Bol. Soc. Paran. Mat. (3s.) **v. 36** 4 (2018): 171–182.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v36i4.33105

On Schouten-van Kampen Connection in Sasakian Manifolds

Gopal Ghosh

ABSTRACT: We study the Schouten-van Kampen connection associated to a Sasakian structure. With the help of the Schouten-van Kampen connection we characterize sasakian manifolds and find certain curvature properties of this connection on Sasakian manifolds. Also we study Ricci solitons on a Sasakian manifold with respect to the Schouten-van Kampen connection. Finally, an illustrative example is given to verify some results.

Key Words: Sasakian manifolds, Schouten-van Kampen connection, η-Einstein manifold, Ricci tensor, Concircular curvature tensor, Ricci soliton.

Contents

2010 *Mathematics Subject Classification:* 53C05, 53C20, 53C25.
Submitted August 16, 2016. Published December 28, 2016

1. Introduction

The Schouten-van Kampen connection have been introduced in the third decade of last century for a study of non-holomorphic manifolds [3,4]. In 2006 Bejancu [5] study Schouten-van Kampen connection on Foliated manifolds. Recently Olszak [9] study Schouten-van Kampen connection on almost(para) contact metric structure and prove some interesting results. In 1960 Sasakian manifolds introduced by Sasaki [6], can be descibed as an odd-dimensional counterpart of Kähler manifolds. The notion of local symmetry of a Riemannian manifolds began with the work of Cartan [10]. The notion of locally symmetry of a Riemannian manifold has been weakend by many authers in several directions. As a weaker version of locally symmetry, in 1977 Takahashi [11] introduced the notion of local ϕ-symmetry on a Sasakian manifold. In this paper we are interested to study Schouten-van Kampen connection and find certain curvature properties of this connection on Sasakian manifold.
A Sasakian manifold is said to be η-Einstein if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where S is the Ricci tensor of type $(0,2)$ and a, b are smooth function. It is known that [2] in an η-Einstein Sasakian manifold the associated scalars are constant.

A transformation of a $(2n+1)$-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation ([27], [28]). A Concircular transformation is always a conformal transformation [28]. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformation that is the concircular geometry, is generalisation of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism [29]. An interesting invariant of a concircular transformation is the concircular curvature tensor $\mathcal{Z}$ with respect to the Levi-Civita connection. It is defined by ([28], [29])

$$\mathcal{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \tag{1.1}$$

where X, Y, Z are differentiable vector fields, R and r are curvature tensor and the scalar curvature with respect to the Levi-Civita connection respectively. A Riemannian manifold with vanishing concircular curvature tensor is of constant curvature. Thus, the concircular curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Definition 1.1. *If the conformal curvature tensor C satisfies [31] $C(X,Y)\xi = 0$ for all differentiable vector fields X, Y on the manifold, then the manifold is called ξ-conformally flat.*

Definition 1.2. *If the conformal curvature tensor C satisfies [31]*

$$\phi^2 C(X,Y)Z = 0, \tag{1.2}$$

for all differentiable vector fields X, Y, Z on the manifold, then the manifold is called ϕ-conformally flat.

In an analogous way we define ξ-concircularly flat and ϕ-concircularly flat Sasakian manifolds.

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton if ([23], [18])

$$(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0,$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a complete vector field on M and λ is a constant. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. For more details we refer to the reader ([24], [25]). Motivated by the above studies we study Sasakian manifolds admitting Schoten-van Kampen connection in the present paper.

This article is arranged as follows: Section 2 is a review of all the necessary background in Sasakian manifolds. In section 3, we obtain the expressions of the curvature tensor and Ricci tensor $\bar{R}$ and $\bar{S}$ with respect to the Schouten-van Kampen connection and then prove some results. Section 4 covers locally symmetric Sasakian manifolds with respect to the Schouten-van Kampen connection. Section 5 deals with ϕ-sectional curvature admitting Schouten-van Kampen connection. Section 6, is devoted to study Ricci semisymmetric Sasakian manifolds and we prove that locally ϕ-Ricci symmetry with respect to $\bar{\nabla}$ and ∇ are equivalent. Next it is shown that a Sasakian manifold admitting Schoten-van Kampen connection is ξ-concircularly flat if and only if the scalar curvature of the manifold vanishes. In this section we also prove that a Sasakian manifold admitting Schouten-van Kampen connection is ξ-concircularly flat if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection. Next in section 8, we prove that if a Sasakian manifold admits Ricci soliton with respect to the Schouten-van Kampen connection then the manifold is an η-Einstein manifold and the Ricci soliton is shrinking, steady or expanding according as $r > 0$, $r = 0$ or $r < 0$. Finaly, we construct an example of a 5-dimensional Sasakian manifold admitting Schouten-van Kampen connection to veryfy some results.

2. Sasakian manifolds

Let M be a $(2n+1)$-dimensional Sasakian manifold with the structure (ϕ, ξ, η, g), where ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is 1-form and g is a Riemannian metric.Then the following relations hold ([1], [7], [12], [13])

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \ g(X, \xi) = \eta(X), \tag{2.3}$$

for all vectors field X, Y.

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.4}$$

for all vectors field X, Y, where ∇ is the Levi-Civita connection of the Riemannian metric. From the above equation it follows that

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

$$(\nabla_X \eta)Y = g(X, \phi Y). \tag{2.6}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q satisfy ([8], [15], [22])

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.7}$$

$$S(X,\xi) = 2n\eta(X), \tag{2.8}$$

$$Q\xi = 2n\xi. \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n\eta(X)\eta(Y). \tag{2.10}$$

Sasakian manifolds have been studied many author such as Boyer [7], Tachibana [8], Tanno [14], Godliński et al [30], De et al [19], Mihai et al ([20], [21]) and many others.

3. Curvature tensor and Ricci tensor with respect to the Schouten-van Kampen connection

The Schouten-van Kampen connection $\bar{\nabla}$ is given by [9],

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi - (\nabla_X \eta)(Y)\xi, \tag{3.1}$$

for any X, Y tangent to M.
With the help of (2.5) and (2.6) , the above equation takes the form,

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X. \tag{3.2}$$

Putting $Y = \xi$ in (3.2) and using (2.1) we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + \eta(X)\xi - X. \tag{3.3}$$

Using (2.5) in (3.3) we get

$$\bar{\nabla}_X \xi = 0.$$

Let R and $\bar{R}$ denote the curvature tensor ∇ and $\bar{\nabla}$ respectively. Then

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.4}$$

Using (3.2) in (3.4) yields

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z - g(Y,\phi Z)X + g(X,\phi Z)\phi Y - \eta(Y)\eta(Z)X \\ &\quad + \eta(X)\eta(Z)Y. \end{aligned} \tag{3.5}$$

Using (2.7) and putting $Z = \xi$ in (3.5) we get

$$\bar{R}(X,Y)\xi = 0.$$

Taking inner product with W of (3.5),

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) - g(Y,\phi Z)g(\phi X,W) + \\ &\quad g(X,\phi Z)g(\phi Y,W) - \eta(Y)\eta(Z)g(X,W) \\ &\quad +\eta(X)\eta(Z)g(Y,W). \end{aligned} \tag{3.6}$$

Let $\{e_1, e_2, e_3, ..., e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = W = e_i$ in (3.6) and taking summation over i, $1 \leqslant i \leqslant (2n+1)$, we obtain

$$\bar{S}(Y,Z) = S(Y,Z) + g(Y,Z) - (2n+1)\eta(Y)\eta(Z), \tag{3.7}$$

where $\bar{S}$ and S are the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively.

Let $\bar{r}$ and r denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇ respectively. Let $\{e_1, e_2, e_3, ..., e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $Y = Z = e_i$ in (3.7) and taking summation over i, $1 \leqslant i \leqslant (2n+1)$, we have

$$\bar{r} = r. \tag{3.8}$$

Therefore we can state the following:

Proposition 3.1. *For a Sasakian manifold M admitting Schouten-van Kampen connection $\bar{\nabla}$*
(i) The curvature tensor $\bar{R}$ of $\bar{\nabla}$ is given by (3.5),
(ii) The Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is given by (3.7),
(iii) The scalar curvature $\bar{r}$ of $\bar{\nabla}$ is given by (3.8),
(iv) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,
(v) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,
(vi) The Ricci tensor $\bar{S}$ is symmetric.

Now suppose that the Sasakian manifold is Ricci flat with respect to the Schouten-van Kampen connection. Then from (3.7) we get

$$S(Y,Z) = -g(Y,Z) + (2n+1)\eta(X)\eta(Y).$$

This leads to the following:

Theorem 3.2. *The manifold M^{2n+1} is Ricci flat with respect to the Schouten-van Kampen connection if and only if M^{2n+1} is an η-Einstein manifold.*

4. Locally symmetreic Sasakian manifolds with respect to the Schouten-van Kampen connection

In this section, we consider locally symmetric Sasakian manifolds with respect to the Schouten-van Kampen connection $\bar{\nabla}$. We have the following theorem:

Theorem 4.1. *Let M be a locally symmetric Sasakian manifold with respect to the Schouten-van Kampen connection $\bar{\nabla}$. Then the manifold is an η-Einstein manifold with respect to the Schouten-van Kampen connection.*

Proof: Let M be a locally symmetric Sasakian manifold with respect to the Schouten-van Kampen connection $\bar{\nabla}$. Then $(\bar{\nabla}_X R)(Y,Z)W = 0$. So by a suitable contraction of this equation, we have

$$(\bar{\nabla}_X \bar{S})(Z,W) = \bar{\nabla}_X \bar{S}(Z,W) - \bar{S}(\bar{\nabla}_X Z, W) - \bar{S}(Z, \bar{\nabla}_X W) = 0.$$

Taking $W = \xi$ in the above equation yields

$$\bar{\nabla}_X \bar{S}(Z,\xi) - \bar{S}(\bar{\nabla}_X Z, \xi) - \bar{S}(Z, \bar{\nabla}_X \xi) = 0. \tag{4.1}$$

Using (3.3) and (3.7) in the (4.1) we obtain

$$\bar{S}(Z, \phi X) = 0. \tag{4.2}$$

Using (3.7) in (4.2) we get

$$S(\phi X, Y) = -g(\phi X, Y). \tag{4.3}$$

Putting $X = \phi X$ and using (2.1) in (4.3) we have

$$S(X,Y) - \eta(X) S(\xi, Y) = -g(X,Y) + \eta(X)\eta(Y). \tag{4.4}$$

Using (2.8) in (4.4) we obtain

$$S(X,Y) = -g(X,Y) + (2n+1)\eta(X)\eta(Y).$$

This completes the proof. □

5. ϕ-sectional curvature of Sasakian manifolds admitting Schouten-van Kampen connection

A plane section in M is called a ϕ-section if there exists a unit vector X in M orthogonal to ξ such that $\{X, \phi X\}$ is an orthonomal basis of the plane section. Then the sectional curvature $K(X, \phi X) = g(R(X, \phi X)\phi X, X)$ is called a ϕ-sectional curvature [1].
Putting $Y = Z = \phi X$ and $W = X$ in (3.6) we get

$$\begin{aligned} g(\bar{R}(X,\phi X)\phi X, X) &= g(R(X,\phi X)\phi X, X) - g(\phi X, \phi^2 X) g(\phi X, X) \\ &\quad + g(X, \phi^2 X) g(\phi^2 X, X) - \eta(Y)\eta(Z) g(X, W) \\ &\quad + \eta(X)\eta(Z) g(Y, W). \end{aligned} \tag{5.1}$$

With the help of (2.1) and (5.1) we obtain

$$g(\bar{R}(X,\phi X)\phi X,X)=g(R(X,\phi X)\phi X,X)+[g(X,X)]^2.$$

Thus we can state the following:

Theorem 5.1. *If the ϕ-sectional curvature of a Sasakian manifold is a constant c with respect to the Levi-Civita connection, then the ϕ-sectional curvature of the manifold with respect to the Schouten-van Kampen connection is $(c+1)$.*

6. Locally ϕ-Ricci symmetry

E. Boeckx, P. Buecken and L. Vanhecke [16] introduced the notion of ϕ-symmetry. In 2008, De and Sarkar [17] studied ϕ-Ricci symmetric Sasakian manifolds. Recently Ghosh [26] studied ϕ-Ricci symmetric almost Kenmotsu manifolds with nullity distribution. A Sasakian manifold M^{2n+1} is said to be ϕ-Ricci symmetric if the Ricci operator satisfies $\phi^2((\nabla_X Q)Y)=0$ for all vector fields X,Y in M and $S(X,Y)=g(QX,Y)$. If X,Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric. From (3.7) we can write

$$\bar{Q}Y=QY+Y-(2n+1)\eta(Y)\xi. \tag{6.1}$$

Now we have

$$(\bar{\nabla}_X\bar{Q})Y=\bar{\nabla}_X\bar{Q}Y-\bar{Q}(\bar{\nabla}_XY). \tag{6.2}$$

Using (3.1) and (6.1) in (6.2) the we get

$$\begin{aligned}(\bar{\nabla}_X\bar{Q})Y &= \bar{\nabla}_XQY+\bar{\nabla}_XY-(2n+1)(\nabla_X\eta(Y))\xi-(2n+1)\eta(Y)\bar{\nabla}_X\xi\\ &\quad -\bar{Q}(\nabla_XY)-\eta(Y)\bar{Q}(\phi X)-g(X,\phi Y)\bar{Q}\xi. \end{aligned}\tag{6.3}$$

Again using (2.6), (3.1) and (6.1) in (6.3) we get

$$\begin{aligned}(\bar{\nabla}_X\bar{Q})Y &= \bar{\nabla}_XQY-(2n+1)((\nabla_X\eta)(Y))\xi+\eta(QY)\phi X-g(\phi X,\phi Y)\xi\\ &\quad +g(X,\phi Y)\xi-\eta(Y)Q(\phi X). \end{aligned}\tag{6.4}$$

Considering X,Y,Z orthogonal to ξ and using (2.1), (3.1) from equation (6.4) it follows that

$$\phi^2(\bar{\nabla}_X\bar{Q})(Y)=\phi^2(\nabla_XQ)(Y).$$

Thus we have the following:

Theorem 6.1. *In a Sasakian manifold locally ϕ-Ricci symmetry with respect to the Schouten-van Kampen connection and the Levi-Civita connection are equivalent.*

7. ξ-Concircularly flat and ϕ-Concircularly flat Sasakian manifolds with respect to the Schouten Van-Kampen connection

In this section we study ξ-Concircularly flat Sasakian manifolds and ϕ-Concircularly flat Sasakian manifolds admitting Schouten-van Kampen connection. Using (1.1) and (3.1) we obtain

$$\begin{aligned}\bar{\mathcal{Z}}(X,Y)Z &= R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y] - g(Y,\phi Z)\phi X \\ &+ g(X,\phi Z)\phi Y - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y. \end{aligned} \tag{7.1}$$

Putting $Z = \xi$, the above equation yields

$$\bar{\mathcal{Z}}(X,Y)\xi = \frac{r}{2n(2n+1)}R(X,Y)\xi.$$

This leads to the following:

Theorem 7.1. *A Sasakian manifold admitting Schouten-van Kampen connection is ξ-concircularly flat if and only if the scalar curvature of the manifold vanishes.*

From (1.1) it follows that

$$\begin{aligned}g(\bar{\mathcal{Z}}(\phi X,\phi Y)\phi Z,\phi W) &= g(\bar{R}(\phi X,\phi Y)\phi Z,\phi W) \\ &- \frac{\bar{r}}{2n(2n+1)}[g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &- g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{7.2}$$

Suppose

$$g(\bar{\mathcal{Z}}(\phi X,\phi Y)\phi Z,\phi W) = 0.$$

Then from (7.2),

$$\begin{aligned}0 &= g(\bar{R}(\phi X,\phi Y)\phi Z,\phi W) - \frac{\bar{r}}{2n(2n+1)}[g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &- g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned} \tag{7.3}$$

Let $\{e_1, e_2, e_3, ..., e_{2n}, \xi\}$ be a local orthonormal basis of the tangent space at a point of the manifold M, then $\{\phi e_1, \phi e_2, \phi e_3, ..., \phi e_{2n}, \xi\}$ is also a local orthonormal basis of the manifold. Putting $X = W = e_i$ in (7.3) and summing over $i = 1$ to $2n$. We obtain

$$\begin{aligned}0 &= \sum_{i=1}^{2n} g(\bar{R}(\phi X,\phi Y)\phi Z,\phi W) - \frac{\bar{r}}{2n(2n+1)}\sum_{i=1}^{2n}[g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &- g(\phi X,\phi Z)g(\phi Y,\phi W)]. \end{aligned}$$

From the above equation it follows that,

$$\bar{S}(\phi Y,\phi Z) - \frac{\bar{r}(2n-1)}{2n(2n+1)}g(\phi Y,\phi Z) = 0. \tag{7.4}$$

Using (3.7) in (7.4) yields

$$S(Y,Z) = -(1-\frac{r(2n-1)}{2n(2n+1)})g(Y,Z) + [(1+2n)-\frac{r(2n-1)}{2n(2n+1)}]\eta(Y)\eta(Z). \tag{7.5}$$

Conversely let if S be of the form (7.5), then obviously

$$g(\bar{\mathcal{Z}}(\phi X,\phi Y)\phi Z,\phi W)=0.$$

Thus we can state:

Theorem 7.2. *A Sasakian manifold admitting Schouten-van Kampen connection is ϕ-concircularly flat if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

8. Ricci solitons

Suppose the Sasakian manifold admits Ricci solitons with respect to the connection $\bar{\nabla}$. Then

$$(\pounds_V g+2\bar{S}+2\lambda g)(X,Y)=0,$$

which implies that

$$g(\bar{\nabla}_X\xi,Y)+g(X,\bar{\nabla}_Y\xi)+2\bar{S}(X,Y)+2\lambda g(X,Y)=0. \tag{8.1}$$

Using (3.3) in (8.1) yields

$$\bar{S}(X,Y)+\lambda g(X,Y)=0. \tag{8.2}$$

Again using (3.7) in (8.2) we have

$$S(X,Y)+g(Y,Z)-(2n+1)\eta(Y)\eta(Z)+\lambda g(Y,Z)=0. \tag{8.3}$$

Putting $X=Y=e_i$ in (8.3), where $\{e_1,e_2,e_3,...,e_{2n+1}\}$ is a local orthonormal basis of the tangent space at a point of the manifold M and taking summation over i, $1\leqslant i\leqslant(2n+1)$ we have

$$\lambda=-\frac{r}{(2n+1)}.$$

Putting $\lambda=-\frac{r}{(2n+1)}$ in (8.3) we get

$$S(X,Y)=(\frac{2r}{2n+1}-1)g(X,Y)+(2n+1)\eta(X)\eta(Y).$$

Thus we have the following:

Theorem 8.1. *If a Sasakian manifold admits Ricci soliton with respect to the Schouten-van Kampen connection then the manifold is an η-Einstein manifold and the Ricci soliton is shrinking, steady or expanding according as $r>0$, $r=0$ or $r<0$.*

9. Example of a 5-dimensional Sasakian manifold admitting Schouten-van Kampen connection

Consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$.
We choose the vector fields
$e_1 = 2(y\frac{\partial}{\partial z} - \frac{\partial}{\partial x})$, $e_2 = 2\frac{\partial}{\partial y}$, $e_3 = -2\frac{\partial}{\partial z}$, $e_4 = 2(v\frac{\partial}{\partial z} - \frac{\partial}{\partial u})$, $e_5 = -2\frac{\partial}{\partial v}$,
which are linearly independent at each point of M.
Let g be the Riemannian metric defined by $g(e_i, e_j) = 0$, $i \neq j$, $i, j = 1, 2, 3, 4, 5$ and
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$
Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.
Let ϕ be the $(1, 1)$-tensor field defined by
$\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$, $\phi e_4 = -e_5$, $\phi e_5 = e_4$.
Using the linearity of ϕ and g, we have
$\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_5$ and $g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$, for any $U, Z \in \chi(M)$. Thus, for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.
Also we have
$[e_1, e_2] = 2e_3, [e_4, e_5] = -2e_3$ and $[e_i, e_j] = 0$ for others i, j.
The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{9.1}$$

Taking $e_3 = \xi$ and using Koszul's formula we get the following
$\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = e_3, \nabla_{e_1} e_3 = -e_2, \nabla_{e_1} e_4 = 0, \nabla_{e_1} e_5 = 0,$
$\nabla_{e_2} e_1 = -e_3, \nabla_{e_2} e_2 = 0, \nabla_{e_2} e_3 = e_1, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_5 = 0,$
$\nabla_{e_3} e_1 = -e_2, \nabla_{e_3} e_2 = e_1, \nabla_{e_3} e_3 = 0, \nabla_{e_3} e_4 = e_5, \nabla_{e_3} e_5 = -e_4,$
$\nabla_{e_4} e_1 = 0, \nabla_{e_4} e_2 = 0, \nabla_{e_4} e_3 = e_5, \nabla_{e_4} e_4 = 0, \nabla_{e_4} e_5 = -e_3,$
$\nabla_{e_5} e_1 = \nabla_{e_5} e_2 = 0, \nabla_{e_5} e_3 = -e_4, \nabla_{e_5} e_4 = -e_3, \nabla_{e_5} e_5 = 0.$
From the above results we see that (ϕ, ξ, η, g) structure satisfies the formula

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

where $\eta(e_3) = 1$. Hence $M(\phi, \xi, \eta, g)$ is a 5-dimensional Sasakian manifold.
Using the above relation in (3.2), we obtain
$\bar{\nabla}_{e_1} e_1 = 0, \bar{\nabla}_{e_1} e_2 = e_3, \bar{\nabla}_{e_1} e_3 = \bar{\nabla}_{e_1} e_4 = \bar{\nabla}_{e_1} e_5 = 0,$
$\bar{\nabla}_{e_2} e_1 = \bar{\nabla}_{e_2} e_2 = \bar{\nabla}_{e_2} e_3 = \bar{\nabla}_{e_2} e_4 = \bar{\nabla}_{e_2} e_5 = 0,$
$\bar{\nabla}_{e_3} e_1 = -e_2, \bar{\nabla}_{e_3} e_2 = e_1, \bar{\nabla}_{e_3} e_3 = \bar{\nabla}_{e_3} e_4 = 0, \bar{\nabla}_{e_3} e_5 = -e_4,$
$\bar{\nabla}_{e_4} e_1 = \bar{\nabla}_{e_4} e_2 = \bar{\nabla}_{e_4} e_3 = \bar{\nabla}_{e_4} e_4 = \bar{\nabla}_{e_4} e_5 = 0,$
$\bar{\nabla}_{e_5} e_1 = \bar{\nabla}_{e_5} e_2 = \bar{\nabla}_{e_5} e_3 = \bar{\nabla}_{e_5} e_4 = \bar{\nabla}_{e_5} e_5 = 0.$
By the above results, we can easily obtain the non-vanishing components of the curvature tensor with respect to the Levi-Civita connection are as follows:

$R(e_1,e_2)e_1 = 3e_2, R(e_1,e_3)e_1 = -e_3, R(e_2,e_4)e_1 = e_5, R(e_2,e_5)e_1 = -e_4,$
$R(e_4,e_5)e_1 = -2e_2, R(e_1,e_2)e_2 = -3e_1, R(e_1,e_4)e_2 = -e_5, R(e_1,e_5)e_2 = e_4,$
$R(e_2,e_3)e_2 = -e_3, R(e_4,e_5)e_2 = 2e_1, R(e_1,e_3)e_3 = e_1, R(e_2,e_3)e_1 = -e_3,$
$R(e_3,e_4)e_3 = -e_4, R(e_3,e_4)e_4 = e_3, R(e_2,e_5)e_4 = e_1, R(e_4,e_5)e_4 = 2e_5,$
$R(e_1,e_2)e_5 = -2e_4, R(e_1,e_4)e_5 = e_2, R(e_2,e_4)e_5 = -e_1, R(e_3,e_5)e_5 = e_3,$
$R(e_4,e_5)e_5 = -e_4.$
Now the component of the curvature tensor with respect to the Schouten-van Kampen connection are as follows:
$\bar{R}(e_1,e_2)e_1 = 2e_2, \bar{R}(e_1,e_2)e_2 = -2e_1, \bar{R}(e_1,e_2)e_4 = -2e_5,$
$\bar{R}(e_1,e_2)e_5 = 2e_4, \bar{R}(e_1,e_3)e_1 = -e_3, \bar{R}(e_4,e_5)e_5 = -2e_4,$
$\bar{R}(e_2,e_3)e_3 = e_2.$
With the help of the above results we get the Ricci tensor are as follows:
$S(e_1,e_1) = -2, S(e_2,e_2) = -3, S(e_3,e_3) = 4, S(e_4,e_4) = 0, S(e_5,e_5) = -1$, and
$\bar{S}(e_1,e_1) = -1, \bar{S}(e_2,e_2) = -2, \bar{S}(e_3,e_3) = 0, \bar{S}(e_4,e_4) = 1, \bar{S}(e_5,e_5) = 0.$
Therefore $r = \sum_{i=1}^{5} S(e_i,e_i) = -2$ and $\bar{r} = \sum_{i=1}^{5} \bar{S}(e_i,e_i) = -2$. Hence Proposition (3.1) is verified.

Acknowledgments

The author is thankful to the referees for their valuable suggestions in the improvement of the paper.

References

1. Blair, D. E., *Contact manifolds in Riemannian geomatry* , Lecture notes in math., springer-verlag, 509, (1976).
2. Blair, D. E., *Riemannian geomatry of contact and Sympletic Manifolds* , Progress in Math. Birkhüser, Boston, 203, (2002).
3. Schouten, J. A. and Van Kampen, E. R., *Zur Einbettungs-und Krümmungstheorie nichtholonomer Gebilde,* Math. Ann., 103, 752-783, (1930).
4. Vrănceanu, G., *Sur quelques points de la théorie des espaces non holonomes,* Bull. Fac. St. Cernăuti, 5, 177-205, (1931).
5. Bejancu, A., *Schouten-van Kampen and Vranceanu connections on Foliated manifolds,* Anale Ştintifice Ale Universitati."AL.I. CUZA" IASI, Tomul LII, Mathematica, 37-60, 2006.
6. Sasaki, S., *On differentiable manifolds with certain structures which are closely related to almost contact structure,* Tohoku Math J.,**12**,456-476, (1960).
7. Boyer, Charles P. and Galicki, K. *Sasakian geomatry* , Oxford Mathematical Monographs, Oxford University Press, Oxford, MR 2382957, Zbl 1155.53002. (2008).
8. Tachibana, S., *On harmonic tensors in compact Sasakian spaces,* Tohoku Math J.(2), 17, 271-284, (1965).
9. Olszak, Z., *The Schouten Van-Kampen affine connection adapted to an almost(para) contact metric structure,* Publications De'linstitut Mathematique, 94(108), 31-42, (2013).
10. Cartan, E., *Sur une classe remarquable d'espaces de Riemannian,* Bull. Soc. Math. France, 54, 214-264, (1926).
11. Takahasi, T., *Sasakian ϕ-symmetryc spaces,* Tohoku Math J., 29, 91-113, (1977).
12. Sasaki, S., *Lectures note on Almost contact manifolds, Part I*, Tohoku University, 1965.
13. Sasaki, S., *Lectures note on Almost contact manifolds, Part II*, Tohoku University, 1967.

14. Tanno, S., *Sasakian manifolds with constant ϕ-holomorphic sectional curvature,* Tohoku Math J., 23, 21-38.(1969).

15. Futaki, A., Ono, H. and Wang, G., *Transverse Kähler geometry of Sasakian manifolds and toric Sasaki Einstein manifolds,* J. Diff. Geom, 83, 585-636, (2009).

16. Boeckx. E., Buecken, P., Vanhecke, L., *ϕ-symmetric contact metric spaces,* Glasgow Math. J., 52, 97-112, (2005).

17. De, U. C., Sarkar, A., *ϕ-Ricci symmetric Sasakian manifolds,* Proceedings of the Jangjeon Mathematical Society, 11, 47-52, (2008).

18. Yildiz, A., de, U. C., Turan, M., *On 3-dimensional f-Kenmotsu manifolds and Ricci soliton,* Ukrainian J. Math.**65**(2013)

19. De, U. C., Gazi, A. K., Özgur, C., *On almost pseudo symmetries of Sasakian manifolds,* Math. Pannon., 19, 81-88, (2008).

20. Hasegawa, I., Mihai, I., *Contact CR-warped product submanifolds in Sasakian manifolds,* Geom. Dedicata, 102, 143-150, (2003).

21. Mihai, I., *Certain submanifolds of a Sasakian manifolds.* Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid,1995), 265-268, World Sci. Publ. River Edge, NJ, 1996.

22. Martelli, D., Sparks, J. and Yau, S. T., *Sasaki Einstein manifolds and volume minimisation,* Commun. Math. Phys., 280, 611-673, (2007).

23. Hamilton, R. S., *The Ricci flow on surfaces,* Contemp. Math., 71, (1988).

24. Ivey, T., *Ricci solitons on compact 3-manifolds,* Different. Geom. Appl., 3, 301-307, (1993).

25. Chow, B., Knopf, D., *The Ricci flow: An introduction,* Math. Surv. and Monogr., 110, (2004).

26. Ghosh, G., *On a semisymmetric non-metric connection in an almost Kenmotsu manifold with nullity distribution,* Ser. Math. Inform. , 31, 245-257, (2016).

27. Kuhnel, W., *Conformal transformations between Einstein spaces, conformal geometry (Bonn, 1985/1986)*, 105-146, ASpects Math., E12, Vieweg, Braunschweig, (1988).

28. Yano, K., *Concircular geometry I. Concircular transformations,* Proc. Imp. Acad. Tokyo, 16, 195-200, (1940).

29. Yano, K., Bochner, S., *Curvature and Betti numbers,* Annals of mathematics studies, 32 Princeton University Press, (1953).

30. Godliński, M., Kopczyński, W., Nurowski, P., *Locally Sasakian manifolds,* Class. Quantum Grav., 17 , (2000).

31. Zhen, G., Cabrerizo, J. L., Ferńndez, L. M., Ferńndez, M., *On ξ- conformally flat contact metric manifolds,* Indian J. Pure Appl. Math., 28, 725-734, (1997).

Gopal Ghosh,
Department of Mathematics,
Bangabasi Evening College,
19, Raj Kumar Chakraborty Sarani,
Kol- 700009, West Bengal, India.
E-mail address: ghoshgopal.pmath@gmail.com