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ON SAINT-VENANT'S PRINCIPLE IN DYNAMIC LINEAR VISCOELASTICITY

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1. Introduction. Sternberg and Al-Khozaie [1] have shown that the von Mises-Sternberg version of Saint-Venant's principle as formulated in [2] is also valid for the linearized theory of quasi-static deformations of isotropic viscoelastic materials. Generalizations of the work of Toupin [3] to certain classes of isotropic viscoelastic materials were carried out by Edelstein [4] and Neapolitan and Edelstein [5]. The asymptotic behaviour of quasi-static solutions in a semi-infinite anisotropic viscoelastic cylinder was studied by Rionero and Chiriţă [6] by using appropriate cross-sectional measures. A comprehensive discussion of literature concerned with Saint-Venant's principle and further up-to-date developments can be found in [7] and [8].

As regards the dynamic theory of linear viscoelasticity we remark that there are no results of Saint-Venant type. Here we endeavour to fill the gap alluded to in the above. This is achieved by using an idea developed for linear elastodynamics in [9].

We emphasize here that the Saint-Venant's principle for classical elastodynamics was originally treated by Flavin and Knops [10] and by Flavin, Knops, and Payne [11, 12]. In fact, in treating special harmonic end loadings on a linear elastic beam, Flavin, Knops, and Payne [10, 11] succeeded in establishing exponential decay of activity away from the excited end provided that the exciting frequency is less than a certain critical frequency, while the investigations of [12] yield interesting inequalities that provide meaningful bounds at all times for the energy contained in that part of the beam whose minimum distance from the loaded end is z. The estimates depend upon z but are independent of time t.

The purpose of the present paper is to investigate the spatial decay of dynamic viscoelastic processes in an anisotropic viscoelastic body subject to nonzero boundary conditions only on a plane portion. To a dynamic viscoelastic process we associate an appropriate energetic measure and then establish a first-order differential inequality which, after integration, leads to a spatial decay estimate controlled by the factor $\exp\{-z/(ct)\}$, where c is a positive constant depending only on the relaxation tensor. The asymptotic behaviour of dynamic viscoelastic processes in a semi-infinite viscoelastic cylinder is also studied and a result of Phragmén-Lindelöf type is established.

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In Sec. 2 we state the basic equations of the dynamic theory of anisotropic linear viscoelasticity and establish two identities involving a dynamic viscoelastic process. In Sec. 3 we set down the relevant constitutive hypotheses and, on this basis, we establish a useful estimate fundamental to the subsequent analysis. Section 4 is devoted to Saint-Venant's principle in a bounded body, while Sec. 5 is concerned with the asymptotic behaviour of a dynamic viscoelastic process in a semi-infinite cylinder.

2. Viscoelastic processes. Let R be a closed, bounded, regular region in three-dimensional space whose boundary ∂R includes a plane portion S_0. Choose Cartesian coordinates x_1, x_2, x_3 so that S_0 lies in the plane $x_3 = 0$, and suppose that R lies in the half-space $x_3 > 0$. Let S_z be the intersection with R of a plane $x_3 = z$, and let

$$R_z = \{\mathbf{x} \in R : x_3 > z\}, \tag{2.1}$$

and let L be the maximum value of x_3 on R.

By an admissible state for R we mean (cf. [13]) an ordered triplet $[\overline{\mathbf{u}}, \overline{\mathbf{E}}, \overline{\mathbf{S}}]$ consisting of a vector displacement field $\overline{\mathbf{u}}$ and symmetric tensor strain and stress fields $\overline{\mathbf{E}}$ and $\overline{\mathbf{S}}$ defined on $\overline{R}$. An admissible process for R is an ordered triplet $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ whose value $[\mathbf{u}, \mathbf{E}, \mathbf{S}](t) = [\mathbf{u}(t), \mathbf{E}(t), \mathbf{S}(t)]$ is an admissible state at each time t.

We suppose that the region R is occupied by an anisotropic and homogeneous viscoelastic medium with relaxation tensor $\mathbf{G}$ and density ρ. We assume the absence of any body forces and that the body is at rest at all times $t < 0$.

Then, by a dynamic viscoelastic process for R corresponding to $\mathbf{G}$, ρ, and null body force, we mean (cf. [13]) an admissible process $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ for R such that

(i) the values of $\mathbf{u}, \mathbf{E}$, and $\mathbf{S}$ are continuous fields on $\overline{R}$;

(ii) the displacement $\mathbf{u}$ is a process of class C^2 whose value at each time t is a vector field of class C^2 on $\overline{R}$;

(iii) on R, the processes $\mathbf{u}, \mathbf{E}$, and $\mathbf{S}$ satisfy the equation of motion

$$\operatorname{div} \mathbf{S} = \rho \ddot{\mathbf{u}}, \tag{2.2}$$

the strain-displacement relation

$$\mathbf{E} = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^{\mathrm{T}}), \tag{2.3}$$

and the stress-strain relation

$$\mathbf{S}(t) = \mathbf{G}(0)[\mathbf{E}(t)] + \int_0^t \dot{\mathbf{G}}(t-\tau)[\mathbf{E}(\tau)]\, d\tau. \tag{2.4}$$

In the above relations a superposed dot denotes the derivative with respect to the time variable and div and ∇ represent the divergence and gradient operators, respectively. In what follows, when no confusion may occur, we suppress the dependence upon the spatial variable. The dot $(\cdot)$ will denote the appropriate inner product, while the norm of a vector or a second-order tensor will be defined by

$$|\mathbf{u}| \equiv \sqrt{\mathbf{u} \cdot \mathbf{u}}, \qquad |\mathbf{E}| \equiv \sqrt{\mathbf{E} \cdot \mathbf{E}}. \tag{2.5}$$

Given a dynamic viscoelastic process $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$, the corresponding surface traction $\mathbf{s}$ is defined at every regular point of $\partial R \times [0, t_0)$ by

$$\mathbf{s}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}, t)\mathbf{n}(\mathbf{x}), \tag{2.6}$$

where $\mathbf{n}(\mathbf{x})$ is the unit outward normal to ∂R at $\mathbf{x}$.

Further, we proceed to establish two identities useful to our subsequent analysis.

LEMMA 2.1. Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be a dynamic viscoelastic process for R corresponding to $\mathbf{G}, \rho$, and null body force. We assume that $\mathbf{G}$ is a symmetric tensor twice continuously differentiable on $[0, \infty)$. Then, we have

$$\begin{aligned}\int_0^t \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau)\, d\tau &= \frac{1}{2}\mathbf{E}(t) \cdot \mathbf{G}(t)[\mathbf{E}(t)] - \frac{1}{2}\mathbf{E}(0) \cdot \mathbf{G}(0)[\mathbf{E}(0)] \\ &\quad - \frac{1}{2}\int_0^t \mathbf{E}(\tau) \cdot \dot{\mathbf{G}}(\tau)[\mathbf{E}(\tau)]\, d\tau - \frac{1}{2}\int_0^t [\mathbf{E}(t) - \mathbf{E}(\tau)] \cdot \dot{\mathbf{G}}(t-\tau)[\mathbf{E}(t) - \mathbf{E}(\tau)]\, d\tau \\ &\quad + \frac{1}{4}\int_0^t\int_0^t [\mathbf{E}(s) - \mathbf{E}(\tau)] \cdot \ddot{\mathbf{G}}(|s-\tau|)[\mathbf{E}(s) - \mathbf{E}(\tau)]\, d\tau\, ds.\end{aligned} \tag{2.7}$$

Proof. By using an integration by parts and by means of the constitutive relation (2.4), we get

$$\begin{aligned}\int_0^t \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau)\, d\tau &= \mathbf{E}(t) \cdot \mathbf{S}(t) - \mathbf{E}(0) \cdot \mathbf{S}(0) - \int_0^t \mathbf{E}(\tau) \cdot \dot{\mathbf{S}}(\tau)\, d\tau \\ &= \frac{1}{2}\mathbf{E}(t) \cdot \mathbf{G}(0)[\mathbf{E}(t)] - \frac{1}{2}\mathbf{E}(0) \cdot \mathbf{G}(0)[\mathbf{E}(0)] + \mathbf{E}(t) \cdot \int_0^t \dot{\mathbf{G}}(t-\tau)[\mathbf{E}(\tau)]\, d\tau \\ &\quad - \int_0^t \mathbf{E}(\tau) \cdot \dot{\mathbf{G}}(0)[\mathbf{E}(\tau)]\, d\tau - \int_0^t\int_0^s \mathbf{E}(s) \cdot \ddot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, d\tau\, ds.\end{aligned} \tag{2.8}$$

Further, we notice that

$$\begin{aligned}\mathbf{E}(t) \cdot \int_0^t \dot{\mathbf{G}}(t-\tau)[\mathbf{E}(\tau)]\, d\tau &= -\frac{1}{2}\int_0^t [\mathbf{E}(t) - \mathbf{E}(\tau)] \cdot \dot{\mathbf{G}}(t-\tau)[\mathbf{E}(t) - \mathbf{E}(\tau)]\, d\tau \\ &\quad + \frac{1}{2}\int_0^t \mathbf{E}(\tau) \cdot \dot{\mathbf{G}}(t-\tau)[\mathbf{E}(\tau)]\, d\tau + \frac{1}{2}\mathbf{E}(t) \cdot [\mathbf{G}(t) - \mathbf{G}(0)][\mathbf{E}(t)],\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}-\int_0^t\int_0^s \mathbf{E}(s) \cdot \ddot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, d\tau\, ds &= -\frac{1}{2}\int_0^t\int_0^t \mathbf{E}(s) \cdot \ddot{\mathbf{G}}(|s-\tau|)[\mathbf{E}(\tau)]\, d\tau\, ds \\ &= \frac{1}{4}\int_0^t\int_0^t [\mathbf{E}(s) - \mathbf{E}(\tau)] \cdot \ddot{\mathbf{G}}(|s-\tau|)[\mathbf{E}(s) - \mathbf{E}(\tau)]\, d\tau\, ds \\ &\quad - \frac{1}{2}\int_0^t\int_0^t \mathbf{E}(s) \cdot \ddot{\mathbf{G}}(|s-\tau|)[\mathbf{E}(s)]\, d\tau\, ds.\end{aligned} \tag{2.10}$$

Moreover, for $0 \le s \le t$, we have

$$\int_0^t \ddot{\mathbf{G}}(|s-\tau|)\, d\tau = \int_0^s \ddot{\mathbf{G}}(s-\tau)\, d\tau + \int_s^t \ddot{\mathbf{G}}(\tau - s)\, d\tau = \dot{\mathbf{G}}(s) + \dot{\mathbf{G}}(t-s) - 2\dot{\mathbf{G}}(0). \tag{2.11}$$

If we substitute the relations (2.9)–(2.11) into the relation (2.8), we obtain (2.7) and the proof is complete.

LEMMA 2.2. Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be a dynamic viscoelastic process for R corresponding to $\mathbf{G}, \rho$, and null body force. We assume that $\mathbf{G}$ is a symmetric tensor, continuously differentiable with respect to the time variable on $[0, \infty)$. Then, for any positive number ε, we have

$$\begin{aligned}\int_0^t |\mathbf{S}(\tau)|^2\, d\tau = {}& \frac{1}{2\varepsilon^2}\int_0^t \mathbf{S}(\tau)\cdot[2\mathbf{G}(0)-\mathbf{G}(\tau)][\mathbf{S}(\tau)]\, d\tau \\ &+ \frac{\varepsilon^2}{2}\int_0^t \mathbf{E}(\tau)\cdot[2\mathbf{G}(0)-\mathbf{G}(t-\tau)][\mathbf{E}(\tau)]\, d\tau \\ &- \frac{1}{2}\int_0^t \left[\frac{1}{\varepsilon}\mathbf{S}(\tau)-\varepsilon\mathbf{E}(\tau)\right]\cdot\mathbf{G}(0)\left[\frac{1}{\varepsilon}\mathbf{S}(\tau)-\varepsilon\mathbf{E}(\tau)\right] d\tau \\ &+ \frac{1}{2}\int_0^t\int_0^s \left[\frac{1}{\varepsilon}\mathbf{S}(s)+\varepsilon\mathbf{E}(\tau)\right]\cdot\dot{\mathbf{G}}(s-\tau)\left[\frac{1}{\varepsilon}\mathbf{S}(s)+\varepsilon\mathbf{E}(\tau)\right] d\tau\, ds.\end{aligned} \tag{2.12}$$

Proof. If we take into account the relation (2.4), we can write

$$\int_0^t |\mathbf{S}(\tau)|^2\, d\tau = \int_0^t \mathbf{S}(\tau)\cdot\mathbf{G}(0)[\mathbf{E}(\tau)]\, d\tau + \int_0^t\int_0^s \mathbf{S}(s)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, d\tau\, ds. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}\int_0^t \mathbf{S}(\tau)\cdot\mathbf{G}(0)[\mathbf{E}(\tau)]\, d\tau = {}& -\frac{1}{2}\int_0^t \left[\frac{1}{\varepsilon}\mathbf{S}(\tau)-\varepsilon\mathbf{E}(\tau)\right]\cdot\mathbf{G}(0)\left[\frac{1}{\varepsilon}\mathbf{S}(\tau)-\varepsilon\mathbf{E}(\tau)\right] d\tau \\ &+ \frac{1}{2\varepsilon^2}\int_0^t \mathbf{S}(\tau)\cdot\mathbf{G}(0)[\mathbf{S}(\tau)]\, d\tau + \frac{\varepsilon^2}{2}\int_0^t \mathbf{E}(\tau)\cdot\mathbf{G}(0)[\mathbf{E}(\tau)]\, d\tau,\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\int_0^t\int_0^s \mathbf{S}(s)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, d\tau\, ds \\ &\quad = \frac{1}{2}\int_0^t\int_0^s \left[\frac{1}{\varepsilon}\mathbf{S}(s)+\varepsilon\mathbf{E}(\tau)\right]\cdot\dot{\mathbf{G}}(s-\tau)\left[\frac{1}{\varepsilon}\mathbf{S}(s)+\varepsilon\mathbf{E}(\tau)\right] d\tau\, ds \\ &\quad\quad - \frac{1}{2\varepsilon^2}\int_0^t\int_0^s \mathbf{S}(s)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{S}(s)]\, d\tau\, ds \\ &\quad\quad - \frac{\varepsilon^2}{2}\int_0^t\int_0^s \mathbf{E}(\tau)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, d\tau\, ds,\end{aligned} \tag{2.15}$$

where ε is an arbitrary positive number. Further, we notice that

$$-\int_0^t\int_0^s \mathbf{S}(s)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{S}(s)]\, d\tau\, ds = \int_0^t \mathbf{S}(\tau)\cdot[\mathbf{G}(0)-\mathbf{G}(\tau)][\mathbf{S}(\tau)]\, d\tau. \tag{2.16}$$

If we interchange the orders of integration, as we may, then we deduce

$$\begin{aligned}\int_0^t\int_0^s \mathbf{E}(\tau)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, d\tau\, ds &= \int_0^t\int_\tau^t \mathbf{E}(\tau)\cdot\dot{\mathbf{G}}(s-\tau)[\mathbf{E}(\tau)]\, ds\, d\tau \\ &= \int_0^t \mathbf{E}(\tau)\cdot[\mathbf{G}(t-\tau)-\mathbf{G}(0)][\mathbf{E}(\tau)]\, d\tau.\end{aligned} \tag{2.17}$$

Finally, we substitute the relations (2.14)–(2.17) into the relation (2.13). Thus, we obtain the relation (2.12) and the proof is complete.

3. Constitutive hypotheses and consequences. In this section we set down the basic constitutive hypotheses necessary to develop our analysis on Saint-Venant's principle in dynamic viscoelasticity. Thus, we assume once and for all that $\rho > 0$ and that the relaxation tensor $\mathbf{G}$ is symmetric and twice continuously differentiable on $[0, \infty)$ and satisfies the following hypotheses:

(H1) it is positive-definite in the sense that there exists a positive constant μ_m so that for any symmetric second-order tensor $\mathbf{A}$, we have

$$\mathbf{A} \cdot \mathbf{G}[\mathbf{A}] \geq \mu_m |\mathbf{A}|^2; \tag{3.1}$$

(H2) it is bounded above in the sense that there exists a positive constant μ_M so that

$$\mathbf{A} \cdot \mathbf{G}[\mathbf{A}] \leq \mu_M |\mathbf{A}|^2; \tag{3.2}$$

(H3) its first time derivative is nonpositive, i.e.,

$$\mathbf{A} \cdot \dot{\mathbf{G}}[\mathbf{A}] \leq 0; \tag{3.3}$$

(H4) its second time derivative is nonnegative, i.e.,

$$\mathbf{A} \cdot \ddot{\mathbf{G}}[\mathbf{A}] \geq 0. \tag{3.4}$$

We remark that under constitutive hypotheses like (H1) and (H3), Dafermos [14] studied the stability of a viscoelastic motion.

Now we can state and prove a result that constitutes the basis of our subsequent analysis.

THEOREM 3.1. Let $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be a dynamic viscoelastic process for R corresponding to $\mathbf{G}$, ρ, and null body force. We assume that the relaxation tensor $\mathbf{G}$ is symmetric and twice continuously differentiable on $[0, \infty)$ and satisfies the hypotheses (H1)–(H4). Then, we have

$$\int_0^t |\mathbf{S}(\tau)|^2 \, d\tau \leq 8\mu^* \left\{ \int_0^t \int_0^s \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau) \, d\tau \, ds + \frac{t}{2} \mathbf{E}(0) \cdot \mathbf{G}(0)[\mathbf{E}(0)] \right\}, \tag{3.5}$$

where

$$\mu^* = \frac{\mu_M^2}{\mu_m}. \tag{3.6}$$

Proof. On the basis of the hypotheses (H1), (H3), and (H4), from Lemma 2.1, it follows that

$$\int_0^t \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau) \, d\tau \geq \frac{1}{2} \mu_m |\mathbf{E}(t)|^2 - \frac{1}{2} \mathbf{E}(0) \cdot \mathbf{G}(0)[\mathbf{E}(0)]. \tag{3.7}$$

On the other hand, by means of the hypotheses (H1)–(H3) and Lemma 2.2, we deduce

$$\int_0^t |\mathbf{S}(\tau)|^2 \, d\tau \leq \frac{1}{\varepsilon^2} \mu_M \int_0^t |\mathbf{S}(\tau)|^2 \, d\tau + \varepsilon^2 \mu_M \int_0^t |\mathbf{E}(\tau)|^2 \, d\tau. \tag{3.8}$$

We now choose

$$\varepsilon = \sqrt{2\mu_M}, \tag{3.9}$$

so that (3.8) gives

$$\int_0^t |\mathbf{S}(\tau)|^2 \, d\tau \le 4\mu_M^2 \int_0^t |\mathbf{E}(\tau)|^2 \, d\tau. \tag{3.10}$$

By combining the estimates (3.7) and (3.10) and by taking into account the notation (3.6), we obtain the relation (3.5) and the proof is complete.

REMARK. Since $|\mathbf{n}| = 1$, from the relations (2.6) and (3.5) we get the following estimate:

$$\int_0^t |\mathbf{s}(\tau)|^2 \, d\tau \le 8\mu^* \left\{ \int_0^t \int_0^s \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau) \, d\tau \, ds + \frac{t}{2} \mathbf{E}(0) \cdot \mathbf{G}(0)[\mathbf{E}(0)] \right\}. \tag{3.11}$$

4. Saint-Venant's principle. Throughout this paper we consider the region R occupied by an anisotropic homogeneous linear viscoelastic material that is at rest at all times $t < 0$. For $t > 0$ the body is deformed under nonzero boundary loads only on the end face S_0. Therefore, the remainder of the boundary ∂R is stress-free, so that

$$\mathbf{s} = \mathbf{0} \quad \text{on } (\partial R \backslash S_0) \times [0, \infty). \tag{4.1}$$

No body-force acts. We assume null initial conditions so that we have

$$\mathbf{u}(\cdot, 0) = \mathbf{0}, \qquad \dot{\mathbf{u}}(\cdot, 0) = \mathbf{0} \quad \text{in } \overline{R}. \tag{4.2}$$

It must be emphasized that the initial boundary-value problem for R will be completed by the boundary conditions on the end face S_0. At this stage of our analysis we shall consider dynamic viscoelastic processes $[\mathbf{u}, \mathbf{E}, \mathbf{S}]$ satisfying only the boundary conditions (4.1) and the initial conditions (4.2). For such a dynamic viscoelastic process $\pi = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ we associate the following functional:

$$E_\pi(z, t) = \int_0^t \int_{R_z} \left\{ \frac{1}{2} \rho \dot{\mathbf{u}}(s) \cdot \dot{\mathbf{u}}(s) + \int_0^s \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau) \, d\tau \right\} dV \, ds, \quad 0 \le z \le L, \; t \ge 0, \tag{4.3}$$

where R_z is defined by the relation (2.1). In view of the inequality (3.7) and the initial conditions (4.2), it can be seen from (4.3) that $E_\pi(z, t)$ may be considered as an energetic measure of the dynamic viscoelastic process $\pi = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$.

For such a measure we can establish the following result of Saint-Venant type describing the spatial decay of end effects away from the loaded end face S_0 within the context of dynamic linear viscoelasticity.

THEOREM 4.1. Let $\pi = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be a dynamic viscoelastic process for R corresponding to $\mathbf{G}, \rho$, and the null body force and null initial conditions and to a surface traction field $\mathbf{s}$ that satisfies the condition (4.1). Further, we assume that ρ is strictly positive and that $\mathbf{G}$ is a symmetric tensor twice continuously differentiable on $[0, \infty)$ that satisfies the hypotheses (H1)–(H4). Then

$$E_\pi(z, t) \le E_\pi(0, t) \exp\left\{ -\frac{z}{ct} \right\}, \quad 0 \le z \le L, \; t \ge 0, \tag{4.4}$$

where

$$c = 2\sqrt{\frac{\mu^*}{\rho}} = \frac{2\mu_M}{\sqrt{\rho\mu_m}}. \tag{4.5}$$

Proof. Since $\pi = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ is a dynamic viscoelastic process, from (4.3) it follows that $E_\pi(z,t)$ is a continuously differentiable function with respect to the spatial variable z and, moreover, we have

$$\frac{dE_\pi}{dz}(z,t) = -\int_0^t \int_{S_z} \left\{ \frac{1}{2}\rho\dot{\mathbf{u}}(s)\cdot\dot{\mathbf{u}}(s) + \int_0^s \dot{\mathbf{E}}(\tau)\cdot\mathbf{S}(\tau)d\tau \right\} dA\, ds. \tag{4.6}$$

Further, we multiply the equation of motion (2.2) through by $\dot{\mathbf{u}}$ and integrate over $R_z \times [0,t]$. Then we use the divergence theorem and the relations (4.2), (2.3), (2.6), and (4.1) in order to obtain

$$\frac{1}{2}\int_{R_z} \rho\dot{\mathbf{u}}(t)\cdot\dot{\mathbf{u}}(t)\, dV + \int_0^t \int_{R_z} \dot{\mathbf{E}}(\tau)\cdot\mathbf{S}(\tau)\, dV\, d\tau = \int_0^t \int_{S_z} \dot{\mathbf{u}}(\tau)\cdot\mathbf{s}(\tau)\, dA\, d\tau. \tag{4.7}$$

In view of the relation (4.3), from (4.7) we deduce

$$E_\pi(z,t) = \int_0^t \int_0^s \int_{S_z} \dot{\mathbf{u}}(\tau)\cdot\mathbf{s}(\tau)\, dA\, d\tau\, ds. \tag{4.8}$$

Then, by the arithmetic-geometric mean inequality, the relation (4.8) leads to

$$E_\pi(z,t) \le \frac{t}{2}\int_0^t \int_{S_z} \left\{ \frac{1}{\alpha}\dot{\mathbf{u}}(\tau)\cdot\dot{\mathbf{u}}(\tau) + \alpha\mathbf{s}(\tau)\cdot\mathbf{s}(\tau) \right\} dA\, d\tau, \tag{4.9}$$

where α is a positive number that will be chosen later.

We next use the initial conditions (4.2) and the strain-displacement relation (2.3) to obtain

$$\mathbf{E}(0) = \mathbf{0} \quad \text{in } R. \tag{4.10}$$

Further, we use the estimate (3.11) into (4.9) and take into account (4.10) so that we get

$$E_\pi(z,t) \le \frac{t}{2}\int_0^t \int_{S_z} \left\{ \frac{2}{\alpha\rho}\frac{1}{2}\rho\dot{\mathbf{u}}(s)\cdot\dot{\mathbf{u}}(s) + 8\alpha\mu^* \int_0^s \dot{\mathbf{E}}(\tau)\cdot\mathbf{S}(\tau)\, d\tau \right\} dA\, ds. \tag{4.11}$$

We now equate the coefficients in the above integral, that is, we choose

$$\alpha = \frac{1}{2\sqrt{\rho\mu^*}}, \tag{4.12}$$

and, therefore, (4.11) gives

$$E_\pi(z,t) \le 2t\sqrt{\frac{\mu^*}{\rho}}\int_0^t \int_{S_z} \left\{ \frac{1}{2}\rho\dot{\mathbf{u}}(s)\cdot\dot{\mathbf{u}}(s) + \int_0^s \dot{\mathbf{E}}(\tau)\cdot\mathbf{S}(\tau)\, d\tau \right\} dA\, ds. \tag{4.13}$$

On the basis of the relations (4.5) and (4.6), the estimate (4.13) leads to the following first-order differential inequality:

$$ct\frac{dE_\pi}{dz}(z,t) + E_\pi(z,t) \leq 0. \tag{4.14}$$

By the integration of this inequality we deduce the estimate (4.4) and the proof is complete.

The remainder of this section is devoted to making estimate (4.4) more explicit. The discussion cannot be conducted simultaneously for all types of end boundary data, however, and so we consider only the case when the end face S_0 is subjected to tractions "self-equilibrated" at each instant. Therefore, we complete the initial boundary-value problem by assuming the end boundary conditions

$$\mathbf{s} = \mathbf{p} \quad \text{on } S_0 \times [0,\infty), \tag{4.15}$$

where $\mathbf{p}$ is a prescribed continuous vector field that satisfies

$$\int_{S_0} \mathbf{p}\, dA = \mathbf{0}, \qquad \int_{S_0} \mathbf{x} \times \mathbf{p}\, dA = \mathbf{0}. \tag{4.16}$$

Then we have

THEOREM 4.2. Suppose the hypotheses of Theorem 4.1 hold true. Moreover, we assume the surface traction $\mathbf{s}$ satisfies the boundary condition (4.15) and the traction $\mathbf{p}$ is continuously differentiable with respect to the time variable on $[0,\infty)$ and that it is "self-equilibrated" at each instant. Then

$$E_\pi(0,t) \leq 4b^{-1}\left\{\int_0^t \left(\int_0^s \int_{S_0} \dot{\mathbf{p}}^2(\tau)\, dA\, d\tau\right)^{1/2} ds\right\}^2, \tag{4.17}$$

where b is a positive constant depending only on R, S_0, and the relaxation tensor $\mathbf{G}$.

Proof. The integration over R of the equation of motion (2.2), followed by the use of the divergence theorem and the boundary conditions (4.1) and (4.15), (4.16) and the initial conditions (4.2) lead to

$$\int_R \mathbf{u}\, dV = \mathbf{0}, \qquad \int_R \mathbf{x} \times \mathbf{u}\, dV = \mathbf{0}. \tag{4.18}$$

On this basis and by using an argument similar to that of [15], we deduce that

$$b_0 \int_{S_0} \mathbf{u}(t) \cdot \mathbf{u}(t)\, dA \leq \int_R |\operatorname{sym} \nabla\mathbf{u}(t)|^2\, dV, \tag{4.19}$$

where b_0 is a positive constant depending only on R and S_0. Further, we use the relations (3.7), (4.10), and (4.19) in order to deduce

$$\int_0^t \int_{S_0} \mathbf{u}(\tau) \cdot \mathbf{u}(\tau)\, dA\, d\tau \leq b^{-1} \int_0^t \int_0^s \int_R \dot{\mathbf{E}}(\tau) \cdot \mathbf{S}(\tau)\, dV\, d\tau\, ds, \tag{4.20}$$

where $b = b_0\mu_m/2$.

On the other hand, from the relations (4.8) and (4.15), we get

$$E_\pi(0,t) = \int_0^t \int_0^s \int_{S_0} \dot{\mathbf{u}}(\tau) \cdot \mathbf{p}(\tau)\, dA\, d\tau\, ds. \tag{4.21}$$

By an integration by parts with respect to the time variable, we obtain

$$E_\pi(0,t) = \int_0^t \int_{S_0} \mathbf{u}(\tau) \cdot \mathbf{p}(\tau)\, dA\, d\tau - \int_0^t \int_0^s \int_{S_0} \mathbf{u}(\tau) \cdot \dot{\mathbf{p}}(\tau)\, dA\, d\tau\, ds, \tag{4.22}$$

where use was made of the initial conditions (4.2). By means of Schwarz's inequality, from (4.22) we deduce

$$\begin{aligned} E_\pi(0,t) \le \left(\int_0^t \int_{S_0} \mathbf{u}^2(\tau)\, dA\, d\tau\right)^{1/2} \\ \times \left\{ \left(\int_0^t \int_{S_0} \mathbf{p}^2(\tau)\, dA\, d\tau\right)^{1/2} + \int_0^t \left(\int_0^s \int_{S_0} \dot{\mathbf{p}}^2(\tau)\, dA\, d\tau\right)^{1/2} ds \right\}. \end{aligned} \tag{4.23}$$

Due to the compatibility between the boundary conditions (4.15) and the initial conditions (4.2), it follows that

$$\mathbf{p}(\cdot, 0) = \mathbf{0}, \tag{4.24}$$

so that we have

$$\begin{aligned} \frac{d}{dt}\left(\int_0^t \int_{S_0} \mathbf{p}(\tau) \cdot \mathbf{p}(\tau)\, dA\, d\tau\right) &= 2\int_0^t \int_{S_0} \mathbf{p}(\tau) \cdot \dot{\mathbf{p}}(\tau)\, dA\, d\tau \\ &\le 2\left(\int_0^t \int_{S_0} \mathbf{p}(\tau) \cdot \mathbf{p}(\tau)\, dA\, d\tau\right)^{1/2} \left(\int_0^t \int_{S_0} \dot{\mathbf{p}}(\tau) \cdot \dot{\mathbf{p}}(\tau)\, dA\, d\tau\right)^{1/2}. \end{aligned} \tag{4.25}$$

Thus, an integration leads to

$$\left(\int_0^t \int_{S_0} \mathbf{p}(\tau) \cdot \mathbf{p}(\tau)\, dA\, d\tau\right)^{1/2} \le \int_0^t \left(\int_0^s \int_{S_0} \dot{\mathbf{p}}(\tau) \cdot \dot{\mathbf{p}}(\tau)\, dA\, d\tau\right)^{1/2} ds. \tag{4.26}$$

Finally, we combine the relations (4.3), (4.20), (4.23), and (4.26) in order to obtain the relation (4.17). Thus, the proof is complete.

5. Asymptotic behaviour of dynamic viscoelastic processes in a semi-infinite cylinder. In this section we assume that R is the semi-infinite cylinder $S_0 \times [0, \infty)$. We use a device discovered by Knops [16] in order to examine aspects of the Saint-Venant's principle by means of first-order differential inequalities governing the plane cross-sectional measures. Integration of these inequalities shows that the plane cross-sectional measure either asymptotically grows faster than a certain increasing exponential function or asymptotically decays faster than a certain decreasing exponential function, for each fixed value of the time variable.

Given a dynamic viscoelastic process for R, $\pi = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$, we introduce the following cross-sectional functional

$$I_\pi(z,t) = \int_0^t \int_0^s \int_{S_z} \dot{\mathbf{u}}(\tau) \cdot \mathbf{s}(\tau)\, dA\, d\tau\, ds. \tag{5.1}$$

With these in mind we can state and prove the following result.

THEOREM 5.1. Let the cylindrical region R be occupied by an anisotropic and homogeneous viscoelastic medium for which the relaxation tensor $\mathbf{G}$ is symmetric and twice continuously differentiable on $[0,\infty)$ and satisfies the hypotheses (H1)–(H4). Let $\pi = [\mathbf{u}, \mathbf{E}, \mathbf{S}]$ be a dynamic viscoelastic process for R corresponding to $\mathbf{G}$, ρ, and the null body force and initial data and to a surface traction $\mathbf{s}$ that satisfies

$$\mathbf{s} = \mathbf{0} \quad \text{on } (\partial S_0 \times [0,\infty)) \times [0,\infty). \tag{5.2}$$

Then, for fixed $t > 0$, we have the following alternative:

1°. There exists $z_t \geq 0$ so that $I_\pi(z_t, t) > 0$ and then we have

$$I_\pi(z,t) \geq I_\pi(z_t,t) \exp\left\{\frac{z - z_t}{ct}\right\}, \quad z \geq z_t; \tag{5.3}$$

2°. $I_\pi(z,t) \leq 0$ for all $z \geq 0$, and then we have

$$-I_\pi(z,t) \leq -I_\pi(0,t) \exp\left\{-\frac{z}{ct}\right\}, \quad z \geq 0, \tag{5.4}$$

where the constant c is defined by (4.5).

Proof. We first differentiate the relation (5.1) with respect to z and then use the equation of motion (2.2), the divergence theorem, the boundary condition (5.2), and the initial conditions (4.2) in order to obtain

$$\frac{dI_\pi}{dz}(z,t) = \int_0^t \int_{S_z} \left\{\frac{1}{2}\rho\dot{\mathbf{u}}(s)\cdot\dot{\mathbf{u}}(s) + \int_0^s \dot{\mathbf{E}}(\tau)\cdot\mathbf{S}(\tau)d\tau\right\} dA\, ds. \tag{5.5}$$

Clearly, in view of the relations (4.2), (4.10), (3.7), and (5.5), it follows that $I_\pi(z,t)$ is a nondecreasing function of z on $[0,\infty)$ for all $t \in (0,\infty)$.

Now, we use the arithmetic-geometric mean inequality into the relation (5.1) so that, by means of the relations (3.11) and (4.10), we deduce

$$|I_\pi(z,t)| \leq ct \int_0^t \int_{S_z} \left\{\frac{1}{2}\rho\dot{\mathbf{u}}(s)\cdot\dot{\mathbf{u}}(s) + \int_0^s \dot{\mathbf{E}}(\tau)\cdot\mathbf{S}(\tau)\, d\tau\right\} dA\, ds. \tag{5.6}$$

From the relations (5.5) and (5.6), we get the following first-order differential inequality:

$$|I_\pi(z,t)| \leq ct\frac{dI_\pi}{dz}(z,t). \tag{5.7}$$

Since $I_\pi(z,t)$ is a nondecreasing function with respect to z on $[0,\infty)$, it follows that we have only the two possibilities specified within the statement of the theorem. Let us consider the first. Then the differential inequality (5.7) gives

$$ct\frac{dI_\pi}{dz}(z,t) - I_\pi(z,t) \geq 0, \quad z \geq z_t,\ t > 0,\ t \text{ fixed}. \tag{5.8}$$

By an integration of the differential inequality (5.8) we obtain the inequality (5.3) and the proof of 1° is complete.

Further, we consider the second case. Then the differential inequality (5.7) implies

$$ct\frac{dI_\pi}{dz}(z,t) + I_\pi(z,t) \geq 0, \quad z \geq 0,\ t > 0, \tag{5.9}$$

which leads to the spatial decay estimate (5.4), so that the proof of the theorem is complete.

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