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On rough sets induced by fuzzy relations approach in semigroups

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Abstract: In this paper, we introduce a rough set in a universal set based on cores of successor classes with respect to level in a closed unit interval under a fuzzy relation, and some interesting properties are investigated. Based on this point, we propose a rough completely prime ideal in a semigroup structure under a compatible preorder fuzzy relation, including the rough semigroup and rough ideal. Then we provide sufficient conditions for them. Finally, the relationships between rough completely prime ideals (rough semigroups and rough ideals) and their homomorphic images are verified.

Keywords: Rough sets, Semigroups, Rough semigroups, Rough ideals, Rough completely prime ideals, Fuzzy relations, Compatible preorder fuzzy relations

MSC: 20M12; 20M99

1 Introduction

The Pawlak's rough set theory is a classical tool for assessing the problems and decision problems in many fields with respect to informations and technology. This theory was introduced by Pawlak [1] in 1982. He proposed the concept of Pawlak's rough sets in universal sets based on equivalence classes induced by equivalence relations. For an equivalence relation on a universal set and a non-empty subset of the universal set, the Pawlak's rough set of the non-empty subset is given by mean of a pair of the Pawlak's upper approximation and the Pawlak's lower approximation where the difference between the Pawlak's upper approximation and the Pawlak's lower approximation (The Pawlak's boundary region) is a non-empty set. The Pawlak's upper approximation is the union of all the equivalence classes which have a non-empty intersection with the non-empty subset. The Pawlak's lower approximation is the union of all the equivalence classes which are subset of the non-empty subset. As mentioned above, the Pawlak's rough set model is defined as a mathematical tool with respect to assessments of decisions. This assessment model is an important tool for dealing with algebraic systems [2–14], information sciences [15] and computer sciences [16] etc.

From Pawlak's rough sets induced by equivalence relations, the generalized Pawlak's rough sets using arbitrary binary relations (briefly, binary relations) were introduced by many researchers. In 1998, Yao [17] introduced roughness models using successor neighborhoods induced by binary relations [$SN_\theta(u) := \{u' \in U : (u, u') \in \theta\}$ denotes a successor neighborhood of u induced by a binary relation θ on a universal set U where u is an element in U]. In 2016, Mareay [18] introduced rough sets using cores of successor

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neighborhoods induced by binary relations [$CSN_\theta(u) := \{u' \in U : SN_\theta(u) = SN_\theta(u')\}$ denotes a core of a successor neighborhood of u induced by a binary relation θ on a universal set U where u is an element in U]. If a binary relation on a universal set is an equivalence relation, then the Yao's rough set and the Mareay's rough set are generalizations of the Pawlak's rough set.

The classical fuzzy set theory was introduced by Zadeh [19] in 1965. Based on this point, Zadeh [20, 21] introduced the concept of fuzzy relations in 1971 which it is researched by many researchers in several fields, such as information sciences [22] and decision systems [23] etc.

The semigroup structure (see [24]) is an algebraic system with respect to wide applications, especially the, notions of Pawlak's rough sets in semigroups. For combinations of Pawlak's rough set theory and semigroup theory, Kuroki [4] proposed the notion of rough ideals in semigroups based on congruence classes induced by congruence relations (equivalence relations and compatible relations) in 1997. Thereafter, Xiao and Zhang [7] proposed the notion of rough completely prime ideals in semigroups based on congruence classes induced by congruence relations in 2006. For the combination of Pawlak's rough set theory, fuzzy set theory and semigroup theory, Wang and Zhan [13] introduced the concept of rough semigroups based on congruence relations with respect to fuzzy ideals of semigroups in 2016.

From an interesting idea about generalized rough set models in the sense of Mareay [18], and after providing some preliminaries about some important definitions of fuzzy relations and semigroups in Section 2, we introduce a rough set in a universal set based on cores of successor classes with respect to level in a closed unit interval under a fuzzy relation, and we verify some interesting properties in Section 3. In Section 4, we introduce a rough completely prime ideal in a semigroup structure under a compatible preorder fuzzy relation, including the rough semigroup and rough ideal. Then we provide sufficient conditions for them. In Section 5, we investigate the relationships between rough completely prime ideals (rough semigroups and rough ideals) and their homomorphic images. Finally, we give a conclusion of the work in Section 6.

2 Preliminaries

In this section, we review some important definitions which will be necessary in the subsequent sections. Throughout this paper, U and V denote two non-empty universal sets.

Definition 2.1. [19] A *fuzzy set* of U is defined as a function from U to the closed unit interval $[0, 1]$.

Definition 2.2. [22] Let $\mathcal{F}(U \times V)$ be a family of all fuzzy sets of $U \times V$. An element in $\mathcal{F}(U \times V)$ is referred to as a *fuzzy relation from U to V*. An element in $\mathcal{F}(U \times V)$ is called a *fuzzy relation on U* if $U = V$. For a fuzzy relation $\Theta \in \mathcal{F}(U \times V)$ and elements $u \in U$, $v \in V$, the value of $\Theta(u, v)$ in $[0, 1]$ representing the *membership grade of relation between u and v under Θ*. If $\Theta \in \mathcal{F}(U \times V)$ where $U := \{u_1, u_2, u_3, \ldots, u_m\}$ and $V := \{v_1, v_2, v_3, \ldots, v_n\}$, then the fuzzy relation Θ is represented by the matrix as

$$\begin{pmatrix} \Theta(u_1, v_1) & \Theta(u_1, v_2) & \Theta(u_1, v_3) & \cdots & \Theta(u_1, v_n) \\ \Theta(u_2, v_1) & \Theta(u_2, v_2) & \Theta(u_2, v_3) & \cdots & \Theta(u_2, v_n) \\ \Theta(u_3, v_1) & \Theta(u_3, v_2) & \Theta(u_3, v_3) & \cdots & \Theta(u_3, v_n) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \Theta(u_m, v_1) & \Theta(u_m, v_2) & \Theta(u_m, v_3) & \cdots & \Theta(u_m, v_n) \end{pmatrix}.$$

Definition 2.3. [22] Let Θ be a fuzzy relation from U to V. Θ is called *serial* if for all $u \in U$, there exists $v \in V$ such that $\Theta(u, v) = 1$.

Definition 2.4. [22] Let Θ be a fuzzy relation on U.
(1) Θ is called *reflexive* if for all $u \in U$, $\Theta(u, u) = 1$,
(2) Θ is called *symmetric* if for all $u_1, u_2 \in U$, $\Theta(u_1, u_2) = \Theta(u_2, u_1)$,
(3) Θ is called *transitive* if for all $u_1, u_2 \in U$, $\Theta(u_1, u_2) \geq \vee_{u_3 \in U}(\Theta(u_1, u_3) \wedge \Theta(u_3, u_2))$,

(4) Θ is called a *similarity fuzzy relation* if it is reflexive, symmetric and transitive.

A *semigroup* [24] $(S, *)$ is defined as an algebraic system where S is a non-empty set and $*$ is an associative binary operation on S. Throughout this paper, S denotes a semigroup. A non-empty subset X of S is called a *subsemigroup* [25] of S if $XX \subseteq X$. A non-empty subset X of S is called a *left (right) ideal* [25] of S if $SX \subseteq X$ ($XS \subseteq X$), and if it is both a left ideal and a right ideal of S, then it is called an *ideal* [25]. An ideal X of S is called a *completely prime ideal* [25] of S if for all $s_1, s_2 \in S$, $s_1 s_2 \in X$ implies $s_1 \in X$ or $s_2 \in X$.

Definition 2.5. [25] Let Θ be a fuzzy relations on S. Θ is called *compatible* if for all $s_1, s_2, s_3 \in S$,

$$\Theta(s_1 s_3, s_2 s_3) \geq \Theta(s_1, s_2) \text{ and } \Theta(s_3 s_1, s_3 s_2) \geq \Theta(s_1, s_2).$$

3 Rough sets induced by fuzzy relations

In this section, we construct rough sets induced by fuzzy relations. Then we give the real-world example and some interesting properties.

Definition 3.1. Let $\iota \in [0, 1]$ and let Θ be a fuzzy relation from U to V. For an element $u \in U$,

$$S_\Theta(u; \iota) := \{v \in V : \Theta(u, v) \geq \iota\}$$

is called a *successor class of u with respect to ι-level under Θ*.

Remark 3.2. Let $\iota \in [0, 1]$. If Θ is a serial fuzzy relation from U to V, then $S_\Theta(u; \iota) \neq \emptyset$ for all $u \in U$.

Definition 3.3. Let $\iota \in [0, 1]$ and let Θ be a fuzzy relation from U to V. For an element $u_1 \in U$,

$$CS_\Theta(u_1; \iota) := \{u_2 \in U : S_\Theta(u_1; \iota) = S_\Theta(u_2; \iota)\}$$

is called a *core of the successor class of u_1 with respect to ι-level under Θ*.

We denote by $\mathcal{CS}_\Theta(U; \iota)$ the collection of $CS_\Theta(u; \iota)$ for all $u \in U$.

Directly from Definition 3.3, we can obtain the following Proposition 3.4 below.

Proposition 3.4. *Let $\iota \in [0, 1]$ and let Θ be a fuzzy relation from U to V. Then the following statements hold.*
(1) For all $u \in U$, $u \in CS_\Theta(u; \iota)$.
(2) For all $u_1, u_2 \in U$, $u_2 \in CS_\Theta(u_1; \iota)$ if and only if $CS_\Theta(u_1; \iota) = CS_\Theta(u_2; \iota)$.

The following remark is an immediate consequence of Proposition 3.4.

Remark 3.5. Let $\iota \in [0, 1]$ and let Θ be a fuzzy relation from U to V. Then $\mathcal{CS}_\Theta(U; \iota)$ is the partition of U.

Proposition 3.6. *Let $\iota \in [0, 1]$ and let Θ be a fuzzy relation on U. Then we have the following statements.*
(1) If Θ is reflexive, then $CS_\Theta(u; \iota) \subseteq S_\Theta(u; \iota)$ for all $u \in U$.
(2) If Θ is a similarity fuzzy relation, then $S_\Theta(u; \iota)$ and $CS_\Theta(u; \iota)$ are identical classes for all $u \in U$.

Proof. The proof is straightforward, so we omit it. □

In the following, we give the concept of rough sets induced by fuzzy relations.

Definition 3.7. Let $\iota \in [0, 1]$ and let Θ be a fuzzy relation from U to V. A triple $(U, V, \mathcal{CS}_\Theta(U; \iota))$ is called an *approximation space based on* $\mathcal{CS}_\Theta(U; \iota)$ (briefly, *$\mathcal{CS}_\Theta(U; \iota)$-approximation space*). If $U = V$, then $(U, V, \mathcal{CS}_\Theta(U; \iota))$ is replaced by a pair $(U, \mathcal{CS}_\Theta(U; \iota))$.

Definition 3.8. Let $(U, V, \mathcal{CS}_\Theta(U;\iota))$ be an $\mathcal{CS}_\Theta(U;\iota)$-approximation space. For a non-empty subset X of U, we define three sets as follows:

$\overline{\Theta}(X;\iota) := \bigcup_{u\in U}\{CS_\Theta(u;\iota) : CS_\Theta(u;\iota)\cap X \neq \emptyset\}$,
$\underline{\Theta}(X;\iota) := \bigcup_{u\in U}\{CS_\Theta(u;\iota) : CS_\Theta(u;\iota)\subseteq X\}$ and
$\Theta_{bnd}(X;\iota) := \overline{\Theta}(X;\iota) - \underline{\Theta}(X;\iota)$.

Then

(1) $\overline{\Theta}(X;\iota)$ is called an *upper approximation of X in* $(U, V, \mathcal{CS}_\Theta(U;\iota))$ (briefly, $\mathcal{CS}_\Theta(U;\iota)$-*upper approximation of X*).
(2) $\underline{\Theta}(X;\iota)$ is called a *lower approximation of X in* $(U, V, \mathcal{CS}_\Theta(U;\iota))$ (briefly, $\mathcal{CS}_\Theta(U;\iota)$-*lower approximation of X*).
(3) $\Theta_{bnd}(X;\iota)$ is called a *boundary region of X in* $(U, V, \mathcal{CS}_\Theta(U;\iota))$ (briefly, $\mathcal{CS}_\Theta(U;\iota)$-*boundary region of X*).
(4) If $\Theta_{bnd}(X;\iota)\neq\emptyset$, then $\Theta\mathrm{R}(X;\iota) := (\overline{\Theta}(X;\iota), \underline{\Theta}(X;\iota))$ is called a *rough set of X in* $(U, V, \mathcal{CS}_\Theta(U;\iota))$ (briefly, $\mathcal{CS}_\Theta(U;\iota)$-*rough set of X*).
(5) If $\Theta_{bnd}(X;\iota)=\emptyset$, then X is called a *definable set in* $(U, V, \mathcal{CS}_\Theta(U;\iota))$ (briefly, $\mathcal{CS}_\Theta(U;\iota)$-*definable set*).

According to Definition 3.8, it is easy to prove that

$\overline{\Theta}(X;\iota) := \{u\in U : CS_\Theta(u;\iota)\cap X\neq\emptyset\}$ and
$\underline{\Theta}(X;\iota) := \{u\in U : CS_\Theta(u;\iota)\subseteq X\}$.

Here we present an example as the following.

Example 3.9. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of doctoral students in a mathematical business classroom of a university and let $V = \{v_1, v_2, v_3, v_4\}$ be a set of subjects where

v_1 is business,
v_2 is economics,
v_3 is computer sciences and
v_4 is mathematics.

For a fuzzy relation $\Theta\in\mathcal{F}(U\times V)$ and elements $u\in U$, $v\in V$, the number $\Theta(u,v)$ in the closed unit interval $[0,1]$ is defined as the score of the doctoral student u with respect to the subject v under Θ. The scores of all doctoral students in U with respect to subjects in V under Θ are given as the following matrix.

$$\begin{pmatrix} 0.7 & 0.9 & 0.8 & 0.9\\ 0.8 & 0.9 & 0.7 & 0.9\\ 0.9 & 0.8 & 0.8 & 0.9\\ 0.5 & 0.5 & 0.9 & 0.9\\ 0.9 & 0.9 & 0.6 & 0.9 \end{pmatrix}$$

Let $\iota = 0.9$ be a minimal score level. If an educational measurement committee assign $X := \{u_2, u_3, u_5\}$ which is a set of excellent doctoral students under the global evaluation, then the assessment of X in an $\mathcal{CS}_\Theta(U;0.9)$-approximation space $(U, V, \mathcal{CS}_\Theta(U;0.9))$ is derived by the process as the following.

According to Definition 3.1, it follows that

$S_\Theta(u_1;0.9) := \{v_2, v_4\}$,
$S_\Theta(u_2;0.9) := \{v_2, v_4\}$,
$S_\Theta(u_3;0.9) := \{v_1, v_4\}$,
$S_\Theta(u_4;0.9) := \{v_3, v_4\}$ and
$S_\Theta(u_5;0.9) := \{v_1, v_2, v_4\}$.

According to Definition 3.3, it follows that
$CS_\Theta(u_1; 0.9) := \{u_1, u_2\}$,
$CS_\Theta(u_2; 0.9) := \{u_1, u_2\}$,
$CS_\Theta(u_3; 0.9) := \{u_3\}$,
$CS_\Theta(u_4; 0.9) := \{u_4\}$ and
$CS_\Theta(u_5; 0.9) := \{u_5\}$.
According to Definition 3.8, it follows that
$\overline{\Theta}(X; 0.9) := \{u_1, u_2, u_3, u_5\}$,
$\underline{\Theta}(X; 0.9) := \{u_3, u_5\}$ and
$\Theta_{bnd}(X; 0.9) := \{u_1, u_2\}$.
Therefore $\Theta\underline{R}(X; 0.9) := (\{u_1, u_2, u_3, u_5\}, \{u_3, u_5\})$ is a $\mathcal{CS}_\Theta(U; 0.9)$-rough set of X. Consequently,
(1) u_1, u_2, u_3 and u_5 are possibly excellent doctoral students,
(2) u_3 and u_5 are certainly excellent doctoral students and
(3) for u_1 and u_2 it cannot be determined whether two students are excellent doctoral students or not.

In what follows, Definition 3.10 follows from the example as the union of upper and lower approximations.

Definition 3.10. Let $(U, V, \mathcal{CS}_\Theta(U; \iota))$ be an $\mathcal{CS}_\Theta(U; \iota)$-approximation space and let X be a non-empty subset of U. $\overline{\Theta}(X; \iota)$ is called a *non-empty* $\mathcal{CS}_\Theta(U; \iota)$*-upper approximation of* X *in* $(U, V, \mathcal{CS}_\Theta(U; \iota))$ if $\overline{\Theta}(X; \iota)$ is a non-empty subset of U. Similarly, we can define a non-empty $\mathcal{CS}_\Theta(U; \iota)$-lower approximation. $\Theta\underline{R}(X; \iota)$ is referred to as a *non-empty* $\mathcal{CS}_\Theta(U; \iota)$*-rough set in* $(U, V, \mathcal{CS}_\Theta(U; \iota))$ if $\overline{\Theta}(X; \iota)$ is a non-empty $\mathcal{CS}_\Theta(U; \iota)$-upper approximation and $\underline{\Theta}(X; \iota)$ is a non-empty $\mathcal{CS}_\Theta(U; \iota)$-lower approximation.

Proposition 3.11. *Let* $(U, V, \mathcal{CS}_\Theta(U; \iota))$ *be an* $\mathcal{CS}_\Theta(U; \iota)$*-approximation space. If* X *and* Y *are non-empty subsets of* U*, then we have the following statements.*
(1) $\overline{\Theta}(U; \iota) = U$ *and*
$\underline{\Theta}(U; \iota) = U$.
(2) $\overline{\Theta}(\emptyset; \iota) = \emptyset$ *and*
$\underline{\Theta}(\emptyset; \iota) = \emptyset$.
(3) $X \subseteq \overline{\Theta}(X; \iota)$ *and*
$\underline{\Theta}(X; \iota) \subseteq X$.
(4) $\overline{\Theta}(X \cup Y; \iota) = \overline{\Theta}(X; \iota) \cup \overline{\Theta}(Y; \iota)$ *and*
$\underline{\Theta}(X \cap Y; \iota) = \underline{\Theta}(X; \iota) \cap \underline{\Theta}(Y; \iota)$.
(5) $\overline{\Theta}(X \cap Y; \iota) \subseteq \overline{\Theta}(X; \iota) \cap \overline{\Theta}(Y; \iota)$ *and*
$\underline{\Theta}(X \cup Y; \iota) \supseteq \underline{\Theta}(X; \iota) \cup \underline{\Theta}(Y; \iota)$.
(6) $\underline{\Theta}(X^c; \iota) = (\overline{\Theta}(X; \iota))^c$*, where* X^c *and* $(\overline{\Theta}(X; \iota))^c$ *are complements of* X *and* $\overline{\Theta}(X; \iota)$*, respectively.*
(7) $\overline{\Theta}(\overline{\Theta}(X; \iota); \iota) = \overline{\Theta}(X; \iota)$ *and*
$\underline{\Theta}(\underline{\Theta}(X; \iota); \iota) = \underline{\Theta}(X; \iota)$.
(8) $\overline{\Theta}((\overline{\Theta}(X; \iota))^c; \iota) = (\overline{\Theta}(X; \iota))^c$*, where* $(\overline{\Theta}(X; \iota))^c$ *is a complement of* $\overline{\Theta}(X; \iota)$ *and*
$\underline{\Theta}((\underline{\Theta}(X; \iota))^c; \iota) = (\underline{\Theta}(X; \iota))^c$*, where* $(\underline{\Theta}(X; \iota))^c$ *is a complement of* $\underline{\Theta}(X; \iota)$.
(9) If $X \subseteq Y$*, then* $\overline{\Theta}(X; \iota) \subseteq \overline{\Theta}(Y; \iota)$ *and* $\underline{\Theta}(X; \iota) \subseteq \underline{\Theta}(Y; \iota)$.

Proof. The proof is straightforward, so we omit it. □

Definition 3.12. Let $(U, V, \mathcal{CS}_\Theta(U; \iota))$ be an $\mathcal{CS}_\Theta(U; \iota)$-approximation space and let X be a non-empty subset of U. If $\underline{\Theta}(X; \iota)$ is a non-empty $\mathcal{CS}_\Theta(U; \iota)$-lower approximation of X in $(U, V, \mathcal{CS}_\Theta(U; \iota))$ and $\underline{\Theta}(X; \iota)$ is a proper subset of X, then X is called a *set over a non-empty interior set*.

Proposition 3.13. *Let* $(U, V, \mathcal{CS}_\Theta(U; \iota))$ *be an* $\mathcal{CS}_\Theta(U; \iota)$*-approximation space and let* X *be a non-empty subset of* U*. If* X *is a set over non-empty interior set, then* $\Theta\underline{R}(X; \iota)$ *is a non-empty* $\mathcal{CS}_\Theta(U; \iota)$*-rough set of* X *in* $(U, V, \mathcal{CS}_\Theta(U; \iota))$.

Proof. Suppose that X is a set over a non-empty interior set. Then we have that $\underline{\Theta}(X;\iota)$ is a non-empty $\mathcal{CS}_\Theta(U;\iota)$-lower approximation and $\underline{\Theta}(X;\iota) \subset X$. By Proposition 3.11 (3), we obtain that $\emptyset \neq X \subseteq \overline{\Theta}(X;\iota)$. Thus we get $\overline{\Theta}(X;\iota)$ is a non-empty $\mathcal{CS}_\Theta(U;\iota)$-upper approximation. We shall verify that $\Theta_{bnd}(X;\iota) \neq \emptyset$. Suppose that $\Theta_{bnd}(X;\iota) = \emptyset$. Then we have $\overline{\Theta}(X;\iota) = \underline{\Theta}(X;\iota)$. From Proposition 3.11 (3), once again, it follows that $\underline{\Theta}(X;\iota) = X$, a contradiction. Therefore $\Theta_{bnd}(X;\iota) \neq \emptyset$. Consequently, $\Theta(X;\iota)$ is a non-empty $\mathcal{CS}_\Theta(U;\iota)$-rough set of X. □

Example 3.14. Let $U := \{u_1 = 3, u_2 = 1, u_3 = \frac{1}{3}, u_4 = \frac{1}{9}, u_5 = \frac{1}{27}\}$ and $V := \{v_1 = 2, v_2 = 2\sqrt{3}, v_3 = 6, v_4 = 6\sqrt{3}\}$. Define a fuzzy relation $\Theta \in \mathcal{F}(U \times V)$ by

$$\Theta(u,v) = \begin{cases} \cos uv & \text{if } u \geq v \\ 1 - \sin uv & \text{if } u < v \end{cases}$$

for all $(u,v) \in U \times V$. Then we have the following ranges of Θ.

$$\begin{pmatrix} 0.99452 & 0.81961 & 0.69098 & 0.48232 \\ 0.96510 & 0.93958 & 0.89547 & 0.81961 \\ 0.98836 & 0.97985 & 0.96510 & 0.93958 \\ 0.99612 & 0.99328 & 0.98836 & 0.97985 \\ 0.99871 & 0.99776 & 0.99612 & 0.99328 \end{pmatrix}$$

Let $\iota = 0.95$ and let $X := \{u_2, u_3\}$ be a non-empty subset of U. According to Definition 3.1, it follows that

$S_\Theta(u_1;0.95) := \{v_1\}$,
$S_\Theta(u_2;0.95) := \{v_1\}$,
$S_\Theta(u_3;0.95) := \{v_1, v_2, v_3\}$,
$S_\Theta(u_4;0.95) := \{v_1, v_2, v_3, v_4\}$ and
$S_\Theta(u_5;0.95) := \{v_1, v_2, v_3, v_4\}$.

According to Definition 3.3, it follows that

$CS_\Theta(u_1;0.95) := \{u_1, u_2\}$,
$CS_\Theta(u_2;0.95) := \{u_1, u_2\}$,
$CS_\Theta(u_3;0.95) := \{u_3\}$,
$CS_\Theta(u_4;0.95) := \{u_4, u_5\}$ and
$CS_\Theta(u_5;0.95) := \{u_4, u_5\}$.

Here it is easy to check that $\underline{\Theta}(X;0.95)$ is a non-empty $\mathcal{CS}_\Theta(U;95)$-lower approximation of X, and also $\underline{\Theta}(X;0.95) \subset X$. Note that $X \subseteq \overline{\Theta}(X;0.95)$. Thus we get $\overline{\Theta}(X;0.95) \neq \emptyset$ and $\overline{\Theta}(X;0.95) \neq \underline{\Theta}(X;0.95)$. It follows that $\Theta\underline{R}(X;0.95)$ is a non-empty $\mathcal{CS}_\Theta(U;0.95)$-rough set of X.

Proposition 3.15. *Let $(U, \mathcal{CS}_\Theta(U;\iota))$ be an $\mathcal{CS}_\Theta(U;\iota)$-approximation space and let $(U, \mathcal{CS}_\Psi(U;\kappa))$ be an $\mathcal{CS}_\Psi(U;\kappa)$-approximation space. If $\iota \geq \kappa$ and $\Theta \subseteq \Psi$ where Θ is reflexive and Ψ is transitive, then we have $\overline{\Theta}(X;\iota) \subseteq \overline{\Psi}(X;\kappa)$ for every non-empty subset X of U.*

Proof. Let X be a non-empty subset of U. Then we prove that $\overline{\Theta}(X;\iota) \subseteq \overline{\Psi}(X;\kappa)$. In fact, let $u_1 \in \overline{\Theta}(X;\iota)$. Then $CS_\Theta(u_1;\iota) \cap X \neq \emptyset$. Thus there exists $u_2 \in CS_\Theta(u_1;\iota) \cap X$, and so $S_\Theta(u_1;\iota) = S_\Theta(u_2;\iota)$. Since Θ is reflexive, we have $\Theta(u_2,u_2) = 1 \geq \iota$. Whence $u_2 \in S_\Theta(u_2;\iota) = S_\Theta(u_1;\iota)$. Thus we have $\Theta(u_1,u_2) \geq \iota$. Since $\iota \geq \kappa$ and $\Theta \subseteq \Psi$, we have $\Psi(u_1,u_2) \geq \Theta(u_1,u_2) \geq \kappa$, and so $\Psi(u_1,u_2) \geq \kappa$. Similary, we have $\Psi(u_2,u_1) \geq \kappa$. We shall verify that $S_\Psi(u_1;\kappa) = S_\Psi(u_2;\kappa)$. Now, let $u_3 \in S_\Psi(u_2;\kappa)$. Then $\Psi(u_2,u_3) \geq \kappa$. Since Ψ is transitive, we have

$$\begin{aligned} \Psi(u_1,u_3) &\geq \vee_{u_4 \in U} (\Psi(u_1,u_4) \wedge \Psi(u_4,u_3)) \\ &\geq \Psi(u_1,u_2) \wedge \Psi(u_2,u_3) \\ &\geq \kappa \wedge \kappa \\ &= \kappa. \end{aligned}$$

Hence $\Psi(u_1,u_3) \geq \kappa$. Thus $u_3 \in S_\Psi(u_1;\kappa)$, which yields $S_\Psi(u_2;\kappa) \subseteq S_\Psi(u_1;\kappa)$. Similary, we can prove that $S_\Psi(u_1;\kappa) \subseteq S_\Psi(u_2;\kappa)$. Whence we get $S_\Psi(u_1;\kappa) = S_\Psi(u_2;\kappa)$, and so $u_2 \in CS_\Psi(u_1;\kappa)$. Thus we have that

$u_2 \in CS_\Psi(u_1;\kappa) \cap X$. Hence $CS_\Psi(u_1;\kappa) \cap X \neq \emptyset$, which yields $u_1 \in \overline{\Psi}(X;\kappa)$. Therefore we get that $\overline{\Theta}(X;\iota) \subseteq \overline{\Psi}(X;\kappa)$. □

Proposition 3.16. *Let $(U, \mathcal{CS}_\Theta(U;\iota))$ be an $\mathcal{CS}_\Theta(U;\iota)$-approximation space and let $(U, \mathcal{CS}_\Psi(U;\kappa))$ be an $\mathcal{CS}_\Psi(U;\kappa)$-approximation space. If $\iota \geq \kappa$ and $\Theta \subseteq \Psi$ where Θ is reflexive and Ψ is transitive, then we have $\underline{\Psi}(X;\kappa) \subseteq \underline{\Theta}(X;\iota)$ for every non-empty subset X of U.*

Proof. Let X be a non-empty subset of U. Then we prove that $\underline{\Psi}(X;\kappa) \subseteq \underline{\Theta}(X;\iota)$. Indeed, let $u_1 \in \underline{\Psi}(X;\kappa)$. Then $CS_\Psi(u_1;\iota) \subseteq X$. We shall show that $CS_\Theta(u_1;\iota) \subseteq CS_\Psi(u_1;\kappa)$. Let $u_2 \in CS_\Theta(u_1;\iota)$. Then we have $S_\Theta(u_1;\iota) = S_\Theta(u_2;\iota)$. Since Θ is reflexive, we have that $\Theta(u_1,u_1) = 1 \geq \iota$. Hence $u_1 \in S_\Theta(u_1;\iota)$, and so $u_1 \in S_\Theta(u_2;\iota)$. Thus $\Theta(u_2,u_1) \geq \iota$. By the assumption, we have $\Psi(u_2,u_1) \geq \Theta(u_2,u_1) \geq \kappa$, and so $\Psi(u_2,u_1) \geq \kappa$. Similary, we get that $\Psi(u_1,u_2) \geq \kappa$. We shall prove that $S_\Psi(u_1;\kappa) = S_\Psi(u_2;\kappa)$. Let $u_3 \in S_\Psi(u_2;\kappa)$. Then $\Psi(u_2,u_3) \geq \kappa$. Since Ψ is transitive, we have

$$\begin{aligned}\Psi(u_1,u_3) &\geq \vee_{u_4 \in U}\,(\Psi(u_1,u_4) \wedge \Psi(u_4,u_3))\\ &\geq \Psi(u_1,u_2) \wedge \Psi(u_2,u_3)\\ &\geq \kappa \wedge \kappa\\ &= \kappa.\end{aligned}$$

Thus $\Psi(u_1,u_3) \geq \kappa$, and so $u_3 \in S_\Psi(u_1;\kappa)$. Hence $S_\Psi(u_2;\kappa) \subseteq S_\Psi(u_1;\kappa)$. Similary, we can prove that $S_\Psi(u_1;\kappa) \subseteq S_\Psi(u_2;\kappa)$, which yields $S_\Psi(u_1;\kappa) = S_\Psi(u_2;\kappa)$. Thus we have $u_2 \in CS_\Psi(u_1;\kappa)$, and so $CS_\Theta(u_1;\iota) \subseteq CS_\Psi(u_1;\kappa) \subseteq X$. Therefore $u_1 \in \underline{\Theta}(X;\iota)$. This means that $\underline{\Psi}(X;\kappa) \subseteq \underline{\Theta}(X;\iota)$. □

4 Roughness in semigroups

In this section, we propose the definition of compatible preorder fuzzy relations on semigroups. Then we introduce the roughness in semigroups induced by compatible preorder fuzzy relations. We provide sufficient conditions for them and give some interesting properties and examples.

Definition 4.1. Let Θ be a fuzzy relation on S. Θ is called a *compatible preorder fuzzy relation* if Θ is reflexive, transitive and compatible. An $\mathcal{CS}_\Theta(S;\iota)$-approximation space $(S, \mathcal{CS}_\Theta(S;\iota))$ is called an $\mathcal{CS}_\Theta(S;\iota)$-*approximation space type CPF* if Θ is a compatible preorder fuzzy relation.

Proposition 4.2. *If $(S, \mathcal{CS}_\Theta(S;\iota))$ is an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CPF, then*

$$(CS_\Theta(s_1;\iota))(CS_\Theta(s_2;\iota)) \subseteq CS_\Theta(s_1s_2;\iota)$$

for all $s_1, s_2 \in S$.

Proof. Let s_1, s_2 be two elements in S and let $s_3 \in (CS_\Theta(s_1;\iota))(CS_\Theta(s_2;\iota))$. Then there exist $s_4 \in CS_\Theta(s_1;\iota)$ and $s_5 \in CS_\Theta(s_2;\iota)$ such that $s_3 = s_4s_5$. Thus $S_\Theta(s_1;\iota) = S_\Theta(s_4;\iota)$ and $S_\Theta(s_2;\iota) = S_\Theta(s_5;\iota)$. Hence we get that $S_\Theta(s_1s_2;\iota) = S_\Theta(s_4s_5;\iota)$. Indeed, we suppose that $s_6 \in S_\Theta(s_4s_5;\iota)$. Then we have $\Theta(s_4s_5,s_6) \geq \iota$. Since Θ is reflexive, we have $\Theta(s_4,s_4) = \Theta(s_5,s_5) = 1 \geq \iota$, and so $s_4 \in S_\Theta(s_4;\iota)$ and $s_5 \in S_\Theta(s_5;\iota)$. Whence $s_4 \in S_\Theta(s_1;\iota)$ and $s_5 \in S_\Theta(s_2;\iota)$. Thus $\Theta(s_1,s_4) \geq \iota$ and $\Theta(s_2,s_5) \geq \iota$. Since Θ is transitive and compatible, we have

$$\begin{aligned}\Theta(s_1s_2,s_4s_5) &\geq \vee_{s_7 \in S}\,(\Theta(s_1s_2,s_7) \wedge \Theta(s_7,s_4s_5))\\ &\geq \Theta(s_1s_2,s_4s_2) \wedge \Theta(s_4s_2,s_4s_5)\\ &\geq \Theta(s_1,s_4) \wedge \Theta(s_2,s_5)\\ &\geq \iota \wedge \iota\\ &= \iota.\end{aligned}$$

Hence $\Theta(s_1s_2, s_4s_5) \geq \iota$. Since Θ is transitive, we have

$$\begin{aligned}\Theta(s_1s_2, s_6) &\geq \vee_{s_8 \in S} (\Theta(s_1s_2, s_8) \wedge \Theta(s_8, s_6)) \\ &\geq \Theta(s_1s_2, s_4s_5) \wedge \Theta(s_4s_5, s_6) \\ &\geq \iota \wedge \iota \\ &= \iota.\end{aligned}$$

Thus $\Theta(s_1s_2, s_6) \geq \iota$, and so $s_6 \in S_\Theta(s_1s_2; \iota)$. Hence $S_\Theta(s_4s_5; \iota) \subseteq S_\Theta(s_1s_2; \iota)$. Similarly, we can show that $S_\Theta(s_1s_2; \iota) \subseteq S_\Theta(s_4s_5; \iota)$. Thus $S_\Theta(s_1s_2; \iota) = S_\Theta(s_4s_5; \iota)$, which yields $s_3 \in CS_\Theta(s_1s_2; \iota)$. This implies that $(CS_\Theta(s_1; \iota))(CS_\Theta(s_2; \iota)) \subseteq CS_\Theta(s_1s_2; \iota)$. □

In the following, we give an example to illustrate that the property in Proposition 4.2 is indispensable.

Example 4.3. Let $S := \{s_1, s_2, s_3, s_4, s_5\}$ be a semigroup with multiplication rules defined by Table 1.

Table 1: The multiplication table on S

$\cdot$	s_1	s_2	s_3	s_4	s_5
s_1	s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_3	s_3	s_5
s_3	s_1	s_3	s_3	s_3	s_5
s_4	s_1	s_3	s_3	s_3	s_5
s_5	s_1	s_5	s_5	s_5	s_5

Define the membership grades of relationship between any two elements in S under the fuzzy relation Θ on S as the following.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Then it is easy to check that Θ is a compatible preorder fuzzy relation. For $\iota = 0.9$, successor classes of each elements in S with respect to 0.9-level under Θ are

$S_\Theta(s_1; 0.9) := \{s_1, s_3, s_5\}$,
$S_\Theta(s_2; 0.9) := \{s_2\}$,
$S_\Theta(s_3; 0.9) := \{s_3, s_5\}$,
$S_\Theta(s_4; 0.9) := \{s_4\}$ and
$S_\Theta(s_5; 0.9) := \{s_3, s_5\}$.

Hence cores of successor classes of each elements in S with respect to 0.9-level under Θ are

$CS_\Theta(s_1; 0.9) := \{s_1\}$,
$CS_\Theta(s_2; 0.9) := \{s_2\}$,
$CS_\Theta(s_3; 0.9) := \{s_3, s_5\}$,
$CS_\Theta(s_4; 0.9) := \{s_4\}$ and
$CS_\Theta(s_5; 0.9) := \{s_3, s_5\}$.

Here it is straightforward to verify that $(CS_\Theta(s; 0.9))(CS_\Theta(s'; 0.9)) \subseteq CS_\Theta(ss'; 0.9)$ for all $s, s' \in S$.

Observe that, in Example 4.3, it does not hold in general for the equality case. Now, we consider the following example.

Example 4.4. Let $S := \{s_1, s_2, s_3, s_4, s_5\}$ be a semigroup with multiplication rules defined by Table 2.

Table 2: The multiplication table on S

$\cdot$	s_1	s_2	s_3	s_4	s_5
s_1	s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_2	s_2	s_2	s_5
s_3	s_1	s_2	s_3	s_2	s_5
s_4	s_1	s_2	s_2	s_4	s_5
s_5	s_1	s_5	s_5	s_5	s_5

Define the membership grades of relationship between any two elements in S under the fuzzy relation Θ on S as the following.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then it is easy to check that Θ is a compatible preorder fuzzy relation. For $\iota = 0.9$, successor classes of each elements in S with respect to 0.9-level under Θ are

$S_\Theta(s_1; 0.9) := \{s_1, s_5\}$,
$S_\Theta(s_2; 0.9) := \{s_2, s_3, s_4\}$,
$S_\Theta(s_3; 0.9) := \{s_2, s_3, s_4\}$,
$S_\Theta(s_4; 0.9) := \{s_2, s_3, s_4\}$ and
$S_\Theta(s_5; 0.9) := \{s_5\}$.

Hence cores of successor classes of each elements in S with respect to 0.9-level under Θ are

$CS_\Theta(s_1; 0.9) := \{s_1\}$,
$CS_\Theta(s_2; 0.9) := \{s_2, s_3, s_4\}$,
$CS_\Theta(s_3; 0.9) := \{s_2, s_3, s_4\}$,
$CS_\Theta(s_4; 0.9) := \{s_2, s_3, s_4\}$ and
$CS_\Theta(s_5; 0.9) := \{s_5\}$.

Here it is straightforward to check that $(CS_\Theta(s; 0.9))(CS_\Theta(s'; 0.9)) = CS_\Theta(ss'; 0.9)$ for all $s, s' \in S$. Based on this point, the property can be considered as a special case of Proposition 4.2. This example leads to the following definition.

Definition 4.5. Let $(S, \mathcal{CS}_\Theta(S; \iota))$ be an $\mathcal{CS}_\Theta(S; \iota)$-approximation space type CPF. The collection $\mathcal{CS}_\Theta(S; \iota)$ is called *complete induced by* Θ (briefly, Θ*-complete*) if for all $s_1, s_2 \in S$,

$$(CS_\Theta(s_1; \iota))(CS_\Theta(s_2; \iota)) = CS_\Theta(s_1 s_2; \iota).$$

Definition 4.6. Let $(S, \mathcal{CS}_\Theta(S; \iota))$ be an $\mathcal{CS}_\Theta(S; \iota)$-approximation space type CPF. If $\mathcal{CS}_\Theta(S; \iota)$ is complete induced by Θ, then Θ is called a *complete fuzzy relation*. $(S, \mathcal{CS}_\Theta(S; \iota))$ is called an $\mathcal{CS}_\Theta(S; \iota)$*-approximation space type CF* if Θ is complete.

Proposition 4.7. *If* $(S, \mathcal{CS}_\Theta(S; \iota))$ *is an* $\mathcal{CS}_\Theta(S; \iota)$*-approximation space type CPF, then*

$$(\overline{\Theta}(X; \iota))(\overline{\Theta}(Y; \iota)) \subseteq \overline{\Theta}(XY; \iota),$$

for every non-empty subsets X, Y *of* S.

Proof. Let X and Y be two non-empty subsets of S. Suppose that $s_1 \in (\overline{\Theta}(X;\iota))(\overline{\Theta}(Y;\iota))$. Then there exist $s_2 \in \overline{\Theta}(X;\iota)$ and $s_3 \in \overline{\Theta}(Y;\iota)$ such that $s_1 = s_2s_3$. Thus we have that $CS_\Theta(s_2;\iota) \cap X \neq \emptyset$ and $CS_\Theta(s_3;\iota) \cap Y \neq \emptyset$. Then there exist $s_4, s_5 \in S$ such that $s_4 \in CS_\Theta(s_2;\iota) \cap X$ and $s_5 \in CS_\Theta(s_3;\iota) \cap Y$. From Proposition 4.2, it follows that $s_4s_5 \in (CS_\Theta(s_2;\iota))(CS_\Theta(s_3;\iota)) \subseteq CS_\Theta(s_2s_3;\iota)$ and $s_4s_5 \in XY$. Thus $CS_\Theta(s_2s_3;\iota) \cap XY \neq \emptyset$, which yields $s_1 = s_2s_3 \in \overline{\Theta}(XY;\iota)$. Therefore $(\overline{\Theta}(X;\iota))(\overline{\Theta}(Y;\iota)) \subseteq \overline{\Theta}(XY;\iota)$. □

Proposition 4.8. *If $(S, \mathcal{CS}_\Theta(S;\iota))$ is an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF, then*

$$(\underline{\Theta}(X;\iota))(\underline{\Theta}(Y;\iota)) \subseteq \underline{\Theta}(XY;\iota),$$

for every non-empty subsets X, Y of S.

Proof. Let X and Y be two non-empty subsets of S and let $s_1 \in (\underline{\Theta}(X;\iota))(\underline{\Theta}(Y;\iota))$. Then there exist $s_2 \in \underline{\Theta}(X;\iota)$ and $s_3 \in \underline{\Theta}(Y;\iota)$ such that $s_1 = s_2s_3$, and so $CS_\Theta(s_2;\iota) \subseteq X$ and $CS_\Theta(s_3;\iota) \subseteq Y$. Since Θ is complete, we get $CS_\Theta(s_2s_3;\iota) = CS_\Theta(s_2;\iota)CS_\Theta(s_3;\iota) \subseteq XY$. Thus $CS_\Theta(s_2s_3;\iota) \subseteq XY$. Hence $s_1 = s_2s_3 \in \underline{\Theta}(XY;\iota)$. Therefore $(\underline{\Theta}(X;\iota))(\underline{\Theta}(Y;\iota)) \subseteq \underline{\Theta}(XY;\iota)$. □

We consider the following example.

Example 4.9. According to Example 4.4, suppose that $X := \{s_1, s_4, s_5\}$ is a subset of S. Then we have $\overline{\Theta}(X;\iota) = S$ and $\underline{\Theta}(X;\iota) := \{s_1, s_5\}$. Here it is easy to verify that $\overline{\Theta}(X;\iota)$ and $\underline{\Theta}(X;\iota)$ are subsemigroups, ideals and completely prime ideals of S. Moreover, we also have $\Theta_{bnd}(X;\iota)$ is a non-empty set. For the existence of subsemigroups, ideals and completely prime ideals of S under compatible preorder fuzzy relations in this example, we give the following definition.

Definition 4.10. Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CPF and let X be a non-empty subset of S. A non-empty $\mathcal{CS}_\Theta(S;\iota)$-upper approximation $\overline{\Theta}(X;\iota)$ of X in $(S, \mathcal{CS}_\Theta(S;\iota))$ is called an $\mathcal{CS}_\Theta(S;\iota)$-*upper approximation semigroup* if it is a subsemigroup of S. A non-empty $\mathcal{CS}_\Theta(S;\iota)$-lower approximation $\underline{\Theta}(X;\iota)$ of X in $(S, \mathcal{CS}_\Theta(S;\iota))$ is called a $\mathcal{CS}_\Theta(S;\iota)$-*lower approximation semigroup* if it is a subsemigroup of S. A non-empty $\mathcal{CS}_\Theta(S;\iota)$-rough set $\Theta\underline{R}(X;\iota)$ of X in $(S, \mathcal{CS}_\Theta(S;\iota))$ is called a $\mathcal{CS}_\Theta(S;\iota)$-*rough semigroup* if $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation semigroup and $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation semigroup.

Similarly, we can define $\mathcal{CS}_\Theta(S;\iota)$-rough (completely prime) ideals.

Theorem 4.11. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CPF. If X is a subsemigroup of S, then $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation semigroup.*

Proof. Suppose that X is a subsemigroup of S. Then $XX \subseteq X$. By Proposition 3.11 (3), we obtain that $\emptyset \neq X \subseteq \overline{\Theta}(X;\iota)$. Hence $\overline{\Theta}(X;\iota)$ is a non-empty $\mathcal{CS}_\Theta(S;\iota)$-upper approximation. From Proposition 3.11 (9), it follows that $\overline{\Theta}(XX;\iota) \subseteq \overline{\Theta}(X;\iota)$. By Proposition 4.7, we obtain that

$$(\overline{\Theta}(X;\iota))(\overline{\Theta}(X;\iota)) \subseteq \overline{\Theta}(XX;\iota) \subseteq \overline{\Theta}(X;\iota).$$

Hence $\overline{\Theta}(X;\iota)$ is a subsemigroup of S. Thus $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation semigroup. □

Theorem 4.12. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is a subsemigroup of S with $\underline{\Theta}(X;\iota) \neq \emptyset$, then $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation semigroup.*

Proof. Suppose that X is a subsemigroup of S. Then $XX \subseteq X$. Obviously, $\underline{\Theta}(X;\iota)$ is a non-empty $\mathcal{CS}_\Theta(S;\iota)$-lower approximation. From Proposition 3.11 (9), it follows that $\underline{\Theta}(XX;\iota) \subseteq \underline{\Theta}(X;\iota)$. By Proposition 4.8, we obtain that

$$(\underline{\Theta}(X;\iota))(\underline{\Theta}(X;\iota)) \subseteq \underline{\Theta}(XX;\iota) \subseteq \underline{\Theta}(X;\iota).$$

Thus $\underline{\Theta}(X;\iota)$ is a subsemigroup of S. Therefore $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation semigroup. □

The following corollary is an immediate consequence of Proposition 3.13, Theorem 4.11 and Theorem 4.12.

Corollary 4.13. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is a subsemigroup of S over a non-empty interior set, then $\Theta\underline{R}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-rough semigroup.*

Observe that, in Corollary 4.13, the converse is not true in general. We present an example as the following.

Example 4.14. According to Example 4.4, suppose that $X := \{s_3, s_4, s_5\}$ is a subset of S, then we have $\overline{\Theta}(X;0.9) := \{s_2, s_3, s_4, s_5\}$ and $\underline{\Theta}(X;0.9) := \{s_5\}$. Thus we see that $\Theta_{bnd}(X;0.9) \neq \emptyset$. Hence it is straightforward to check that $\overline{\Theta}(X;0.9)$ is an $\mathcal{CS}_\Theta(S;0.9)$-upper approximation semigroup and $\underline{\Theta}(X;0.9)$ is a $\mathcal{CS}_\Theta(S;0.9)$-lower approximation semigroup. However, X is not a subsemigroup of S. Consequently, $\Theta\underline{R}(X;0.9)$ is a $\mathcal{CS}_\Theta(S;0.9)$-rough semigroup, but X is not a subsemigroup of S.

Theorem 4.15. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CPF. If X is an ideal of S, then $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation ideal.*

Proof. Suppose that X is an ideal of S. Then $SX \subseteq X$. From Proposition 3.11 (9), it follows that $\overline{\Theta}(SX;\iota) \subseteq \overline{\Theta}(X;\iota)$. By Proposition 3.11 (1), we obtain that $\overline{\Theta}(S;\iota) = S$. From Proposition 4.7, it follows that

$$S(\overline{\Theta}(X;\iota)) = (\overline{\Theta}(S;\iota))(\overline{\Theta}(X;\iota)) \subseteq \overline{\Theta}(SX;\iota) \subseteq \overline{\Theta}(X;\iota).$$

Hence $\overline{\Theta}(X;\iota)$ is a left ideal of S.

Similarly, we can prove that $\overline{\Theta}(X;\iota)$ is a right ideal of S. Therefore we have $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation ideal. □

Theorem 4.16. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is an ideal of S with $\underline{\Theta}(X;\iota) \neq \emptyset$, then $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation ideal.*

Proof. Suppose that X is an ideal of S. Then $SX \subseteq X$. From Proposition 3.11 (9), it follows that $\underline{\Theta}(SX;\iota) \subseteq \underline{\Theta}(X;\iota)$. By Proposition 3.11 (1), we obtain that $\underline{\Theta}(S;\iota) = S$. From Proposition 4.8, it follows that

$$S(\underline{\Theta}(X;\iota)) = (\underline{\Theta}(S;\iota))(\underline{\Theta}(X;\iota)) \subseteq \underline{\Theta}(SX;\iota) \subseteq \underline{\Theta}(X;\iota).$$

Thus $\underline{\Theta}(X;\iota)$ is a left ideal of S.

Similarly, we can prove that $\underline{\Theta}(X;\iota)$ is a right ideal of S. Thus $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation ideal. □

The following corollary is an immediate consequence of Proposition 3.13, Theorem 4.15 and Theorem 4.16.

Corollary 4.17. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is an ideal of S over a non-empty interior set, then $\Theta\underline{R}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-rough ideal.*

Observe that, in Corollary 4.17, the converse is not true in general. We present an example as the following.

Example 4.18. According to Example 4.4, if $X := \{s_1, s_3, s_5\}$ is a subset of S, then we have $\overline{\Theta}(X;0.9) = S$ and $\underline{\Theta}(X;0.9) := \{s_1, s_5\}$. Thus we see that $\Theta_{bnd}(X;0.9) \neq \emptyset$. Obviously, $\overline{\Theta}(X;0.9)$ is an $\mathcal{CS}_\Theta(S;0.9)$-upper approximation ideal, and it is straightforward to check that $\underline{\Theta}(X;0.9)$ is a $\mathcal{CS}_\Theta(S;0.9)$-lower approximation ideal. However, X is not an ideal of S. Consequently, $\Theta\underline{R}(X;0.9)$ is a $\mathcal{CS}_\Theta(S;0.9)$-rough ideal, but X is not an ideal of S.

Theorem 4.19. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is a completely prime ideal of S, then $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal.*

Proof. We prove that $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal. In fact, since X is an ideal of S, by Theorem 4.15, we have that $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation ideal. Let $s_1, s_2 \in S$ such that $s_1 s_2 \in \overline{\Theta}(X;\iota)$. Then by the Θ-complete property of $\mathcal{CS}_\Theta(S;\iota)$, we get

$$(CS_\Theta(s_1;\iota))(CS_\Theta(s_2;\iota)) \cap X = CS_\Theta(s_1 s_2;\iota) \cap X \neq \emptyset.$$

Thus there exist $s_3 \in CS_\Theta(s_1;\iota)$ and $s_4 \in CS_\Theta(s_2;\iota)$ such that $s_3s_4 \in X$. Since X is a completely prime ideal, we have $s_3 \in X$ or $s_4 \in X$. Hence we have $CS_\Theta(s_1;\iota) \cap X \neq \emptyset$ or $CS_\Theta(s_2;\iota) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Theta}(X;\iota)$ or $s_2 \in \overline{\Theta}(X;\iota)$. Therefore $\overline{\Theta}(X;\iota)$ is a completely prime ideal of S. As a consequence, $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal. □

Theorem 4.20. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is a completely prime ideal of S with $\underline{\Theta}(X;\iota) \neq \emptyset$, then $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation completely prime ideal.*

Proof. Since X is an ideal of S, by Theorem 4.16, $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation ideal. Let $s_1, s_2 \in S$ such that $s_1s_2 \in \underline{\Theta}(X;\iota)$. Since Θ is complete, we have

$$(CS_\Theta(s_1;\iota))(CS_\Theta(s_2;\iota)) = CS_\Theta(s_1s_2;\iota) \subseteq X.$$

Now, we suppose that $s_1 \notin \underline{\Theta}(X;\iota)$. Then $CS_\Theta(s_1;\iota)$ is not a subset of X. Thus there exists $s_3 \in CS_\Theta(s_1;\iota)$ but $s_3 \notin X$. For each $s_4 \in CS_\Theta(s_2;\iota)$,

$$s_3s_4 \in (CS_\Theta(s_1;\iota))(CS_\Theta(s_2;\iota)) \subseteq X.$$

Whence $s_3s_4 \in X$. Since X is a completely prime ideal and $s_3 \notin X$, we have $s_4 \in X$. Thus $CS_\Theta(s_2;\iota) \subseteq X$, which yields $s_2 \in \underline{\Theta}(X;\iota)$. Hence we get $\underline{\Theta}(X;\iota)$ is a completely prime ideal of S. Therefore $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation completely prime ideal. □

The following corollary is an immediate consequence of Proposition 3.13, Theorem 4.19 and Theorem 4.20.

Corollary 4.21. *Let $(S, \mathcal{CS}_\Theta(S;\iota))$ be an $\mathcal{CS}_\Theta(S;\iota)$-approximation space type CF. If X is a completely prime ideal of S over a non-empty interior set, then $\Theta\underline{R}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-rough completely prime.*

Observe that, in Corollary 4.21, the converse is not true in general. We present an example as the following.

Example 4.22. According to Example 4.4, if $X := \{s_1, s_2, s_5\}$ is a subset of S, then we have $\overline{\Theta}(X;0.9) = S$ and $\underline{\Theta}(X;0.9) := \{s_1, s_5\}$. Thus we see that $\Theta_{bnd}(X;0.9) \neq \emptyset$. Obviously, $\overline{\Theta}(X;0.9)$ is an $\mathcal{CS}_\Theta(S;0.9)$-upper approximation completely prime ideal, and it is straightforward to check that $\underline{\Theta}(X;0.9)$ is a $\mathcal{CS}_\Theta(S;0.9)$-lower approximation completely prime ideal. Here we can verify that X is an ideal of S, but it is not a completely prime ideal of S since $s_3s_4 = s_2 \in X$ but $s_3 \notin X$ and $s_4 \notin X$. As a consequence, $\Theta\underline{R}(X;0.9)$ is a $\mathcal{CS}_\Theta(S;0.9)$-rough completely prime ideal, but X is not a completely prime ideal of S.

5 Homomorphic images of roughness in semigroups

In this section, we investigate the relationships between rough semigroups (resp. rough ideals, rough completely prime ideals) and their homomorphic images. Throughout this section, T denotes a semigroup.

Proposition 5.1. *Let f be an epimorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. Then the following statements hold.*
(1) For all $s_1, s_2 \in S$, $s_1 \in CS_\Theta(s_2;\iota)$ if and only if $f(s_1) \in CS_\Psi(f(s_2);\iota)$.
(2) $f(\overline{\Theta}(X;\iota)) = \overline{\Psi}(f(X);\iota)$ for every non-empty subset X of S.
(3) $f(\underline{\Theta}(X;\iota)) \subseteq \underline{\Psi}(f(X);\iota)$ for every non-empty subset X of S.
(4) If f is injective, then $f(\underline{\Theta}(X;\iota)) = \underline{\Psi}(f(X);\iota)$ for every non-empty subset X of S.
(5) If Ψ is a compatible preorder fuzzy relation, then Θ is a compatible preorder fuzzy relation.

Proof. (1) Let $s_1, s_2 \in S$ be such that $s_1 \in CS_\Theta(s_2;\iota)$. Then $f(s_1), f(s_2) \in T$ and $S_\Theta(s_1;\iota) = S_\Theta(s_2;\iota)$. In the following, we shall prove that $S_\Psi(f(s_1);\iota) = S_\Psi(f(s_2);\iota)$. Let $t_1 \in S_\Psi(f(s_1);\iota)$. Then $\Psi(f(s_1), t_1) \geq \iota$. Since f is surjective, there exists $s_3 \in S$ such that $f(s_3) = t_1$. Whence $\Psi(f(s_1), f(s_3)) \geq \iota$, and so $\Theta(s_1, s_3) \geq \iota$. Thus

$s_3 \in S_\Theta(s_1;\iota)$. Whence we have $s_3 \in S_\Theta(s_2;\iota)$. Hence $\Theta(s_2, s_3) \geq \iota$, and so $\Psi(f(s_2), f(s_3)) \geq \iota$. Thus $t_1 = f(s_3) \in S_\Psi(f(s_2);\iota)$. Then we have $S_\Psi(f(s_1);\iota) \subseteq S_\Psi(f(s_2);\iota)$. Similarly, we can show that $S_\Psi(f(s_2);\iota) \subseteq S_\Psi(f(s_1);\iota)$. Therefore $S_\Psi(f(s_1);\iota) = S_\Psi(f(s_2);\iota)$. As a consequence, $f(s_1) \in CS_\Psi(f(s_2);\iota)$.

Conversely, it is easy to verify that $s_1 \in CS_\Theta(s_2;\iota)$ whenever $f(s_1) \in CS_\Psi(f(s_2);\iota)$ for all $s_1, s_2 \in S$.

(2) Let X be a non-empty subset of S. We verify firstly that $f(\overline{\Theta}(X;\iota)) = \overline{\Psi}(f(X);\iota)$. Suppose that $t_1 \in f(\overline{\Theta}(X;\iota))$. Then there exists $s_1 \in \overline{\Theta}(X;\iota)$ such that $f(s_1) = t_1$. Therefore we have $CS_\Theta(s_1;\iota) \cap X \neq \emptyset$. Thus there exists $s_2 \in S$ such that $s_2 \in CS_\Theta(s_1;\iota)$ and $s_2 \in X$. By the argument (1), we obtain that $f(s_2) \in CS_\Psi(f(s_1);\iota)$ and $f(s_2) \in f(X)$. Then we have $CS_\Psi(f(s_1);\iota) \cap f(X) \neq \emptyset$, and so $t_1 = f(s_1) \in \overline{\Psi}(f(X);\iota)$. Thus we have $f(\overline{\Theta}(X;\iota)) \subseteq \overline{\Psi}(f(X);\iota)$.

On the other hand, let $t_2 \in \overline{\Psi}(f(X);\iota)$. Then there exists $s_3 \in S$ such that $f(s_3) = t_2$, and so $CS_\Psi(f(s_3);\iota) \cap f(X) \neq \emptyset$. Thus there exists $s_4 \in X$ such that $f(s_4) \in f(X)$ and $f(s_4) \in CS_\Psi(f(s_3);\iota)$. By the argument (1), we get that $s_4 \in CS_\Theta(s_3;\iota)$, and so we have $CS_\Theta(s_3;\iota) \cap X \neq \emptyset$. Hence $s_3 \in \overline{\Theta}(X;\iota)$, and so $t_2 = f(s_3) \in f(\overline{\Theta}(X;\iota))$. Thus we get $\overline{\Psi}(f(X);\iota) \subseteq f(\overline{\Theta}(X;\iota))$. This implies that $f(\overline{\Theta}(X;\iota)) = \overline{\Psi}(f(X);\iota)$.

(3) Let X be a non-empty subset of S. Let $t_1 \in f(\underline{\Theta}(X;\iota))$. Then there exists $s_1 \in \underline{\Theta}(X;\iota)$ such that $f(s_1) = t_1$. Thus we get $CS_\Theta(s_1;\iota) \subseteq X$. We shall prove that $CS_\Psi(t_1;\iota) \subseteq f(X)$. Let $t_2 \in CS_\Psi(t_1;\iota)$. Then there exist $s_2 \in S$ such that $f(s_2) = t_2$. Thus we have $f(s_2) \in CS_\Psi(f(s_1);\iota)$. By the argument (1), we obtain that $s_2 \in CS_\Theta(s_1;\iota)$, and so $s_2 \in X$. Hence we have $t_2 = f(s_2) \in f(X)$, and Thus $CS_\Psi(t_1;\iota) \subseteq f(X)$. Therefore we have $t_1 \in \underline{\Psi}(f(X);\iota)$. As a consequence, $f(\underline{\Theta}(X;\iota)) \subseteq \underline{\Psi}(f(X);\iota)$.

(4) Let X be a non-empty subset of S. We only need to prove that $\underline{\Psi}(f(X);\iota) \subseteq f(\underline{\Theta}(X;\iota))$. Suppose that $t_1 \in \underline{\Psi}(f(X);\iota)$. Then there exists $s_1 \in S$ such that $f(s_1) = t_1$. Thus we have $CS_\Psi(f(s_1);\iota) \subseteq f(X)$. We shall show that $CS_\Theta(s_1;\iota) \subseteq X$. Let $s_2 \in CS_\Theta(s_1;\iota)$. Then by the argument (1), we have $f(s_2) \in CS_\Psi(f(s_1);\iota)$. Hence $f(s_2) \in f(X)$. Thus there exists $s_3 \in X$ such that $f(s_3) = f(s_2)$. By the assumption, we have $s_2 \in X$, and so $CS_\Theta(s_1;\iota) \subseteq X$. Hence $s_1 \in \underline{\Theta}(X;\iota)$, and so $t_1 = f(s_1) \in f(\underline{\Theta}(X;\iota))$. Thus $\underline{\Psi}(f(X);\iota) \subseteq f(\underline{\Theta}(X;\iota))$.

By the argument (3), we get $f(\underline{\Theta}(X;\iota)) \subseteq \underline{\Psi}(f(X);\iota)$. Consequently, $f(\underline{\Theta}(X;\iota)) = \underline{\Psi}(f(X);\iota)$.

(5) The proof is straightforward, so we omit it. □

Proposition 5.2. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If Ψ is complete, then Θ is complete.*

Proof. Let s_1, s_2 be two elements in S and let $s_3 \in CS_\Theta(s_1s_2;\iota)$. Then by Proposition 5.1 (1), we get that $f(s_3) \in CS_\Psi(f(s_1s_2);\iota)$. Since f is a homomorphism and Ψ is complete, we have

$$f(s_3) \in CS_\Psi(f(s_1s_2);\iota) = CS_\Psi(f(s_1)f(s_2);\iota) = (CS_\Psi(f(s_1);\iota))(CS_\Psi(f(s_2);\iota)).$$

Thus there exist $t_1 \in CS_\Psi(f(s_1);\iota)$ and $t_2 \in CS_\Psi(f(s_2);\iota)$ such that $f(s_3) = t_1t_2$. Since f is surjective, there exist $s_4, s_5 \in S$ such that $f(s_4) = t_1$ and $f(s_5) = t_2$. From

$$f(s_4)f(s_5) = f(s_3) \in (CS_\Psi(f(s_1);\iota))(CS_\Psi(f(s_2);\iota)),$$

it follows that $f(s_4) \in CS_\Psi(f(s_1);\iota)$ and $f(s_5) \in CS_\Psi(f(s_2);\iota)$. By Proposition 5.1 (1), we obtain that $s_4 \in CS_\Theta(s_1;\iota)$ and $s_5 \in CS_\Theta(s_2;\iota)$. Since f is a homomorphism, we have $f(s_3) = f(s_4)f(s_5) = f(s_4s_5)$. Since f is injective, we get $s_3 = s_4s_5$. Thus we get that $s_3 \in CS_\Theta(s_1;\iota)CS_\Theta(s_2;\iota)$. Therefore we have $CS_\Theta(s_1s_2;\iota) \subseteq CS_\Theta(s_1;\iota)CS_\Theta(s_2;\iota)$.

On the other hand, by Proposition 4.2 and Proposition 5.1 (5), $CS_\Theta(s_1;\iota)CS_\Theta(s_2;\iota) \subseteq CS_\Theta(s_1s_2;\iota)$. Thus $CS_\Theta(s_1;\iota)CS_\Theta(s_2;\iota) = CS_\Theta(s_1s_2;\iota)$. Hence $\mathcal{CS}_\Theta(S;\iota)$ is Θ-complete. Therefore Θ is complete. □

Theorem 5.3. *Let f be an epimorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$ type CPF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation semigroup if and only if $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_\Psi(T;\iota)$-upper approximation semigroup.*

Proof. Suppose that $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation semigroup. Then by Proposition 5.1 (2),

$$(\overline{\Psi}(f(X);\iota))(\overline{\Psi}(f(X);\iota)) = (f(\overline{\Theta}(X;\iota)))(f(\overline{\Theta}(X;\iota)))$$

$$\begin{aligned}&=f((\overline{\Theta}(X;\iota))(\overline{\Theta}(X;\iota)))\\&\subseteq f(\overline{\Theta}(X;\iota))\\&=\overline{\Psi}(f(X);\iota).\end{aligned}$$

Hence $\overline{\Psi}(f(X);\iota)$ is a subsemigroup of T. Thus $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_{\Psi}(T;\iota)$-upper approximation semigroup.

Conversely, let $s_1 \in (\overline{\Theta}(X;\iota))(\overline{\Theta}(X;\iota))$. From Proposition 5.1 (2), it follows that

$$\begin{aligned}f(s_1) \in &f((\overline{\Theta}(X;\iota))(\overline{\Theta}(X;\iota)))\\&=(f(\overline{\Theta}(X;\iota)))(f(\overline{\Theta}(X;\iota)))\\&=(\overline{\Psi}(f(X);\iota))(\overline{\Psi}(f(X);\iota))\\&\subseteq\overline{\Psi}(f(X);\iota)\\&=f(\overline{\Theta}(X;\iota)).\end{aligned}$$

Thus there exists $s_2 \in \overline{\Theta}(X;\iota)$ such that $f(s_1) = f(s_2)$. Hence we have $CS_{\Theta}(s_2;\iota) \cap X \neq \emptyset$. From Proposition 3.4 (1), it follows that $f(s_1) \in CS_{\Psi}(f(s_2);\iota)$. By Proposition 5.1 (1), we obtain that $s_1 \in CS_{\Theta}(s_2;\iota)$. From Proposition 3.4 (2), it follows that $CS_{\Theta}(s_1;\iota) = CS_{\Theta}(s_2;\iota)$. Thus we have $CS_{\Theta}(s_1;\iota) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Theta}(X;\iota)$. Hence we have that $(\overline{\Theta}(X;\iota))(\overline{\Theta}(X;\iota)) \subseteq \overline{\Theta}(X;\iota)$. Thus $\overline{\Theta}(X;\iota)$ is a subsemigroup of S. Therefore $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_{\Theta}(S;\iota)$-upper approximation semigroup. □

Theorem 5.4. *Let f be an isomorphism from S in $(S, \mathcal{CS}_{\Theta}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Psi}(T;\iota))$ type CPF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_{\Theta}(S;\iota)$-lower approximation semigroup if and only if $\underline{\Psi}(f(X);\iota)$ is a $\mathcal{CS}_{\Psi}(T;\iota)$-lower approximation semigroup.*

Proof. By Proposition 5.1 (4) and using the similar method in the proof of Theorem 5.3, we can prove that the statement holds. □

The following corollary is an immediate consequence of Theorems 5.3 and 5.4.

Corollary 5.5. *Let f be an isomorphism from S in $(S, \mathcal{CS}_{\Theta}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Psi}(T;\iota))$ type CPF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\Theta R(X;\iota)$ is a $\mathcal{CS}_{\Theta}(S;\iota)$-rough semigroup if and only if $\Psi R(f(X);\iota)$ is a $\mathcal{CS}_{\Psi}(T;\iota)$-rough semigroup.*

Theorem 5.6. *Let f be an epimorphism from S in $(S, \mathcal{CS}_{\Theta}(S;\iota))$ to T in $(T, \mathcal{CS}_{\Psi}(T;\iota))$ type CPF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_{\Theta}(S;\iota)$-upper approximation ideal if and only if $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_{\Psi}(T;\iota)$-upper approximation ideal.*

Proof. Suppose that $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_{\Theta}(S;\iota)$-upper approximation ideal. Then we have $S\overline{\Theta}(X;\iota) \subseteq \overline{\Theta}(X;\iota)$. Whence we have $f(S\overline{\Theta}(X;\iota)) \subseteq f(\overline{\Theta}(X;\iota))$. By Proposition 5.1 (2), we obtain that

$$T\overline{\Psi}(f(X);\iota) = f(S\overline{\Theta}(X;\iota)) \subseteq f(\overline{\Theta}(X;\iota)) = \overline{\Psi}(f(X);\iota).$$

Hence $\overline{\Psi}(f(X);\iota)$ is a left ideal of T. Similarly, we can prove that $\overline{\Psi}(f(X);\iota)$ is a right ideal of T. Thus $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_{\Psi}(T;\iota)$-upper approximation ideal.

Conversely, we suppose that $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_{\Psi}(T;\iota)$-upper approximation ideal. Then we have $T\overline{\Psi}(f(X);\iota) \subseteq \overline{\Psi}(f(X);\iota)$. Now, let $s_1 \in S\overline{\Theta}(X;\iota)$. From Proposition 5.1 (2), it follows that

$$f(s_1) \in f(S\overline{\Theta}(X;\iota)) = T\overline{\Psi}(f(X);\iota) \subseteq \overline{\Psi}(f(X);\iota) = f(\overline{\Theta}(X;\iota)).$$

Thus there exists $s_2 \in \overline{\Theta}(X;\iota)$ such that $f(s_1) = f(s_2)$, and so $CS_{\Theta}(s_2;\iota) \cap X \neq \emptyset$. By Proposition 3.4 (1), we obtain that $f(s_1) \in CS_{\Psi}(f(s_2);\iota)$. By Proposition 5.1 (1), we obtain $s_1 \in CS_{\Theta}(s_2;\iota)$. From Proposition 3.4 (2), it follows that $CS_{\Theta}(s_1;\iota) = CS_{\Theta}(s_2;\iota)$. Hence we have $CS_{\Theta}(s_1;\iota) \cap X \neq \emptyset$, and so $s_1 \in \overline{\Theta}(X;\iota)$. Thus $S\overline{\Theta}(X;\iota) \subseteq \overline{\Theta}(X;\iota)$. Whence $\overline{\Theta}(X;\iota)$ is a left ideal of S. Similarly, we can prove that $\overline{\Theta}(X;\iota)$ is a right ideal of S. Therefore $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_{\Theta}(S;\iota)$-upper approximation ideal. □

Theorem 5.7. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$ type CPF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation ideal if and only if $\underline{\Psi}(f(X);\iota)$ is a $\mathcal{CS}_\Psi(T;\iota)$-lower approximation ideal.*

Proof. By Proposition 5.1 (4) and using the similar method in the proof of Theorem 5.6, we can prove that the statement holds. □

The following corollary is an immediate consequence of Theorems 5.6 and 5.7.

Corollary 5.8. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$ type CPF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\Theta\dot{R}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-rough ideal if and only if $\Psi\dot{R}(f(X);\iota)$ is a $\mathcal{CS}_\Psi(T;\iota)$-rough ideal.*

Theorem 5.9. *Let f be an epimorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$ type CF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal if and only if $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_\Psi(T;\iota)$-upper approximation completely prime ideal.*

Proof. Assume that $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal. Let $t_1, t_2 \in T$ be such that $t_1t_2 \in \overline{\Psi}(f(X);\iota)$. Thus there exist $s_1, s_2 \in S$ such that $f(s_1) = t_1$ and $f(s_2) = t_2$. Hence we have $CS_\Psi(f(s_1)f(s_2);\iota) \cap f(X) \neq \emptyset$. Since Ψ is complete, we have

$$(CS_\Psi(f(s_1);\iota))(CS_\Psi(f(s_2);\iota)) \cap f(X) = CS_\Psi(f(s_1)f(s_2);\iota) \cap f(X) \neq \emptyset.$$

Then there exist $f(s_3) \in CS_\Psi(f(s_1);\iota)$ and $f(s_4) \in CS_\Psi(f(s_2);\iota)$ such that $f(s_3)f(s_4) \in f(X)$, and so $f(s_3s_4) \in f(X)$. Then there exists $s_5 \in X$ such that $f(s_3s_4) = f(s_5)$. By Proposition 5.1 (1), we obtain that $s_3 \in CS_\Theta(s_1;\iota)$ and $s_4 \in CS_\Theta(s_2;\iota)$. From Propositions 4.2 and 5.1 (5), we get that $s_3s_4 \in CS_\Theta(s_1s_2;\iota)$. By Proposition 3.4 (2), we obtain that $CS_\Theta(s_1s_2;\iota) = CS_\Theta(s_3s_4;\iota)$. Note that $f(s_3s_4) \in CS_\Psi(f(s_3s_4);\iota)$. Then $f(s_5) \in CS_\Psi(f(s_3s_4);\iota)$. By Proposition 5.1 (1), once again, we get that $s_5 \in CS_\Theta(s_3s_4;\iota) = CS_\Theta(s_1s_2;\iota)$. Thus $CS_\Theta(s_1s_2;\iota) \cap X \neq \emptyset$, and so $s_1s_2 \in \overline{\Theta}(X;\iota)$. Since $\overline{\Theta}(X;\iota)$ is a completely prime ideal of S, we have $s_1 \in \overline{\Theta}(X;\iota)$ or $s_2 \in \overline{\Theta}(X;\iota)$. Hence we have $f(s_1) \in f(\overline{\Theta}(X;\iota))$ or $f(s_2) \in f(\overline{\Theta}(X;\iota))$. From Proposition 5.1 (2), we get $f(s_1) \in \overline{\Psi}(f(X);\iota)$ or $f(s_2) \in \overline{\Psi}(f(X);\iota)$, which yields $t_1 \in \overline{\Psi}(f(X);\iota)$ or $t_2 \in \overline{\Psi}(f(X);\iota)$. Thus $\overline{\Psi}(f(X);\iota)$ is a completely prime ideal of T. Therefore $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_\Psi(T;\iota)$-upper approximation completely prime ideal.

Conversely, we suppose that $\overline{\Psi}(f(X);\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal. Now, let s_6, s_7 be elements in S such that $s_6s_7 \in \overline{\Theta}(X;\iota)$. Then $f(s_6s_7) \in f(\overline{\Theta}(X;\iota))$. By Proposition 5.1 (2), we obtain that

$$f(s_6)f(s_7) = f(s_6s_7) \in f(\overline{\Theta}(X;\iota)) = \overline{\Psi}(f(X);\iota).$$

Thus $f(s_6) \in \overline{\Psi}(f(X);\iota)$ or $f(s_7) \in \overline{\Psi}(f(X);\iota)$. Now, we consider the following two cases.

Case 1. If $f(s_6) \in \overline{\Psi}(f(X);\iota)$, then we have $f(s_6) \in f(\overline{\Theta}(X;\iota))$ since Proposition 5.1 (2). Thus there exists $s_8 \in \overline{\Theta}(X;\iota)$ such that $f(s_6) = f(s_8)$. Whence $CS_\Theta(s_8;\iota) \cap X \neq \emptyset$. By Proposition 3.4 (1), we obtain that $f(s_8) \in CS_\Psi(f(s_8);\iota)$. Thus $f(s_6) \in CS_\Psi(f(s_8);\iota)$. By Proposition 5.1 (1), we have $s_6 \in CS_\Theta(s_8;\iota)$. From Proposition 3.4 (2), it follows that $CS_\Theta(s_6;\iota) = CS_\Theta(s_8;\iota)$. Thus we have $CS_\Theta(s_6;\iota) \cap X \neq \emptyset$, and so $s_6 \in \overline{\Theta}(X;\iota)$.

Case 2. If $f(s_7) \in \overline{\Psi}(f(X);\iota)$, then $s_7 \in \overline{\Theta}(X;\iota)$ since the proof is similar to that the case above.

As a consequence, $\overline{\Theta}(X;\iota)$ is an $\mathcal{CS}_\Theta(S;\iota)$-upper approximation completely prime ideal. □

Theorem 5.10. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$ type CF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\underline{\Theta}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-lower approximation completely prime ideal if and only if $\underline{\Psi}(f(X);\iota)$ is a $\mathcal{CS}_\Psi(T;\iota)$-lower approximation completely prime ideal.*

Proof. By Proposition 5.1 (4) and using the similar method as in the proof of Theorem 5.9, we can prove that the statement holds. □

The following corollary is an immediate consequence of Theorems 5.9 and 5.10.

Corollary 5.11. *Let f be an isomorphism from S in $(S, \mathcal{CS}_\Theta(S;\iota))$ to T in $(T, \mathcal{CS}_\Psi(T;\iota))$ type CF, where Θ is defined by for all $s_1, s_2 \in S$, $\Theta(s_1, s_2) = \Psi(f(s_1), f(s_2))$. If X is a non-empty subset of S, then $\Theta\underline{R}(X;\iota)$ is a $\mathcal{CS}_\Theta(S;\iota)$-rough completely prime ideal if and only if $\Psi\underline{R}(f(X);\iota)$ is a $\mathcal{CS}_\Psi(T;\iota)$-rough completely prime ideal.*

6 Conclusions

In the present paper, we proposed rough sets in universal sets based on cores of successor classes with respect to level in closed unit intervals under fuzzy relations. Then we gave the real world example and proved some interesting properties. Based on this point, we gave a definition of a non-empty rough set in a universal set. Then we derived a sufficient condition of the such set. We introduced concepts of rough semigroups, rough ideals and rough completely prime ideals in semigroups under compatible preorder fuzzy relations. Then we derived sufficient conditions for them. We proved the relationships between rough semigroups (resp. rough ideals and rough completely prime ideals) and their homomorphic images.

Finally, we hope that the definitions and results of rough sets in universal sets and semigroup structures using fuzzy relations under mathematical principles in this paper may provide a powerful tool for assessment problems and decision problems in several fields with respect to informations and technology.

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