6. In the three cases there are two edge disjoint triangles, which contradicts Lemma 3.14.

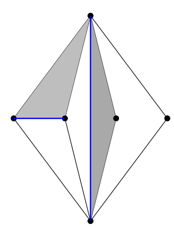

FIGURE 6. The three non-isomorphic graphs $K_{2,4}$ plus two edges

As a consequence, either i) $[G_i] = K_4$, or ii) $[G_i] = K_{2,\ell} \cup \{e\}$. In ii), by (Claim 1), there is another maximal G_j such that $e \in E(G_j)$. We observe that $[G_j]$ is bipartite. Otherwise, there exists a subgraph as the one shown in Figure 7, again a contradiction to Lemma 3.14.(b).

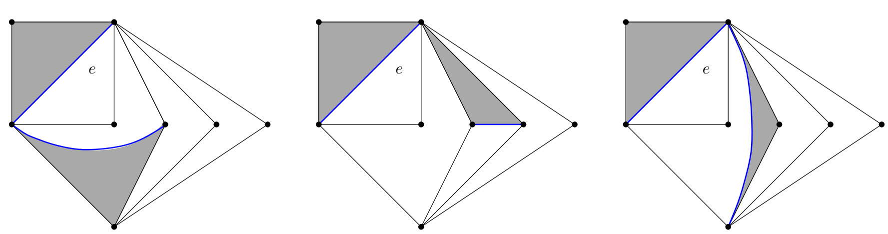

FIGURE 7. The two $[G_i]$'s with a common edge are non bipartite

Thus, in i) we have that $[G_i] = K_4$ and in ii) we have that the induced subgraph with vertex set $V(G_i) \cup V(G_j)$ is a double-$K_{2,(r,s)}$ graph. We just have to prove that in both cases the graph has no more vertices. We denote $H = K_4$ in i) and H double-$K_{2,(r,s)}$ in ii). We observe that every two vertices in H can be joined by two (non necessarily disjoint) paths: an odd path of length ≥ 3 and an even path. Assume that there is another vertex v at distance one of $u \in V(H)$. Since the graph is biconnected there exists a path from v to a vertex of $V(H) - \{u\}$ and we take the shortest one. Hence, we have a path from u to another vertex of $u' \in V(H)$ that only has its two endpoints in $V(H)$. If this path is odd, there is an even cycle of length ≥ 6, a contradiction to Lemma 3.14. If this path is even, then there is even cycle of length ≥ 6 or one of the G_i's involved is not a maximal $K_{2,\ell}$ subgraph, a contradiction arises for both cases.

($\Longleftarrow$) We have exactly one non-bipartite block. As a consequence, every element in the Graver basis (and, hence, every minimal generator and every element in the universal Gröbner basis) corresponds to a walk entirely contained in a block. Thus, by Theorem 3.7 it suffices to prove that K_4, $K_{2,\ell} \cup \{e\}$, a necklace-$K_{2,\ell}$ and a double-$K_{2(r,s)}$ graph give rise to generalized robust toric ideals generated by quadrics. Clearly I_{K_4} is generalized robust and generated by quadrics (see Example 3.16) and, since $I_{K_{2,\ell} \cup \{e\}} = I_{K_{2,\ell}}$ (they are both prime ideals, have the same height and one is contained in the other, so they are equal), so is $I_{K_{2,\ell} \cup \{e\}}$. The remaining two cases follow from Propositions 3.10 and 3.12, and the proof is complete. $\square$

In Figure 8 we see an example of a graph G whose corresponding toric ideal I_G is a generalized robust ideal and is generated by quadrics. The graph G consists of five blocks; two cut edges, a $K_{2,4}$, a $K_{2,6}$ and exactly one non bipartite block which is a K_4. The existence of the K_4 as a subgraph, as we show in the next section, implies that the ideal I_G is not robust.

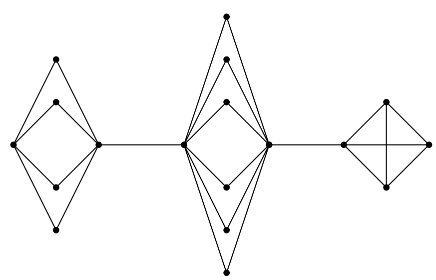

FIGURE 8. A graph G such that I_G is quadratic generalized robust and not quadratic robust

3.4. **Quadratic robust graphs.** The goal of this section is to prove Theorem 3.17, where we characterize the graphs providing robust toric ideals generated by quadrics. This result completes Corollary 5.2 of Boocher et al. in [12]. The main ingredients for proving the present theorem are Theorem 3.13 and the following property:

Theorem 3.15. [41, Theorem 5.10] *Let I_A be a robust toric ideal. Then I_A has a unique minimal system of generators.*

From the above result it follows that a toric ideal I_A with a unique minimal system of generators (i.e., $M_A = \mathcal{M}_A$) is robust if and only if it is a generalized robust

([41, Corollary 5.13]). From [38] we know that I_G is generated by indispensable binomials (i.e., it has a unique minimal system of generators) if and only if no closed walk w such that B_w is a minimal generator of I_G has an F_4 (being the concept of F_4 rather technical, we refer to [38] for its definition). In the particular case that I_G is generated by quadrics, the existence of an F_4 in a closed walk w providing a minimal generator B_w is equivalent to the existence of a subgraph K_4. Hence, the only obstruction for a toric ideal generated by quadrics to have a unique minimal set of generators is the existence of a subgraph K_4 (or, equivalently, having clique number ≥ 4). We work out the example of $G = K_4$ in detail to show that I_{K_4} is generalized robust but not robust.

Example 3.16. Consider the complete graph on four vertices K_4 on the vertex set $V(K_4) = \{v_1, v_2, v_3, v_4\}$ and on the edge set $E(K_4) = \{e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, e_3 = \{v_3, v_4\}, e_4 = \{v_4, v_1\}, f_1 = \{v_1, v_3\}, f_2 = \{v_2, v_4\}\}$ (see Figure 9). It is clear that we have exactly three 4-cycles $w_1 = (v_1, v_2, v_3, v_4, v_1)$, $w_2 = (v_1, v_2, v_4, v_3, v_1)$ and $w_3 = (v_1, v_4, v_2, v_3, v_1)$.

The corresponding ideal I_{K_4} is generated by the three binomials:

$$I_{K_4} = \langle B_{w_1} = e_1e_3 - e_2e_4, B_{w_2} = e_1e_3 - f_1f_2, B_{w_3} = e_2e_4 - f_1f_2\rangle.$$

Obviously, none of $B_{w_1}, B_{w_2}, B_{w_3}$ is indispensable since

$$B_{w_i} \in \langle B_{w_j}, B_{w_k}\rangle, \text{ for all distinct } i, j, k \text{ where } i, j, k \in \{1, 2, 3\}.$$

Thus, the ideal has three different Markov bases;

$$M_1 = \{B_{w_1}, B_{w_2}\}, M_2 = \{B_{w_1}, B_{w_3}\}, M_3 = \{B_{w_2}, B_{w_3}\}.$$

The universal Markov basis of the ideal is $\mathcal{M}_{K_4} = \{B_{w_1}, B_{w_2}, B_{w_3}\}$. It is easy to check that the universal Gröbner basis of the ideal I_{K_4} is $\mathcal{U}_{K_4} = \{B_{w_1}, B_{w_2}, B_{w_3}\}$. It follows that the quadratic ideal I_{K_4} is generalized robust but not robust.

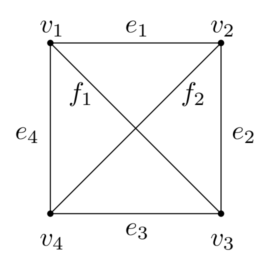

FIGURE 9. The graph K_4 in Example 3.16.

Putting all together we have that if I_G is generated by quadrics, then I_G is robust if and only if I_G is generalized robust and does not have K_4 as subgraph. Thus, from Theorem 3.17 we deduce the following structural result.

Theorem 3.17. *Let G be a non bipartite graph. The ideal I_G is robust and is generated by quadrics if and only if all the blocks of G are bipartite except one which is either a $K_{2,\ell} \cup \{e\}$ or a double-$K_{2,(r,s)}$ or a necklace-$K_{2,\ell}$ graph. The bipartite blocks of the graph G are of type $K_{2,\ell}$ or cut edges.*

Remark 3.18. In [40] Sullivant introduces and studies the notion of strongly robust toric ideals. The motivation for studying strongly robust toric ideals comes from algebraic statistics. A toric ideal is strongly robust if its Graver basis coincides with its set of indispensable binomials, for more see [40]. From [41] one has that the notion of strongly robust and robust coincide for toric ideals of graphs. It follows that Theorem 3.17, also characterizes completely the strongly robust quadratic toric ideals of graphs.

Example 3.19. In Figure 10 we consider a graph G which consists of six blocks; two cut edges, a $K_{2,2}$, a $K_{2,4}$, a $K_{2,6}$ and one necklace-$K_{2,\ell}$ graph. By Theorem 3.13 and Theorem 3.17 the corresponding ideal I_G is both a robust and a generalized robust toric ideal and is generated by quadrics. The graph in Figure 8 contains a K_4 as a subgraph. The corresponding ideal is a quadratic generalized robust toric ideal, however it is not robust because of the existence of a K_4.

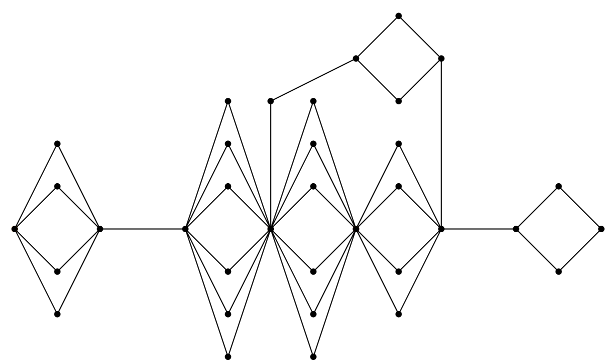

FIGURE 10. A graph G such that I_G is both quadratic robust and quadratic generalized robust

4. Generalized robust ideals and numerical semigroups

4.1. **Numerical semigroups having a complete intersection initial ideal.** A numerical semigroup is a submonoid $\mathcal{S}$ of $(\mathbb{N}, +)$ with finite complement. Every numerical semigroup has a unique minimal generating set $A = \{a_1, \ldots, a_m\} \subseteq \mathbb{Z}^+$ of relatively prime integers, which is always finite. The number m of elements of A is usually called the *embedding dimension* of $\mathcal{S}$. The only numerical semigroup with embedding dimension 1 is $\mathcal{S} = \mathbb{N}$, here $A = \{1\}$ and $I_\mathcal{S}$ is the zero ideal. From now on we assume that $\mathcal{S} \subsetneq \mathbb{N}$ and, as a consequence, its embedding dimension is at least two. When we write $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ we implicitly mean the numerical semigroup $\mathcal{S} = \{\sum_{i=1}^m \alpha_i a_i \,|\, \alpha_i \in \mathbb{N}\}$ with minimal set of generators $A = \{a_1, \ldots, a_m\}$, and we call $I_\mathcal{S} := I_A$ the toric ideal of the corresponding semigroup.

By Krull's dimension theorem, any set of generators of an ideal $J \subseteq \mathbb{K}[\mathbf{x}]$ has at least $\mathrm{ht}(J)$ elements and is a complete intersection when equality occurs. Whenever $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ is a numerical semigroup, then $I_\mathcal{S}$ has height $m - 1$ and $I_\mathcal{S}$ is a complete intersection if and only if $\mu(I_\mathcal{S}) = m - 1$ or, in other words, if it can be generated by a set of $m - 1$ binomials.

Given a monomial ordering $\prec$, we have that $\mathrm{ht}(\mathrm{in}_\prec(I_\mathcal{S})) = \mathrm{ht}(I_\mathcal{S})$ and that $\mu(\mathrm{in}_\prec(I_\mathcal{S})) \geq \mu(I_\mathcal{S})$. Hence, whenever there exists a monomial ordering such that

$\mathrm{in}_\prec(I_\mathcal{S})$ is a complete intersection, then so is $I_\mathcal{S}$. In Theorem 4.7 we characterize all numerical semigroups having a complete intersection initial ideal.

A numerical semigroup $\mathcal{S}$ with minimal generating set $A = \{a_1, \ldots, a_m\}$ is said to be *free for the arrangement* $a_1, \ldots, a_m$ if

$$\mathrm{lcm}(a_i, \gcd(a_{i+1}, \ldots, a_m)) \in \langle a_{i+1}, \ldots, a_m \rangle, \text{ for all } i \in \{1, \ldots, m-1\}. \tag{8}$$

Equivalently, if $\mu_i := \mathrm{lcm}(a_i, \gcd(a_{i+1}, \ldots, a_m))$, the numerical semigroup $\mathcal{S}$ is free for the arrangement $a_1, \ldots, a_m$ if there exist $\alpha_{(i,i+1)}, \ldots, \alpha_{(i,m)} \in \mathbb{N}$ such that $\mu_i = \alpha_{(i,i+1)} a_{i+1} + \cdots + \alpha_{(i,m)} a_m$ for all $i \in \{1, \ldots, m-1\}$. We say that $\mathcal{S}$ is *free* if it is free for an arrangement of its minimal generating set.

Example 4.1. Consider the numerical semigroup $\mathcal{S} = \langle a_1, a_2, a_3, a_4 \rangle$ with $a_1 = 8$, $a_2 = 9$, $a_3 = 10$, $a_4 = 12$. We have that $\mathcal{S}$ is not free for the arrangement a_1, a_2, a_3, a_4 because $\mathrm{lcm}(a_1, \gcd(a_2, a_3, a_4)) = 8 \notin \langle a_2, a_3, a_4 \rangle$. However, $\mathcal{S}$ is free for the arrangement $a_2 = 9$, $a_3 = 10$, $a_1 = 8$, $a_4 = 12$. Indeed,

- $\mathrm{lcm}(a_2, \gcd(a_1, a_3, a_4)) = 18 = a_1 + a_3 \in \langle a_1, a_3, a_4 \rangle$,
- $\mathrm{lcm}(a_3, \gcd(a_1, a_4)) = 20 = a_1 + a_4 \in \langle a_1, a_4 \rangle$, and
- $\mathrm{lcm}(a_1, a_4) = 24 = 2a_4 \in \langle a_4 \rangle$.

Thus, $\mathcal{S}$ is a free numerical semigroup.

Equivalently, this notion can be inductively defined as follows: a numerical semigroup $\mathcal{S}$ is free if either $\mathcal{S} = \langle 1 \rangle = \mathbb{N}$ or there exists an arrangement $A = \{a_1, \ldots, a_m\}$ of its minimal generators such that $da_1 \in \langle a_2, \ldots, a_m \rangle$ and the numerical semigroup $\mathcal{S}' = \langle a_2/d, \ldots, a_m/d \rangle$ is free, where $d = \gcd(a_2, \ldots, a_m)$. The following result can be found in several places in the literature, see, e.g., [7, Lemma 2.1 and Proposition 2.3] or [32, Lemma 3.2].

Proposition 4.2. *Let $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ be a numerical semigroup and set $d := \gcd(a_2, \ldots, a_m)$. If $da_1 = \alpha_2 a_2 + \cdots + \alpha_m a_m$ with $\alpha_2, \ldots, \alpha_m \in \mathbb{N}$, then*

$$I_\mathcal{S} = I_{\mathcal{S}'} \cdot \mathbb{K}[x_1, \ldots, x_m] + \langle x_1^d - x_2^{\alpha_2} \cdots x_m^{\alpha_m} \rangle,$$

where $\mathcal{S}' = \langle a_2/d, \ldots, a_m/d \rangle$ and $I_{\mathcal{S}'} \subseteq \mathbb{K}[x_2, \ldots, x_m]$.

Applying inductively Proposition 4.2 we get the following result, which explains how to use condition (8) to construct a minimal set of generators of $I_\mathcal{S}$ when $\mathcal{S}$ is free.

Proposition 4.3. *Let $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ be a free semigroup for the arrangement $a_1, \ldots, a_m$. Consider $\beta_i := \mathrm{lcm}(a_i, \gcd(a_{i+1}, \ldots, a_m))/a_i$ and $\alpha_{(i,j)} \in \mathbb{N}$ so that*

$$\mathrm{lcm}(a_i, \gcd(a_{i+1}, \ldots, a_m)) = \sum_{j=i+1}^{m} \alpha_{(i,j)} a_j \text{ for all } i \in \{1, \ldots, m-1\}.$$

Then, $I_\mathcal{S} = \langle f_1, \ldots, f_{m-1} \rangle$, where $f_i = x_i^{\beta_i} - \prod_{j=i+1}^{m} x_j^{\alpha_{(i,j)}}$ for all $i \in \{1, \ldots, m-1\}$.

As a direct consequence of Proposition 4.3, every free semigroup has a set of generators consisting of $m-1$ binomials and, thus, it is a complete intersection. When $m = 2$ every numerical semigroup is free. For $m = 3$, Herzog proved that $\mathcal{S}$ is free if and only if $I_\mathcal{S}$ is a complete interesection, see [27]. For $m \geq 4$, there are complete intersection semigroups which are not free. For example $\mathcal{S} = \langle 10, 14, 15, 21 \rangle$ is not free for any arrangement of the generators and $I_\mathcal{S}$ is a complete intersection;

indeed, one can check that $\{x_1^3 - x_3^2, x_2^3 - x_4^2, x_1^2x_3 - x_2x_4\}$ is a Markov basis for $I_{\mathcal{S}}$ (and it is also the universal Markov basis of $I_{\mathcal{S}}$).

In this section we will use the following general fact about toric ideals, which we write here only for numerical semigroups (see, e.g., [39] or [47]). Let $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ be a numerical semigroup and consider the morphism of groups

$$\rho : \mathbb{Z}^m \longrightarrow \mathbb{Z}, \text{ induced by } \rho(\mathbf{e}_i) = a_i, \ \forall i \in \{1, \ldots, m\},$$

being $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ the canonical basis of $\mathbb{Z}^m$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^m$ and consider the binomial $f = \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$, we set $\widetilde{f} := \mathbf{u} - \mathbf{v} \in \mathbb{Z}^m$. We have that $f \in I_{\mathcal{S}}$ if and only if $\widetilde{f} \in \ker(\rho)$. Moreover, we have the following:

Proposition 4.4. *Let $\{f_1, \ldots, f_r\}$ be a set of binomials. If $I_{\mathcal{S}} = \langle f_1, \ldots, f_r \rangle$, then $\ker(\rho) = \langle \widetilde{f_1}, \ldots, \widetilde{f_r} \rangle$.*

From the following two propositions easily follows the proof of Theorem 4.7, which is the main result of this subsection.

Proposition 4.5. *Let $\mathcal{S}$ be a numerical semigroup with minimal generating set $A = \{a_1, \ldots, a_m\}$. Then, $\mathcal{S}$ is free for the arrangement $a_1, \ldots, a_m$ if and only if the reduced Gröbner basis with respect to the lexicographic order with $x_1 \succ \cdots \succ x_m$ has $m - 1$ elements.*

Proof. ($\Longrightarrow$) Suppose that $\mathcal{S} = \langle a_1, \ldots, a_m \rangle$ is free for the ordering $a_1, \ldots, a_m$. By Proposition 4.3 we have that

$$I_{\mathcal{S}} = \langle f_1, \ldots, f_{m-1} \rangle, \text{ where } f_i = x_i^{\beta_i} - \prod_{j=i+1}^{m} x_j^{\alpha_{(i,j)}}, \text{ for some } \beta_i, \alpha_{(i,j)} \in \mathbb{N}.$$

Considering $\prec$ the lexicographic order with $x_1 \succ \cdots \succ x_m$ we observe that $\mathrm{in}_{\prec}(f_i) = x_i^{\beta_i}$ for all $i \in \{1, \ldots, m-1\}$. Since the initial forms are pairwise prime, then $\mathcal{G} = \{f_1, \ldots, f_{m-1}\}$ is a Gröbner basis of $I_{\mathcal{S}}$ for $\prec$ and, hence, the reduced Gröbner basis with respect to $\prec$ has $m-1$ elements.

($\Longleftarrow$) Consider $\prec$ the lexicographic order with $x_1 \succ \cdots \succ x_m$ and let $\mathcal{G}$ be the corresponding reduced Gröbner basis of $I_{\mathcal{S}}$. For all $i \in \{1, \ldots, m-1\}$ we have that $x_i^{a_{i+1}} - x_{i+1}^{a_i} \in I_{\mathcal{S}}$ and, hence, $x_i^{a_{i+1}} \in \mathrm{in}_{\prec}(I_{\mathcal{S}})$. As a consequence, $\mathrm{in}_{\prec}(I_{\mathcal{S}}) = \langle x_1^{b_1}, \ldots, x_{m-1}^{b_{m-1}} \rangle$, with $b_1, \ldots, b_{m-1} \in \mathbb{Z}^+$ and $\mathcal{G} = \{g_1, \ldots, g_{m-1}\}$ with $g_i = x_i^{b_i} - M_i$, being M_i a monomial not involving the variables $x_1, \ldots, x_i$.

Our next goal is to prove that $b_1 = \gcd(a_2, \ldots, a_m)$ from which we conclude that $\mathrm{lcm}(a_1, \gcd(a_2, \ldots, a_m)) \in \langle a_2, \ldots, a_m \rangle$.

Set $B := \gcd(a_2, \ldots, a_m)$, we observe that

$$\mathbb{Z}Ba_1 = \mathbb{Z}a_1 \cap (\sum_{j=2}^{m} \mathbb{Z}a_j)$$

and, in particular,

$$Ba_1 = \sum_{i=2}^{m} \gamma_j a_j \text{ for some } \gamma_j \in \mathbb{Z}, \tag{9}$$

Since $g_1 \in I_{\mathcal{S}}$, it follows that

$$b_1 a_1 = \deg_A(M_1) \in \mathbb{Z}a_1 \cap (\sum_{j=2}^{m} \mathbb{Z}a_j)$$

and, then, B divides b_1. Moreover, following the notation of Proposition 4.4, we have that

$$\ker(\rho) = \langle \widetilde{g}_1, \ldots, \widetilde{g}_{m-1} \rangle \text{ where } \widetilde{g}_i = b_i e_i - \sum_{j>i} c_{(i,j)} e_j \in \mathbb{Z}^m$$

for some $c_{(i,j)} \in \mathbb{N}$ and, from (9), we deduce that

$$Be_1 - \sum_{j=2}^{m} \gamma_j e_j \in \ker(\rho).$$

Since $\widetilde{g}_1$ is the only element among $\widetilde{g}_1, \ldots, \widetilde{g}_{m-1}$ with a nonzero first entry, we conclude that b_1 divides B.

We have thus already proved that

$$b_1 = \gcd(a_2, \ldots, a_m) \text{ and } g_1 = x_1^{b_1} - M_1 \in I_{\mathcal{S}}.$$

Then,

$$a_1 \gcd(a_2, \ldots, a_m) = \deg_A(M_1) \in \langle a_2, \ldots, a_m \rangle.$$

Since $\gcd(a_1, \gcd(a_2, \ldots, a_m) = 1$ it follows that

$$a_1 \gcd(a_2, \ldots, a_m) = \operatorname{lcm}(a_1, \gcd(a_2, \ldots, a_m)) \in \langle a_2, \ldots, a_m \rangle.$$

Finally, it suffices to observe that

$$\mathcal{G} \cap \mathbb{K}[x_2, \ldots, x_m] = \{g_2, \ldots, g_{m-1}\}$$

is the reduced Gröbner basis of $I_{\mathcal{S}} \cap \mathbb{K}[x_2, \ldots, x_m] = I_{\mathcal{S}'}$ with respect to the lexicographic order with $x_2 \succ \cdots \succ x_m$ and proceed inductively to get the result. □

Proposition 4.6. *Let $\mathcal{S}$ be a numerical semigroup with minimal generating set $A = \{a_1, \ldots, a_m\}$. If $\mathcal{G} = \{g_1, \ldots, g_{m-1}\}$ is a Gröbner basis of $I_{\mathcal{S}}$ with respect to a monomial ordering $\prec$, then $\mathcal{G}$ is also a Gröbner basis with respect to a certain lexicographic monomial ordering $\prec_{lex}$.*

Proof. Since $\mathcal{G}$ has $m-1$ elements, then $\mu(\mathrm{in}_\prec(I_{\mathcal{S}})) = m-1$. For all $i, j \in \{1, \ldots, m\}$ with $i \neq j$, we have that $x_i^{a_j} - x_j^{a_i} \in I_{\mathcal{S}}$ and, hence, either $x_i^{a_j}$ or $x_j^{a_i}$ belongs to the initial ideal $\mathrm{in}_\prec(I_{\mathcal{S}})$. As a consequence, we may assume (after reindexing the variables if necessary) that $\mathrm{in}_\prec(I_{\mathcal{S}}) = \langle x_1^{b_1}, \ldots, x_{m-1}^{b_{m-1}} \rangle$ for some $b_1, \ldots, b_{m-1} \in \mathbb{Z}^+$ and that $g_i = x_i^{b_i} - M_i$, where M_i is a monomial not involving the variable x_i, in which $i = 1, \ldots, m-1$.

Claim: there exists an $\ell \in \{1, \ldots, m-1\}$ such that x_ℓ does not divide M_i for all $i \in \{1, \ldots, m-1\}$.

Proof of the claim: Assume by contradiction that the claim does not hold, i.e., for all $\ell \in \{1, \ldots, m-1\}$ there exists $i \in \{1, \ldots, m-1\}$ such that x_ℓ divides M_i. We consider the simple directed graph with vertex set $\{1, \ldots, m-1\}$ and arc set $\{(j, i) \,|\, 1 \leq i, j \leq m-1 \text{ and } x_j \text{ divides } M_i\}$. Then, the out-degree of every vertex is greater or equal to one and, thus, there is a directed cycle in the graph. Assume, without loss of generality, that the cycle is $(1, 2, \ldots, r, 1)$ with $r \leq m-1$. This implies that $g_i \in \langle x_{i-1}, x_i \rangle$, $\forall i = 1, \ldots, r-1$ and $x_0 = x_r$, thus:

$$\langle g_1, \ldots, g_r \rangle \subsetneq \langle x_1, \ldots, x_r \rangle,$$

and so

$$I_{\mathcal{S}} \subsetneq H := \langle x_1, \ldots, x_r, g_{r+1}, \ldots, g_{m-1} \rangle,$$

but this is not possible because $I_{\mathcal{S}}$ is prime and

$$m-1=\mathrm{ht}(I_{\mathcal{S}})<\mathrm{ht}(H)\le m-1.$$

Hence, x_ℓ only appears in g_ℓ. Assume without loss of generality that $\ell=1$. Proceeding as before, one can prove that there exists an $\ell'\in\{2,\dots,m\}$ such that x'_ℓ does not divide M_i for all $i\in\{2,\dots,m-1\}$. Iterating this idea one gets that, after reindexing the variables if necessary, the variables $x_1,\dots,x_i$ do not divide M_i. Hence, taking $\prec_{lex}$ the lexicographic order with $x_1\succ_{lex}\cdots\succ_{lex}x_m$ one has that $\mathrm{in}_{\prec_{lex}}(g_i)=x_i^{b_i}$. Since they are all relatively prime, then $\mathcal{G}$ is also a Gröbner basis for $\prec_{lex}$. $\square$

Now, we can prove the main result of this subsection.

Theorem 4.7. *Let $\mathcal{S}$ be a numerical semigroup. Then, $\mathcal{S}$ is free if and only if it has a Gröbner basis with $m-1$ elements.*

Proof. ($\Longrightarrow$) Follows directly from Proposition 4.5.

($\Longleftarrow$) Assume that $\mathcal{G}=\{g_1,\dots,g_{m-1}\}$ is a Gröbner basis of $I_{\mathcal{S}}$ with respect to a monomial ordering $\prec$. By Proposition 4.6, $\mathcal{G}$ is also a Gröbner basis with respect to a certain lexicographic monomial ordering $\prec_{lex}$. The result follows from Proposition 4.5. $\square$

Let us illustrate these result with some examples.

Example 4.8. Consider the numerical semigroup $\mathcal{S}=\langle a_1,a_2,a_3,a_4\rangle$ with $a_1=8$, $a_2=9$, $a_3=10$, $a_4=12$ of Example 4.1. Since $\mathcal{S}$ is not free for the arrangement a_1,a_2,a_3,a_4, Proposition 4.5 assures that the reduced Gröbner basis with respect to the lexicographic order with $x_1\succ x_2\succ x_3\succ x_4$ has more than 3 elements. Indeed, it has 8 elements. Nevertheless, $\mathcal{S}$ is free for the arrangement a_2,a_3,a_1,a_4 and, again by Proposition 4.5, we know that the reduced Gröbner basis with respect to the lexicographic order with $x_2\succ x_3\succ x_1\succ x_4$ has 3 elements. Indeed, it is $\{x_2^2-x_1x_3,\, x_3^2-x_1x_4, x_1^3-x_4^2\}$, which also is a Markov basis of $I_{\mathcal{S}}$ (and it is also the universal Markov basis of $I_{\mathcal{S}}$).

Example 4.9. Consider the numerical semigroup $\mathcal{S}=\langle a_1,a_2,a_3,a_4\rangle$ with $a_1=10$, $a_2=14$, $a_3=15$, $a_4=21$. We know that $\mathcal{S}$ is not free and $I_{\mathcal{S}}$ is a complete intersection. Thus, by Theorem 4.7, we conclude that $I_{\mathcal{S}}$ cannot be minimally generated by a Gröbner basis.

4.2. **Numerical semigroups defining robust - generalized robust toric ideals.** In [24], García-Sánchez, Ojeda and Rosales studied a family of affine submonoids of $\mathbb{N}^n$ which they called *semigroups with a unique Betti element.*

Definition 4.10. *An affine monoid $\mathcal{S}$ with minimal generating set A has a unique Betti element if and only if the A-degrees of all the binomials in a Markov basis of I_A coincide.*

In [24, Theorem 6], the authors characterized semigroups with a unique Betti element as those where $\mathcal{C}_A=Gr_A$, i.e., the set of circuits of I_A coincides with the Graver basis. Moreover, in this family of semigroups one has that every circuit is a minimal generator of I_A. As a consequence of these two facts and Proposition 2.2, one has that all the toric bases coincide ($\mathcal{C}_A=\mathcal{M}_A=\mathcal{U}_A=Gr_A$) and one directly derives the following result.

Proposition 4.11. *Every affine monoid with a unique Betti element defines a generalized robust toric ideal.*

The converse of this result does not hold for general toric ideals (in Section 3 one can find examples of generalized robust toric ideals of graphs not having a unique Betti element). Nevertheless, in the following result, which is the main one in this section, we aim at proving that the converse of Proposition 4.11 holds for numerical semigroups.

Theorem 4.12. *A numerical semigroup defines a generalized robust toric ideal if and only if it has a unique Betti element.*

In the proof of this result, we handle with Betti divisible numerical semigroups. This is a family of numerical semigroups studied in [23] that contains those with a unique Betti element.

Definition 4.13. *A numerical semigroup $\mathcal{S}$ (and more generally an affine monoid) with minimal generating set A is Betti divisible if the A-degrees of all the binomials in a Markov basis of I_A are ordered by divisibility.*

Our strategy for proving Theorem 4.12 is the following. We first study the case $m = 3$ and prove that whenever $\mathcal{S} = \langle a_1, a_2, a_3\rangle$ satisfies that $\mathcal{C}_{I_\mathcal{S}} \subseteq \mathcal{M}_{I_\mathcal{S}}$, then it is Betti divisible (Proposition 4.15). Then, we move on to the case of a numerical semigroup $\mathcal{S} = \langle a_1, \ldots, a_m\rangle$ defining a generalized robust toric ideal. Since $I_\mathcal{S}$ is generalized robust (i.e. $\mathcal{M}_{I_\mathcal{S}} = \mathcal{U}_{I_\mathcal{S}}$), by Theorem 2.2 we have that $\mathcal{C}_{I_\mathcal{S}} \subseteq \mathcal{M}_{I_\mathcal{S}}$. By Proposition 2.4, we have that for all $A' \subseteq \{a_1, \ldots, a_m\}$, if we take $\mathcal{S}' = \langle A'\rangle$, then $\mathcal{C}_{I_{\mathcal{S}'}} \subseteq \mathcal{M}_{I_{\mathcal{S}'}}$. In particular, if we take A' a set of three elements, by Proposition 4.15, we have that $\mathcal{S}'$ is Betti divisible. By conveniently choosing the set A' and using the fact that $\mathcal{U}_{I_\mathcal{S}} = \mathcal{M}_{I_\mathcal{S}}$ we will conclude that $\mathcal{S}$ has a unique Betti element.

We introduce some concepts and results that we will use in the proof. Firstly, we have that the set of the circuits of the toric ideal of a numerical semigroup is given by the following result (see, e.g., [39, Chapter 4] or [29, Lemma 2.8]).

Lemma 4.14. *Let $\mathcal{S} = \langle a_1, \ldots, a_m\rangle \subseteq \mathbb{N}$ be a numerical semigroup. Then,*

$$\mathcal{C}_{I_\mathcal{S}} = \left\{ q_{i,j} := x_i^{a_j/\gcd(a_i,a_j)} - x_j^{a_i/\gcd(a_i,a_j)} \mid 1 \le i, j \le m,\, i \neq j \right\},$$

In the forthcoming we will use the concept of critical binomial, which was introduced by Eliahou [22] and later studied in [2] and [29], among others. Let $\mathcal{S} = \langle a_1, \ldots, a_m\rangle$ be a numerical semigroup, one sets

$$n_i = \min \left\{ b \in \mathbb{Z}^+ \mid b a_i \in \sum_{j \in \{1,\ldots,m\}\setminus\{i\}} \mathbb{N} a_j \right\}, \text{ for } i = 1, \ldots, m.$$

Write

$$n_i a_i = \sum_{j\in\{1,\ldots,m\}\setminus\{i\}} \beta_j a_j, \text{ with } \beta_j \in \mathbb{N},$$

the binomials

$$g_i := x_i^{n_i} - \prod_{j\in\{1,\ldots,m\}\setminus\{i\}} x_j^{\beta_j} \text{ and } -g_i$$

of A-degree $n_i a_i$ are called *critical binomials with respect to x_i*.

As we mentioned before, a key point in the proof is the case of embedding dimension $m = 3$. Three-generated numerical semigroups and their toric ideals have been extensively studied in the literature. Here we will recall some results concerning them that we will use later; one can find restatements of these results and their proofs in [6, 27]. Let $\mathcal{S} = \langle a_1, a_2, a_3\rangle$ be a numerical semigroup, then $2 \leq \mu(I_{\mathcal{S}}) \leq 3$ and the A-degrees of $I_{\mathcal{S}}$ are $\{n_1a_1, n_2a_2, n_3a_3\}$. Moreover, $\mu(I_{\mathcal{S}}) = 2$ or, equivalently, $I_{\mathcal{S}}$ is a complete intersection if and only if there exist $1 \leq i < j \leq 3$ such that $n_ia_i = n_ja_j$. Clearly $\mathcal{S}$ has a unique Betti element if and only if $n_1a_1 = n_2a_2 = n_3a_3$.

Proposition 4.15. *Let $\mathcal{S} = \langle a_1, a_2, a_3\rangle$ be a numerical semigroup. If $\mathcal{C}_{I_{\mathcal{S}}} \subseteq \mathcal{M}_{I_{\mathcal{S}}}$, then $\mathcal{S}$ is Betti divisible.*

Proof. Suppose that $n_1a_1 \leq n_2a_2 \leq n_3a_3$. We assume that $\mathcal{C}_{I_{\mathcal{S}}} \subseteq \mathcal{M}_{I_{\mathcal{S}}}$ and let us see that $n_1a_1 = n_2a_2$ and that they both divide n_3a_3.

Claim 1: $n_1a_1 = n_2a_2$.

Proof of claim 1: if $n_1a_1 < n_2a_2$, we write $n_1a_1 = \alpha_2a_2 + \alpha_3a_3$ with $\alpha_2, \alpha_3 \in \mathbb{N}$. Since $n_1a_1 < n_2a_2 \leq n_3a_3$ it follows that both α_2 and α_3 are nonzero. Take the critical binomial $f := x_1^{n_1} - x_2^{\alpha_2}x_3^{\alpha_3} \in I_{\mathcal{S}}$. Consider now the circuit

$$q_{1,2} = x_1^{a_2/\gcd(a_1,a_2)} - x_2^{a_1/\gcd(a_1,a_2)} \in \mathcal{C}_{I_{\mathcal{S}}} \subseteq \mathcal{M}_{I_{\mathcal{S}}}.$$

We have that $a_2/\gcd(a_1, a_2) > n_1$ (otherwise, $a_2/\gcd(a_1, a_2) = n_1$, then n_1a_1 is a multiple of a_2 and $n_1a_1 \geq n_2a_2$, a contradiction). Hence,

$$q_{1,2} - x_1^{a_2/\gcd(a_1,a_2)-m_1} f = x_2h$$

for some $h \in I_{\mathcal{S}}$. Thus, $q_{1,2} \in \langle x_1, \ldots, x_m\rangle \cdot I_{\mathcal{S}}$ and, by Proposition 2.1, $q_{1,2} \notin \mathcal{M}_{I_{\mathcal{S}}}$, a contradiction.

Claim 2: n_3a_3 is a multiple of both a_1 and a_2.

Proof of claim 2: Assume n_3a_3 is not a multiple of a_1, then $n_1a_1 = n_2a_2 < n_3a_3 < \operatorname{lcm}(a_1, a_3)$. Take a critical binomial $f := x_3^{n_3} - x_1^{\gamma_1}x_2^{\gamma_2} \in I_{\mathcal{S}}$ with respect to x_3. We observe that $n_3 < a_1/\gcd(a_1, a_3)$. Also, we may assume that $\gamma_1 > 0$, otherwise we have that $\gamma_2 > n_2$ and we consider $f = x_3^{n_3} - x_1^{n_1}x_2^{\gamma_2-n_2} \in I_{\mathcal{S}}$. As a consequence, the circuit

$$q_{3,1} = x_3^{a_1/\gcd(a_1,a_3)} - x_1^{a_3/\gcd(a_1,a_3)} = x_3^{a_1/\gcd(a_1,a_3)-n_3} f + x_1h,$$

for some $h \in I_{\mathcal{S}}$. Thus, $q_{3,1} \in \langle x_1, \ldots, x_m\rangle \cdot I_{\mathcal{S}}$ and, by Proposition 2.1, $q_{3,1} \notin \mathcal{M}_{I_{\mathcal{S}}}$, a contradiction. Hence, n_3a_3 is a multiple of a_1. A similar argument proves that n_3a_3 is also a multiple of a_2.

Now, by (Claim 1), we have that $n_1a_1 = n_2a_2 = \operatorname{lcm}(a_1, a_2)$ and, by (Claim 2), n_3a_3 is a multiple of $\operatorname{lcm}(a_1, a_2)$. Thus, $\mathcal{S}$ is Betti divisible and the result holds. □

Now we can proceed to the proof of the main theorem of this subsection.

Proof of Theorem 4.12 ($\Longleftarrow$) This is a particular case of Proposition 4.11.

($\Longrightarrow$) Let $\mathcal{S} = \langle a_1, \ldots, a_m\rangle$ be a numerical semigroup defining a generalized robust toric ideal. Let us prove that it has a unique Betti element. Assume that $n_1a_1 \leq n_2a_2 \leq \cdots \leq n_ma_m$.

Claim 1: $n_1a_1 = n_2a_2$.

Let us first see that n_1a_1 is a multiple of a_i for some $i \in \{2, \ldots, m\}$. Assume, by contradiction that this is not true. We write

$$n_1a_1 = \sum_{k=2}^{m} \alpha_k a_k \text{ with } \alpha_2, \ldots, \alpha_m \in \mathbb{N},$$

consider

$$f := x_1^{n_1} - \prod_{j=2}^{m} x_j^{\alpha_j} \in I_{\mathcal{S}}$$

and observe that at least two of the α_i's are nonzero. We take $s \in \{2, \ldots, m\}$ such that $\alpha_s \neq 0$, and we are going to see that the circuit

$$q_{s,1} = x_1^{a_s/\gcd(a_1,a_s)} - x_s^{a_1/\gcd(a_1,a_s)}$$

is not a minimal binomial, which contradicts the hypothesis. We have that $n_1 < a_s/\gcd(a_1, a_s)$. We set

$$h := q_{s,1} - x_1^{\frac{a_s}{\gcd(a_1,a_s)} - n_1} f \in I_{\mathcal{S}},$$

and we have that

$$h = x_1^{\frac{a_s}{\gcd(a_1,a_s)} - n_1} \prod_{j=2}^{m} x_j^{\alpha_j} - x_s^{\frac{a_1}{\gcd(a_1,a_s)}} \neq 0$$

and the two monomials apprearing in h are multiples of x_s. Then, we have that $h = x_s h'$ for some $h' \in I_{\mathcal{S}}$. We conclude that $q_{s,1} \in \langle x_1, \ldots, x_m \rangle \cdot I_{\mathcal{S}}$ and, by Proposition 2.1, $q_{s,1} \notin \mathcal{M}_{I_{\mathcal{S}}}$, a contradiction.

So far we have seen that $n_1 a_1$ is a multiple of a_i for some $i \in \{2, \ldots, m\}$. In particular, we have that $n_1 a_1 = \mathrm{lcm}(a_1, a_i) \geq n_i a_i$. Thus, $n_1 a_1 = n_2 a_2 = n_i a_i$ and the claim follows.

Claim 2: $n_1 a_1 = n_k a_k$ for all $k \in \{3, \ldots, m\}$.

Take $k \in \{3, \ldots, m\}$ and let us prove that $n_1 a_1 = n_k a_k$. Assume by contradiction that $n_1 a_1 < n_k a_k$. Set $n'_k = \min\{b \in \mathbb{Z}^+ \mid b a_k \in \langle a_1, a_2 \rangle\}$, we have that $n_1 a_1 < n_k a_k \leq n'_k a_k$. We consider the semigroup $\mathcal{S}' = \langle a_1, a_2, a_k \rangle$. Since $I_{\mathcal{S}}$ is generalized robust, then $\mathcal{U}_{I_{\mathcal{S}}} \subseteq \mathcal{M}_{I_{\mathcal{S}}}$ and, by Proposition 2.4.(a), we have $\mathcal{U}_{I_{\mathcal{S}'}} \subseteq \mathcal{M}_{I_{\mathcal{S}'}}$. Now, applying Proposition 4.15 we get that $\mathcal{S}'$ is Betti divisible. Since the Betti elements of $\mathcal{S}'$ are $n_1 a_1 = n_2 a_2 < n'_k a_k$ we have that $n_1 a_1 \mid n'_k a_k$ and, as a consequence, $n'_k a_k = \mathrm{lcm}(a_1, a_k) = b\, n_1 a_1 = b\, n_2 a_2$ for some $b \geq 2$.

Now we have that the circuit

$$q_{1,k} = x_1^{a_k/\gcd(a_1,a_k)} - x_k^{a_1/\gcd(a_1,a_k)} = x_1^{b\, n_1} - x_k^{n'_k} \in \mathcal{C}_{I_{\mathcal{S}}} \subseteq \mathcal{U}_{I_{\mathcal{S}}} = \mathcal{M}_{I_{\mathcal{S}}}.$$

Set $p := x_1^{n_1} x_2^{(b-1)n_2} - x_k^{n'_k} \in I_{\mathcal{S}}$, the equality

$$p = q_{1,k} - x_1^{(b-1)n_1}(x_1^{n_1} - x_2^{n_2})$$

implies that $p \in \mathcal{M}_{I_{\mathcal{S}}}$ (see Proposition 2.1). Nevertheless, $x_1^{n_1} - x_2^{n_2} \in I_{\mathcal{S}}$ and, hence, for every monomial order $\prec$, either $x_1^{n_1}$ or $x_2^{n_2}$ belongs to $\mathrm{in}_{\prec}(I_{\mathcal{S}})$. In both cases the monomial $x_1^{n_1} x_2^{(b-1)n_2} \in \mathrm{in}_{\prec}(I_{\mathcal{S}})$ but is not a minimal generator of $\mathrm{in}_{\prec}(I_{\mathcal{S}})$. Thus, $p \notin \mathcal{U}_{I_{\mathcal{S}}}$, a contradiction.

From (Claim 1) and (Claim 2) we conclude that $\mathcal{S}$ has a unique Betti element. $\square$

The shape of the generators of a numerical semigroup with a unique Betti element is given by the following result (see [29] or [24, Example 12]):

Proposition 4.16. *A numerical semigroup $\mathcal{S} = \langle a_1, \ldots, a_m \rangle \subseteq \mathbb{N}$ has a unique Betti element if and only if there exist $d_1, \ldots, d_m \geq 2$ pairwise prime integers such that $a_i = (\prod_{j=1}^{m} d_j)/d_i$ for all $i \in \{1, \ldots, m\}$.*

Hence, one can easily construct examples of generalized robust toric ideals of numerical semigroups with arbitrarily large embedding dimension. Since numerical semigroups with a unique Betti element only have a unique minimal set of generators when $m = 2$, we directly get the following.

Corollary 4.17. *A numerical semigroup $\mathcal{S}$ defines a robust toric ideal if and only if $\mathcal{S}$ is 2-generated.*

5. Conclusions and open questions

In this paper we have described different families of robust and generalized robust toric ideals. In the context of toric ideals of graphs generated by quadrics, the family of generalized robust toric ideals is only slightly bigger than the family of robust ideals. Nevertheless, for toric ideals of numerical semigroups the situation changes: while only principal ideals are robust, there are generalized robust ideals with arbitrarily large number of generators.

In [41], the author asks if $\mathcal{M}_A = Gr_A$ for generalized robust toric ideals and verifies it for toric ideals of graphs. As a consequence of Theorem 4.12, we provide an affirmative answer in the case of numerical semigroups. For toric ideals of graphs the equality $\mathcal{M}_A = Gr_A$ characterizes generalized robustness (Theorem 3.3). However, this is not true for numerical semigroups. Indeed, consider the (Betti divisible) numerical semigroup $\mathcal{S} = \langle a_1, a_2, a_3\rangle$ with $a_1 = 10 = 2 \cdot 5$, $a_2 = 12 = 2^2 \cdot 3$, $a_3 = 15 = 3 \cdot 5$. By Proposition 4.16 we have that $\mathcal{S}$ is not a semigroup with a unique Betti element and, thus, it is not generalized robust (Theorem 4.12). Nevertheless, the equality $\mathcal{M}_A = Gr_A = \{x_1^3 - x_3^2, x_2^5 - x_1^6, x_2^5 - x_1^3x_3^2, x_2^5 - x_3^4\}$ holds. The same example also shows that the containment $\mathcal{M}_A \subseteq \mathcal{U}_A$, which holds for toric ideals of graphs, does not always work for toric ideals of numerical semigroups.

By [11, Proposition 2.5], we have that robustness property is preserved under an elimination of variables. However, we do not know if the same result is true when we replace robustness by generalized robustness. By Proposition 2.4.(a) we know that whenever I_A is generalized robust and $A' \subseteq A$, then $\mathcal{U}_{A'} \subseteq \mathcal{M}_{A'}$, but we do not know if equality holds.

Question 5.1. Let I_A be a generalized robust toric ideal and $A' \subseteq A$, is $I_{A'}$ generalized robust?

In the present paper we give a positive answer to this question for: (1) toric ideals of graphs (see Corollary 3.4), and (2) toric ideals of numerical semigroups (this follows as a consequence of Theorem 4.12 and Proposition 4.16).

In Theorem 4.7 we characterize when the toric ideal of a numerical semigroup has a complete intersection initial ideal. It would be interesting to seek the answer to the same question for toric ideals of graphs. Since having a complete intersection initial ideal implies that the ideal itself is a complete intersection, the class of graphs that we are looking for is a subfamily of the one described in [44].

Open problem 5.2. Characterize when the toric ideal I_G of a graph G has a complete intersection initial ideal.

Also, in Theorem 4.7, we proved that free numerical semigroups have an initial ideal such that $\mu(I_\mathcal{S}) = \mu(\mathrm{in}_\prec(I_\mathcal{S}))$. There are further families of numerical semigroups with the same property. For example when $\mathcal{S}$ is generated by an arithmetic sequence of integers (see, e.g., [25]).

Open problem 5.3. Characterize the numerical semigroups such that

$$\mu(I_{\mathcal{S}}) = \mu(\mathrm{in}_{\prec}(I_{\mathcal{S}}))$$

for a monomial order $\prec$.

We have verified when the equality $\mathcal{M}_{I_{\mathcal{S}}} = \mathcal{U}_{I_{\mathcal{S}}}$ occurs for a numerical semigroup $\mathcal{S}$. It would be interesting to characterize when equality or containment of other toric bases holds. For example, we say that a toric ideal I_A is a *circuit ideal* if it is generated by its set of circuits (see [10, 30] for a deeper study of circuit ideals).

Open problem 5.4. Characterize the numerical semigroups $\mathcal{S}$ such that $I_{\mathcal{S}}$ is a circuit ideal.

We do not know the answer to this question even if we add the hypothesis of $\mathcal{S}$ being a complete intersection. Indeed, whenever $\mathcal{S}$ is a Betti divisible numerical semigroup, then it is a complete intersection and $I_{\mathcal{S}}$ is a circuit ideal (see [23, Section 7]). However, there are further examples of complete intersection numerical semigroups such that $I_{\mathcal{S}}$ is generated by $\mathcal{C}_{I_{\mathcal{S}}}$. For example, consider the numerical semigroup $\mathcal{S} = \langle a_1, a_2, a_3, a_4\rangle$ with $a_1 = 390$, $a_2 = 546$, $a_3 = 770$, $a_4 = 1155\rangle$. Then, $I_{\mathcal{S}}$ is minimally generated by $\{x_1^7 - x_2^5, x_3^3 - x_4^2, x_2^{55} - x_4^{26}\}$ and, thus, it is a complete intersection and a circuit ideal. However, the Betti degrees of the generators are $\beta_1 = 7a_1 = 5a_2 = 2730$, $\beta_2 = 3a_3 = 2a_4 = 2310$ and $\beta_3 = 55a_2 = 26a_4 = 30030$ and, hence, it is not Betti divisible. Interestingly, in the context of toric ideals of graphs, every complete intersection ideal is a circuit ideal (by [43, Theorem 5.1]).

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Data Availability Statements
All data generated or analysed during this study are included in this published article (and its supplementary information files).

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