On second order differential equations with highly oscillatory forcing terms

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We present a method to compute efficiently solutions of systems of ordinary differential equations that possess highly oscillatory forcing terms. This approach is based on asymptotic expansions in inverse powers of the oscillatory parameter, and features two fundamental advantages with respect to standard ODE solvers: firstly, the construction of the numerical solution is more efficient when the system is highly oscillatory, and secondly, the cost of the computation is essentially independent of the oscillatory parameter. Numerical examples are provided, motivated by problems in electronic engineering.

Keywords: Highly oscillatory problems, Ordinary differential equations, Modulated Fourier expansions, Numerical analysis

1. General setting

In this paper we are concerned with second order ordinary differential equations with highly oscillatory forcing terms. More explicitly, we are considering equations of the form

$$y''(t) - R(y(t))y'(t) + S(y(t)) = f_\omega(t), \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{1.1}$$

for $t \geq 0$, where the forcing term $f_\omega(t)$ can be expressed as a modulated Fourier expansion (MFE), that is

$$f_\omega(t) = \sum_{m=-\infty}^{\infty} \alpha_m(t) e^{im\omega t}, \tag{1.2}$$

We will further assume that $R(y)$ and $S(y)$ are analytic, which ensures the existence and uniqueness of the solution $y(t)$. This setting includes some differential equations with important applications, in particular the *Van der Pol* oscillator:

$$y''(t) - \mu \left[1 - y^2(t)\right] y'(t) + y(t) = f_\omega(t) \quad t \geq 0, \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{1.3}$$

where $f_\omega(t)$ is of the form (1.2) and $\mu > 0$ is given. The standard *forced Van der Pol* oscillator is given by

$$y''(t) - \mu \left[1 - y^2(t)\right] y'(t) + y(t) = A \sin \omega t, \qquad y(0) = y_0, \quad y'(0) = y_0',$$

which is clearly a special case of (1.3), with $\alpha_{-1} = -\alpha_1 = \mathrm{i}A/2$.

Another important example belonging to this type of differential equation is the *Duffing oscillator*:

$$y''(t) + ky'(t) + ay(t) + by(t)^3 = f_\omega(t), \qquad y(0) = y_0, \quad y'(0) = y_0', \tag{1.4}$$

where $d > 0$ is the damping constant, $b > 0$ corresponds to the so called hard spring case and $b < 0$ to the soft spring case.

Two particular examples of forcing terms are of importance in electronic engineering:

$$f_\omega(t) = c_1 \sin \omega_1 t, \qquad f_\omega(t) = c_1 \sin \omega_1 t + c_2 \sin \omega_2 t \sin \omega_1 t, \tag{1.5}$$

where $\omega_1 \gg \omega_2 \gg 1$ and $c_1, c_2 \neq 0$ are constants. The first example represents a simple sinusoidal signal (possibly highly oscillatory), whereas the second one corresponds to an AM modulated signal (if $c_1 = 0$ we have a double-sideband suppressed carrier AM Modulation). The presence of two different frequencies is motivated by the fact that RF communications circuits are marked by the presence of signals with widely varying time scales. In particular, for modulated signals low-frequency information (in this case given by ω_2) is superimposed on a high-frequency carrier (given by ω_1), so that aerials of practical dimensions can be employed. In addition, different signals can be modulated onto carriers of different frequencies, thereby enabling a large number of radio transmitters to transmit at the same time.

In both the Van der Pol and Duffing equations, if the forcing term $f_\omega(t)$ has period $T > 0$, the existence of a non-constant T-periodic solution to the forced equation is known, see for instance Farkas (1994). These two equations have been extensively studied, notably in the context of singular perturbation theory, where the damping parameter is supposed to be small, see for instance Jordan & Smith (2007) or the classical reference Bogoliubov & Mitropolsky (1961). Both equations have been widely used as well in the modelling of electronic circuits, for instance in Hilborn (2000), Pulch (2005) and Volos *et al.* (2007).

In this paper, we investigate the properties and computation of solutions of this type of equations when the forcing term is highly oscillatory, that is, when $\omega \gg 1$. Similarly to what happens in the case of linear systems with nonlinear highly oscillatory forcing terms (Condon *et al.* 2009a, 2009b; Condon *et al.* 2009), the oscillatory nature of the solution imposes a very small stepsize on standard numerical methods for ODEs, thereby rendering them exceedingly expensive.

Our approach is a combination of asymptotic and numerical techniques: asymptotic expansion in inverse powers of the oscillatory parameter ω provides a convenient and fast-converging representation of the solution (especially for large values of ω), while numerical discretization of *nonoscillatory* differential equations generates expansion coefficients in an efficient way.

In Sections 2 and 3 we present the construction of our asymptotic-numerical solvers, and the explicit derivation of the first few terms. As we shall see, the first few terms in our asymptotic expansion of the solution of the forced ODE preserve the bandwidth of the original input, which is important for efficiency issues. However, as we progress to higher order terms we will find that the bandwidth of the solution of the ODE increases, a phenomenon that we call *blossoming*. This is an unavoidable consequence of the nonlinearity of the differential equation, but we are able to quantify the increase in the number of frequencies in Section 4.

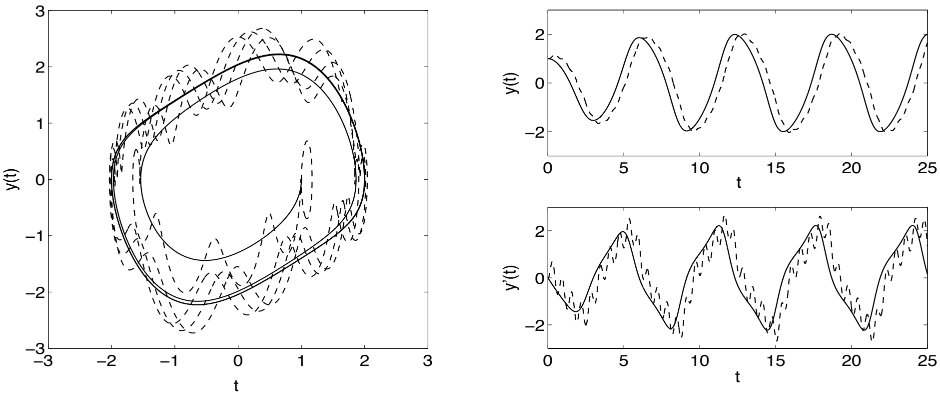

Figure 1. On the left, the limit cycles of the unforced (solid line) and forced (dashed line) van der Pol oscillators. On the right, the trajectories of the unforced (solid line) and forced (dashed line) van der Pol oscillator. Here $y(0) = 1$, $y'(0) = 1$, $\mu = \frac{1}{2}$, $\omega = 10$ and $\alpha_{-1} = 5\mathrm{i}/2$, $\alpha_1 = -5\mathrm{i}/2$, otherwise $\alpha_m = 0$.

Before we commence with the theory, we display in Fig.1 the limit cycle and the trajectories of the unforced (solid line) and forced (dashed line) oscillators. Looking at the limit cycle it is clear that forcing induces oscillations. However, a closer look at the trajectories indicates that these oscillations follow a pattern. While they are hardly visible in $y(t)$, the variable $y'(t)$ exhibits significant oscillations. This observation is critical to our analysis.

2. An asymptotic-numerical solver

We seek to represent $y(t)$ as an asymptotic series in inverse powers of the oscillatory parameter ω:

$$y(t) \sim \sum_{r=0}^{\infty} \frac{\psi_r(t)}{\omega^r}, \tag{2.1}$$

where each term $\psi_r(t)$ has the form of a modulated Fourier series,

$$\psi_r(t) = \sum_{m=-\infty}^{\infty} p_{r,m}(t)\mathrm{e}^{\mathrm{i}m\omega t}, \qquad r \geq 0. \tag{2.2}$$

As already explained in Condon *et al.* (2009b), modulated Fourier expansions provide a natural framework for solving differential equations with highly oscillatory forcing terms. This type of expansions have already been used in the context of Hamiltonian systems, see Hairer *et al.* (2006, Ch. XIII) and Cohen *et al.* (2003), and also as basic ingredient in the Heterogenous Multiscale Method, see Sanz-Serna (2009).

It is important to observe that we need to impose $p_{0,m}(t) \equiv 0$ and $p_{1,m}(t) \equiv 0$ for $m \neq 0$, since otherwise differentiation with respect to t would produce positive powers of ω, which are not present in the original equation (1.1). For this reason, we make the following *ansatz*:

$$y(t) \sim p_{0,0}(t) + \frac{1}{\omega} p_{1,0}(t) + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} p_{r,m}(t) \mathrm{e}^{\mathrm{i}m\omega t}. \tag{2.3}$$

Note also that in this setting the oscillations in $y(t)$ have amplitude of order $1/\omega^2$, whereas in $y'(t)$ they are of order $1/\omega$, consistently with the behaviour that can be observed in the previous example.

Following the theory presented in Condon *et al.* (2009), we proceed to expand the different terms in the ODE and identify those multiplying equal powers of ω. In order to do so, we first observe that term by term differentiation in (2.3) gives

$$\begin{aligned} y'(t) \sim {} & p'_{0,0}(t) + \frac{1}{\omega}\left[p'_{1,0}(t) + \mathrm{i} \sum_{m=-\infty}^{\infty} m p_{2,m}(t) \mathrm{e}^{\mathrm{i}m\omega t} \right] & (2.4) \\ & + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \left[p'_{r,m}(t) + \mathrm{i}m p_{r+1,m}(t) \right] \mathrm{e}^{\mathrm{i}m\omega t}, \\ y''(t) = {} & p''_{0,0}(t) - \sum_{m=-\infty}^{\infty} m^2 p_{2,m}(t) \mathrm{e}^{\mathrm{i}m\omega t} & (2.5) \\ & + \frac{1}{\omega}\left\{ p''_{1,0}(t) + \sum_{m=-\infty}^{\infty} \left[2\mathrm{i}m p'_{2,m}(t) - m^2 p_{3,m}(t) \right] \mathrm{e}^{\mathrm{i}m\omega t} \right\} \\ & + \sum_{r=2}^{\infty} \frac{1}{\omega^r} \sum_{m=-\infty}^{\infty} \left[p''_{r,m}(t) + 2\mathrm{i}m p'_{r+1,m}(t) - m^2 p_{r+2,m}(t) \right] \mathrm{e}^{\mathrm{i}m\omega t}. \end{aligned}$$

Since we have assumed that $R(y)$ and $S(y)$ are analytic, we can expand in Taylor series about the function $p_{0,0}(t)$:

$$R(y) \sim R(p_{0,0}) + \sum_{s=1}^{\infty} \frac{1}{\omega^s} \sum_{n=1}^{s} \frac{R^{(n)}(p_{0,0}(t))}{n!} \sum_{\boldsymbol{k} \in \mathbb{I}_{n,s}} \chi_{k_1} \cdots \chi_{k_n},$$

where

$$\chi_k(t) = \sum_{m=-\infty}^{\infty} p_{k,m}(t) \mathrm{e}^{\mathrm{i}m\omega t} \tag{2.6}$$

and

$$\mathbb{I}_{n,s} = \{(l_1, \ldots, l_n) \in \mathbb{N}^n \, : \, |\boldsymbol{l}| = s\},$$

with the standard notation for multi-indices $|\boldsymbol{l}| = l_1 + l_2 + \ldots + l_n$. A similar formula applies to $S(y)$.

For simplicity of notation, in the sequel we suppress the dependence on t of the different terms in the expansion.

(*a*) *Separation of orders of magnitude*

We now attempt to separate the $\mathcal{O}(\omega^{-r})$ term for $r \geq 0$ in the differential equation. Firstly, the contribution of $y''(t)$ is

$$\sum_{m=-\infty}^{\infty} (p''_{r,m} + 2\mathrm{i}mp'_{r+1,m} - m^2 p_{r+2,m})\mathrm{e}^{\mathrm{i}m\omega t}, \tag{2.7}$$

while the term $S(y)$ yields

$$S(y) \sim S(p_{0,0}) + \sum_{n=1}^{r} \frac{S^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \chi_{k_1} \cdots \chi_{k_n}. \tag{2.8}$$

The most complicated expression is given by $-R(y)y'$. Combining both expansions, we need to extract the $\mathcal{O}(\omega^{-r})$ term from the product

$$\begin{aligned}
&-\left[R(p_{0,0}) + \sum_{s=1}^{\infty} \frac{1}{\omega^s} \sum_{n=1}^{s} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \chi_{k_1} \cdots \chi_{k_n}\right] \\
&\times \left[p'_{0,0} + \frac{1}{\omega}\left(p'_{1,0} + \mathrm{i} \sum_{m=-\infty}^{\infty} mp_{2,m}\mathrm{e}^{\mathrm{i}m\omega t}\right) + \sum_{q=2}^{\infty} \frac{1}{\omega^q} \sum_{m=-\infty}^{\infty} (p'_{q,m} + \mathrm{i}mp_{q+1,m})\mathrm{e}^{\mathrm{i}m\omega t}\right].
\end{aligned}$$

The outcome is

$$\begin{aligned}
&- R(p_{0,0}) \sum_{m=-\infty}^{\infty} (p'_{r,m} + \mathrm{i}mp_{r+1,m})\mathrm{e}^{\mathrm{i}m\omega t} \\
&- p'_{0,0} \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \chi_{k_1} \cdots \chi_{k_n} \\
&- \left(p'_{1,0} + \mathrm{i} \sum_{m=-\infty}^{\infty} mp_{2,m}\mathrm{e}^{\mathrm{i}m\omega t}\right) \sum_{n=1}^{r-1} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r-1}} \chi_{k_1} \cdots \chi_{k_n} \\
&- \sum_{s=1}^{r-2} \sum_{n=1}^{s} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \chi_{k_1} \cdots \chi_{k_n} \sum_{m=-\infty}^{\infty} (p'_{r-s,m} + \mathrm{i}mp_{r-s+1,m})\mathrm{e}^{\mathrm{i}m\omega t}.
\end{aligned} \tag{2.9}$$

Note that the last term is nil for $r = 2$. Putting (2.7), (2.8) and (2.9) together, we obtain the whole contribution of the $\mathcal{O}(\omega^{-r})$ level.

(*b*) *Separation of frequencies*

Next we want to separate the different frequencies within each $\mathcal{O}(\omega^{-r})$ term. We first observe that

$$\begin{aligned}
\chi_{k_1} \cdots \chi_{k_n} &= \sum_{l_1=-\infty}^{\infty} \cdots \sum_{l_n=-\infty}^{\infty} p_{k_1,l_1} \cdots p_{k_n,l_n} \mathrm{e}^{\mathrm{i}(l_1+\cdots+l_n)\omega t} \\
&= \sum_{m=-\infty}^{\infty} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1} \cdots p_{k_n,l_n} \mathrm{e}^{\mathrm{i}m\omega t},
\end{aligned}$$

where

$$\mathbb{K}_{n,m} = \{(l_1, \ldots, l_n) \in \mathbb{Z}^n \,:\, |\boldsymbol{l}| = m\}.$$

Observe that, unlike the multi-indices in $\mathbb{I}_{n,s}$, in this case the components can also be nonpositive integers.

Likewise,

$$\begin{aligned}
&\chi_{k_1} \cdots \chi_{k_n} \sum_{j=-\infty}^{\infty} (p'_{r-s,j} + \mathrm{i} j p_{r-s+1,j}) \mathrm{e}^{\mathrm{i} j\omega t} \\
&= \sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,q}} p_{k_1,l_1} \cdots p_{k_n,l_n} [p'_{r-s,m-q} + \mathrm{i}(m-q) p_{r-s+1,m-q}] \mathrm{e}^{\mathrm{i} m\omega t} \\
&= \sum_{m=-\infty}^{\infty} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} p_{k_1,l_1} \cdots p_{k_n,l_n} (p'_{r-s,l_{n+1}} + \mathrm{i} l_{n+1} p_{r-s+1,l_{n+1}}) \mathrm{e}^{\mathrm{i} m\omega t}
\end{aligned}$$

and

$$\chi_{k_1} \cdots \chi_{k_n} \sum_{j=-\infty}^{\infty} j p_{2,j} \mathrm{e}^{\mathrm{i} j\omega t} = \sum_{m=-\infty}^{\infty} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_n,l_n} p_{2,l_{n+1}} \mathrm{e}^{\mathrm{i} m\omega t}.$$

Combining everything, we obtain

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty} (p''_{r,m} + 2\mathrm{i} m p'_{r+1,m} - m^2 p_{r+2,m}) \mathrm{e}^{\mathrm{i} m\omega t} \\
&- R(p_{0,0}) \sum_{m=-\infty}^{\infty} (p'_{r,m} + \mathrm{i} m p_{r+1,m}) \mathrm{e}^{\mathrm{i} m\omega t} \\
&- \sum_{m=-\infty}^{\infty} \left[p'_{0,0} A_r[R] + p'_{1,0} A_{r-1}[R] + \mathrm{i} B_r[R] + C_r[R] + D_r[R] - A_r[S]\right] \mathrm{e}^{\mathrm{i} m\omega t} = 0,
\end{aligned}$$

where

$$\begin{aligned}
A_r[f] &= \sum_{n=1}^{r} \frac{f^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1} \cdots p_{k_n,l_n}, \\
B_r[f] &= \sum_{n=1}^{r-1} \frac{f^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r-1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_n,l_n} p_{2,l_{n+1}}, \\
C_r[f] &= \sum_{s=1}^{r-2} \sum_{n=1}^{s} \frac{f^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} p_{k_1,l_1} \cdots p_{k_n,l_n} p'_{r-s,l_{n+1}}, \\
D_r[f] &= \mathrm{i} \sum_{s=1}^{r-2} \sum_{n=1}^{s} \frac{f^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}},
\end{aligned}$$

This can be somewhat simplified, because

$$
\begin{aligned}
&\mathrm{i}B_r[R] + D_r[R] \\
&= \mathrm{i}\sum_{n=1}^{r-1} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r-1} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1}\cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}} \\
&= \mathrm{i}\sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1}\cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}},
\end{aligned}
$$

the last line being justified by the fact that $l_{n+1}p_{1,l_{n+1}} = 0$. Moreover, since, for $j \in \{0,1\}$ we have $p'_{j,l_{n+1}} = 0$ (unless $l_{n+1} = 0$), we obtain

$$
\begin{aligned}
&p'_{0,0}A_r[R] + p'_{1,0}A_{r-1}[R] + C_r[R] \\
&= \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} p_{k_1,l_1}\cdots p_{k_n,l_n} p'_{r-s,l_{n+1}}.
\end{aligned}
$$

Therefore, separating frequencies, we have for every $m \in \mathbb{Z}$

$$
\begin{aligned}
&p''_{r,m} + 2\mathrm{i}mp'_{r+1,m} - m^2 p_{r+2,m} - R(p_{0,0})(p'_{r,m} + \mathrm{i}mp_{r+1,m}) \\
=&\sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} p_{k_1,l_1}\cdots p_{k_n,l_n} p'_{r-s,l_{n+1}} \\
&+\mathrm{i}\sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1}\cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}} \\
&-\sum_{n=1}^{r} \frac{S^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}.
\end{aligned}
\tag{2.10}
$$

Further simplification is possible, using the following results:

Proposition 2.1. *For every $r \geq 1$ and $m \in \mathbb{Z}$ it is true that*

$$
\begin{aligned}
&R(p_{0,0})p'_{r,m} + \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} p_{k_1,l_1}\cdots p_{k_n,l_n} p'_{r-s,l_{n+1}} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\right].
\end{aligned}
\tag{2.11}
$$

Proposition 2.2. *For every $r \geq 1$ and $m \in \mathbb{Z}$ it is true that*

$$
\begin{aligned}
&R(p_{0,0})mp_{r+1,m} + \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1}\cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}} \\
&= m\sum_{n=1}^{r+1} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}.
\end{aligned}
\tag{2.12}
$$

The proofs of both propositions are relegated to the Appendix.

If we substitute (2.11) and (2.12) into (2.10), the outcome is

$$
\begin{aligned}
&p''_{r,m} + 2\mathrm{i}mp'_{r+1,m} - m^2 p_{r+2,m} \qquad (2.13)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\right]\\
&+ \mathrm{i}m \sum_{n=1}^{r+1} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\\
&- \sum_{n=1}^{r} \frac{S^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}.
\end{aligned}
$$

An important observation is that in each step we obtain from (2.13) both an ODE for $p_{r,0}(t)$, which is *nonoscillatory* since there is no dependence on ω, and a recursion for $p_{r+2,m}(t)$, $m \neq 0$. More precisely, since $p_{r,0}(t)$ terms on the right feature only for $n = 1$, we have

$$
\begin{aligned}
\mathcal{L}[p_{r,0}] = &\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=2}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,0}} p_{k_1,l_1}\cdots p_{k_n,l_n}\right]\\
&- \sum_{n=2}^{r} \frac{S^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,0}} p_{k_1,l_1}\cdots p_{k_n,l_n},
\end{aligned} \qquad (2.14)
$$

where

$$
\mathcal{L}[y] = y'' - \frac{\mathrm{d}}{\mathrm{d}t}\left[R(p_{0,0})y\right] + S'(p_{0,0})y
$$

is the linearisation of the original ODE $y'' - R(y)y' + S(y) = 0$ about $y = p_{0,0}$.

Likewise, for $m \neq 0$, we have a recursion for the $p_{r,m}(t)$ terms,

$$
\begin{aligned}
m^2 p_{r+2,m} = p''_{r,m} + 2\mathrm{i}mp'_{r+1,m} &- \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\right]\\
&- \mathrm{i}m \sum_{n=1}^{r+1} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\\
&+ \sum_{n=1}^{r} \frac{S^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}.
\end{aligned} \qquad (2.15)
$$

The initial conditions for the ODE (2.14) are determined by imposing

$$
y(0) = p_{0,0}(0) = y_0, \qquad y'(0) = p'_{0,0}(0) = y'_0, \qquad (2.16)
$$

consequently the rest of the $p_{r,m}(t)$ coefficients, together with their derivatives, should be equal to 0 when $t = 0$, that is

$$p_{1,0}(0) = 0, \qquad p'_{1,0}(0) = -\mathrm{i} \sum_{m=-\infty}^{\infty} m p_{2,m}(0), \tag{2.17}$$

$$p_{r,0}(0) = -\sum_{m\neq 0} p_{r,m}(0), \quad r \geq 2, \tag{2.18}$$

$$p'_{r,0}(0) = -\sum_{m\neq 0} p'_{r,m}(0) - \mathrm{i} \sum_{m=-\infty}^{\infty} m p_{r+1,m}(0), \quad r \geq 2.$$

In the next section we present the first few terms of the expansion, computed using the differential equation and the recursion presented above.

3. Construction of the asymptotic expansion

(a) *The zeroth term*

The zeroth term, corresponding to $r = 0$, is readily available from the differential equation:

$$p''_{0,0} - R(p_{0,0})p'_{0,0} + S(p_{0,0}) = \alpha_0(t),$$

together with the initial conditions (2.16). It is also possible to show that

$$p_{2,m}(t) = -\frac{\alpha_m(t)}{m^2}, \qquad m \neq 0, \tag{3.1}$$

directly in terms of the modulated Fourier coefficients of the forcing term. We thus deduce that

$$p_{1,0}(0) = 0, \qquad p'_{1,0}(0) = \mathrm{i} \sum_{m\neq 0} \frac{\alpha_m(0)}{m}, \tag{3.2}$$

in accordance with (2.17). These initial conditions will be used to solve the ODE for $p_{1,0}(t)$, which is given by the analysis of the $\mathcal{O}(\omega^{-1})$ terms.

(b) *The first term*

We now look at the $\mathcal{O}\big(\omega^{-1}\big)$ terms and separate scales. We obtain a differential equation for $p_{1,0}(t)$:

$$\mathcal{L}[p_{1,0}] = p''_{1,0} - \frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{1,0}] + S'(p_{0,0})p_{1,0} = 0,$$

and a recursion for the next level:

$$p_{3,m} = \frac{\mathrm{i}}{m^3}[R(p_{0,0})\alpha_m - 2\alpha'_m], \qquad m \neq 0. \tag{3.3}$$

Moreover, it follows from (2.18) that

$$p_{2,0}(0) = -\sum_{m\neq 0} \frac{\alpha_m(0)}{m^2}, \qquad p'_{2,m}(0) = -\sum_{m\neq 0} \frac{1}{m^2}[\alpha'_m(0) - R(y_0)\alpha_m(0)]. \tag{3.4}$$

(c) The second term

When $r=2$ we need the sets

$$\mathbb{I}_{1,2}=\{(2)\},\quad \mathbb{I}_{2,2}=\{(1,1)\},$$
$$\mathbb{I}_{1,3}=\{(3)\},\quad \mathbb{I}_{2,3}=\{(1,2),(2,1)\},\quad \mathbb{I}_{3,3}=\{(1,1,1)\}.$$

The differential equation for $p_{2,0}(t)$ reads

$$\begin{aligned} & p_{2,0}''-\frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{2,0}]+S'(p_{0,0})p_{2,0} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[R'(p_{0,0})\sum_{k\in\mathbb{I}_{2,2}}\sum_{l\in\mathbb{K}_{2,0}} p_{k_1,l_1}p_{k_2,l_2}\right]-\frac{1}{2}S''(p_{0,0})\sum_{k\in\mathbb{I}_{2,2}}\sum_{l\in\mathbb{K}_{2,0}} p_{k_1,l_1}p_{k_2,l_2}. \end{aligned}$$

However, note that because $p_{1,m}(t)\equiv 0$ when $m\neq 0$, we have the simplification

$$\sum_{k\in\mathbb{I}_{2,2}}\sum_{l\in\mathbb{K}_{2,0}} p_{k_1,l_1}p_{k_2,l_2}=p_{1,0}^2(t),$$

therefore

$$p_{2,0}''-\frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{2,0}]+S'(p_{0,0})p_{2,0}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[R'(p_{0,0})p_{1,0}^2\right]-\frac{1}{2}S''(p_{0,0})p_{1,0}^2.$$

The recursion for $p_{4,m}(t)$, $m\neq 0$, reads

$$\begin{aligned} m^2p_{4,m} &= p_{2,m}''+2\mathrm{i}mp_{3,m}'-\frac{\mathrm{d}}{\mathrm{d}t}\left[R(p_{0,0})p_{2,m}+\frac{1}{2}R'(p_{0,0})\sum_{l=-\infty}^{\infty}p_{1,l}p_{1,m-l}\right] \\ &\quad -\mathrm{i}m\left\{R(p_{0,0})p_{3,m}+\tfrac{1}{2}R'(p_{0,0})\left[\sum_{l=-\infty}^{\infty}p_{1,l}p_{2,m-l}+\sum_{l=-\infty}^{\infty}p_{2,l}p_{1,m-l}\right]\right. \\ &\quad \left.+\frac{1}{6}R''(p_{0,0})\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}p_{1,l_1}p_{1,l_2}p_{1,m-l_1-l_2}\right\} \\ &\quad +S'(p_{0,0})p_{2,m}+\frac{1}{2}S''(p_{0,0})\sum_{l=-\infty}^{\infty}p_{1,l}p_{1,m-l}. \end{aligned}$$

Again, it is possible to simplify this expression, noting that

$$\sum_{l=-\infty}^{\infty}p_{1,l}p_{1,m-l}=\sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty}p_{1,l_1}p_{1,l_2}p_{1,m-l_1-l_2}=0$$

and

$$\sum_{l=-\infty}^{\infty}p_{1,l}p_{2,m-l}=\sum_{l=-\infty}^{\infty}p_{2,l}p_{1,m-l}=p_{1,0}p_{2,m}.$$

Therefore we obtain

$$\begin{aligned} p_{4,m} &= \frac{1}{m^2}\left\{p_{2,m}''+2\mathrm{i}mp_{3,m}'-\frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{2,m}]\right. \\ &- \left.\mathrm{i}m[R(p_{0,0})p_{3,m}+R'(p_{0,0})p_{1,0}p_{2,m}]+S'(p_{0,0})p_{2,m}\right\}. \end{aligned} \tag{3.5}$$

(d) The third term

There is an important reason to consider the case $r = 3$ in detail, and it is related to the blossoming phenomenon that we consider in the next section. With the same simplifications that we applied before, the ODE for $p_{3,0}(t)$ reads

$$\begin{aligned} & p''_{3,0} - \frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{3,0}] + S'(p_{0,0})p_{3,0} \\ & = \frac{\mathrm{d}}{\mathrm{d}t}\left[R'(p_{0,0})p_{1,0}p_{2,0} + \frac{1}{6}R''(p_{0,0})p_{1,0}^3\right] - \frac{1}{2}S''(p_{0,0})p_{1,0}p_{2,0} - \frac{1}{6}S'''(p_{0,0})p_{1,0}^3. \end{aligned}$$

The recursion for $p_{5,m}(t)$, $m \neq 0$, can be deduced from (2.15), setting $r = 3$. Again using the fact that $p_{1,m}(t) \equiv 0$ except when $m = 0$, the expressions can be considerably simplified to yield

$$\begin{aligned} p_{5,m} = \frac{1}{m^2}\Bigg\{ & p''_{3,m} + 2\mathrm{i}mp'_{4,m} - \frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{3,m} + R'(p_{0,0})p_{1,0}p_{2,m}] \\ & - \mathrm{i}m\left[R(p_{0,0})p_{4,m} + R'(p_{0,0})\left(p_{1,0}p_{3,m} + \frac{1}{2}\sum_{l=-\infty}^{\infty} p_{2,l}p_{2,m-l}\right)\right. \\ & \left. + \frac{1}{2}R''(p_{0,0})p_{1,0}^2p_{2,m}\right] + S(p_{0,0})p_{3,m} + S'(p_{0,0})p_{1,0}p_{2,m}\Bigg\} \end{aligned}$$

It is clear that the process can be continued, at the price of increasingly more complicated algebra. However, it is easy to derive more expansion terms using a symbolic algebra package, and it is worth noticing that in most important examples the functions $R(y)$ and $S(y)$ are quite elementary, and hence some terms in the previous expressions are identically 0.

4. Band-limited input and blossoming

The stage $r = 3$ in the previous computations has special significance. We note that the forcing terms in (1.5) share a common feature, namely that they are clearly *band limited*, since the number of frequencies is finite. In the case of the AM modulated signal, this follows from the elementary identities

$$\sin\omega_2 t \sin\omega_1 t = \frac{1}{2}\cos(\omega_2 - \omega_1)t - \frac{1}{2}\cos(\omega_2 + \omega_1)t$$

and

$$\sin\omega_2 t \cos\omega_1 t = \frac{1}{2}\sin(\omega_2 + \omega_1)t + \frac{1}{2}\sin(\omega_2 - \omega_1)t.$$

In this way, the spectrum of the forcing term will contain the carrier frequency ω_1 (unless we implement a suppressed carrier modulation) and the two sidebands $\omega_1 \pm \omega_2$, together with the negative frequencies $-\omega_1$ and $-\omega_1 \pm \omega_2$.

If the original oscillator is *band limited*, i.e., there exists $\varrho \in \mathbb{N}$ such that $\alpha_m = 0$ for $|m| \geq \varrho + 1$, then it is clear from our narrative that $p_{2,m}, p_{3,m}, p_{4,m} = 0$ for $|m| \geq \varrho + 1$. In other words, the relevant modulated Fourier expansions stay band limited with the same original bandwidth ϱ. However, things change with regard to $p_{5,m}$. The fact that $p_{2,m}$ is band limited implies that

$$\sum_{l=-\infty}^{\infty} p_{2,l}p_{2,m-l} = \sum_{l=-\varrho+\max\{m,0\}}^{\varrho+\min\{m,0\}} p_{2,l}p_{2,m-l}.$$

This leads to non-empty range of summation when

$$\max\{m,0\} - \min\{m,0\} \le 2\varrho,$$

hence $|m| \le 2\varrho$. In other words, $p_{5,m}$ is band limited of bandwidth 2ϱ: we call this phenomenon *blossoming* and note that it has obvious implications in the programming of the method.

The bandwidth of linear systems is the same as of the highly oscillatory input (i.e. there is no blossoming). It is known, however, that nonlinearity might interfere with bandwidth, and our analysis quantifies this phenomenon for equations of type (1.1). Of course, the bandwidth is likely to blossom further as r increases, indicating that resonance shifts energy between frequencies. What is remarkable is that all this occurs only once we hit $p_{5,m}$. Consequently, as long as we do not go beyond $r = 4$, disregarding error of $\mathcal{O}(\omega^{-5})$, we retain the original bandwidth of the forcing term. If ω is large enough, this is likely to be sufficient for virtually all cases of interest.

(*a*) *Blossoming*

How fast does blossoming occur? Let us denote by θ_r the bandwidth of the $\mathcal{O}(\omega^{-r})$ term, therefore

$$\theta_0 = \theta_1 = 0, \qquad \theta_2 = \theta_3 = \theta_4 = \varrho, \qquad \theta_5 = 2\varrho.$$

Before we state a general theorem, let us acquire basic intuition in manipulating the relevant expressions. In order to compute the bandwidth θ_{r+1}, we need to consider terms of the form

$$\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r-1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}, \ \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n},$$

for $n \in \{1,\ldots,r-1\}$ and $n \in \{1,\ldots,r\}$ respectively, but clearly it is enough to consider only the terms in $\mathbb{I}_{n,r}$, since we wish to maximise the bandwidth.

(i) $r = 5$

Now $n \in \{1,\ldots,5\}$. Because of symmetry, we might assume without loss of generality that the entries of $\boldsymbol{k} \in \mathbb{I}_{n,r}$ are ordered monotonically.

$n = 1$**:** The only possible term is $p_{5,m}$, hence the maximal bandwidth that we can attribute to this term is $\theta_5 = 2\varrho$;

$n = 2$**:** We have two monotone choices: $\boldsymbol{k} = (1,4)$ results in $p_{1,l_1}p_{4,l_2}$. But p_{1,l_1} can be nonzero only if $|l_1| \le \theta_1 = 0$ and p_{4,l_2} is nonzero only if $|l_2| \le \theta_4 = \varrho$. Hence the bandwidth is at most ϱ.

The second choice is $\boldsymbol{k} = (2,3)$, whereby the term is $p_{2,l_1}p_{3,l_2}$ and the bandwidth is maximised by $\theta_2 + \theta_3 = 2\varrho$;

$n = 3$**:** Now there are two monotone possibilities, $(1, 1, 3)$ and $(1, 2, 2)$, with maximal bandwidths of $2\theta_1 + \theta_3 = \varrho$ and $\theta_1 + 2\theta_2 = 2\varrho$ respectively;

$n = 4$**:** Just a single monotone choice, $(1, 1, 1, 2)$, leading to $3\theta_1 + \theta_2 = \varrho$;

$n = 5$**:** Only one 5-tuple, $(1, 1, 1, 1, 1)$, and the maximal bandwidth is $5\theta_1 = 0$.

Therefore $\theta_6 = 2\varrho$, the maximum over all possible choices.

(ii) $r = 6$

We now move faster, considering only monotone sequences:

$n = 1$**:** $\boldsymbol{k} = (6)$, hence $\theta_6 = 2\varrho$;

$n = 2$**:** $\boldsymbol{k} = (1, 5)$ yields $\theta_1 + \theta_5 = 2\varrho$, $\boldsymbol{k} = (2, 4)$ results in $\theta_2 + \theta_2 = 2\varrho$ and $\boldsymbol{k} = (3, 3)$ in $2\theta_3 = 2\varrho$;

$n = 3$**:** $\boldsymbol{k} = (1, 1, 4)$ gives $2\theta_1 + \theta_4 = \varrho$, $\boldsymbol{k} = (1, 2, 3)$ results in $\theta_1 + \theta_2 + \theta_3 = 2\varrho$ and $\boldsymbol{k} = (2, 2, 2)$ in $3\theta_2 = 3\varrho$;

$n = 4$**:** $\boldsymbol{k} = (1, 1, 1, 3)$ yields $3\theta_1 + \theta_3 = \varrho$ and $\boldsymbol{k} = (1, 1, 2, 2)$ results in $2\theta_1 + 2\theta_2 = 2\varrho$;

$n = 5$**:** $\boldsymbol{k} = (1, 1, 1, 1, 2)$ and $4\theta_1 + \theta_2 = \varrho$;

$n = 6$**:** $\boldsymbol{k} = (1, 1, 1, 1, 1, 1)$ results in the bandwidth $6\theta_1 = 0$.

Therefore, $\theta_7 = 3\varrho$. Note that the bandwidth is obtained from $n = 3$ and $\boldsymbol{k} = (2, 2, 2)$: this observation is crucial in the proof of the theorem.

Now that we have see a number of examples, we can embark on the proof of the general theorem underlying blossoming:

Theorem 4.1. *It is true that*

$$\theta_r = \left\lfloor \frac{r-1}{2} \right\rfloor \varrho, \qquad r \geq 3. \tag{4.1}$$

Proof. The theorem is certainly true for $r \leq 5$. We continue by induction on r. Thus, we assume that it is true up to $r \geq 2$ and wish to prove it for $r + 1$.

Given $r \geq 2$, the recurrence relation for $p_{r+1,m}$ is a linear combination of terms of the form

$$\sum_{\boldsymbol{k} \in \mathbb{I}_{n,r}} \sum_{\boldsymbol{l} \in \mathbb{K}_{n,m}} p_{k_1,l_1} \cdots p_{k_n,l_n}$$

and their derivatives – as the derivative does not change the bandwidth, we can disregard differentiation in this context. The bandwidth is provided by the largest $m \in \mathbb{N}$ such that

$$\varpi_{\boldsymbol{k},\boldsymbol{l}} = p_{k_1,l_1} \cdots p_{k_n,l_n} \neq 0,$$

where $k_1, \ldots, k_n \in \mathbb{N}$, $|\boldsymbol{k}| = r$, $l_1, \ldots, l_n \in \mathbb{Z}$ and $|\boldsymbol{l}| = m$. Each k_j can contribute at most the bandwidth θ_j, hence altogether the bandwidth of $\varpi_{\boldsymbol{k},\boldsymbol{l}}$ is at most

$$\sum_{j=1}^{n} \theta_{k_j},$$

therefore

$$\theta_{r+1} \le \max_{\substack{\boldsymbol{k}\in\mathbb{I}_{n,r}\\ 1\le n\le r}} \sum_{j=1}^{n} \theta_{k_j}. \tag{4.2}$$

First we observe that

$$\theta_{r+1} \ge \theta_r, \qquad r \ge 2.$$

We deduce this at once by considering $n=1$, whence $\varpi_{r,m} = p_{r,m}$. Next, we prove that

$$\theta_{r+1} \ge \left\lfloor \frac{r}{2} \right\rfloor \varrho.$$

If $r = 2\tilde{r}$ then let us choose $n = \tilde{r}$ and $\boldsymbol{k} = (2,2,\ldots,2) \in \mathbb{N}^{\tilde{r}}$. Since $\theta_2 = \varrho$, we deduce that the bandwidth of $\varpi_{\boldsymbol{k},\boldsymbol{l}}$ is maximised by $\tilde{r}\varrho$, therefore $\theta_{2\tilde{r}+1} \ge \tilde{r}\varrho$. Likewise, if $r = 2\tilde{r}+1$ then we choose $n = \tilde{r}+1$ and $\boldsymbol{k} = (1,2,2,\ldots,2)$. Since $\theta_1 = 0$ and $\theta_2 = \varrho$, it follows again that $\theta_{2\tilde{r}+2} \ge \tilde{r}\varrho$.

In order to prove the reverse inequality

$$\theta_{r+1} \le \left\lfloor \frac{r}{2} \right\rfloor \varrho,$$

let $\boldsymbol{k} \in \mathbb{I}_{n,r}$, $n \in \{1,2,\ldots,r\}$. We may assume without loss of generality that none of the k_js equals one. The reason is as follows. Suppose that s of the k_js are one and denote by $\tilde{\boldsymbol{k}} \in \mathbb{N}^{n-s}$ the vector which we obtain after excising these terms. Since $\theta_1 = 0$, the maximal bandwidths of $\varpi_{\boldsymbol{k},\boldsymbol{l}}$ is the same as the maximal bandwidth of $\varpi_{\tilde{\boldsymbol{k}},\tilde{\boldsymbol{l}}}$ for some $\tilde{\boldsymbol{l}} \in \mathbb{Z}^{n-s}$. But, since $\tilde{k}_1 + \cdots + \tilde{k}_{n-s} + s = r$, the term $\varpi_{\tilde{\boldsymbol{k}},\tilde{\boldsymbol{l}}}$ has already featured while forming $p_{r-s+1,m}$ for some $m \in \mathbb{Z}$. Now, we already know that $\theta_{r+1} \ge \theta_r \ge \cdots \ge \theta_{r-s}$, hence this term can be disregarded and we can assume without loss of generality that $k_j \ne 1$, $j = 1,2,\ldots,n$.

We write

$$\{k_1,\ldots,k_n\} = \{\beta_1,\ldots,\beta_{n_1}\} \cup \{\gamma_1,\ldots,\gamma_{n_2}\} \cup \{\delta_1,\ldots,\delta_{n_3}\},$$

where $\beta_1 = \ldots = \beta_{n_1} = 2$, $\gamma_1,\ldots,\gamma_{n_2}$ are even, $\gamma_j = 2\tilde{\gamma}_j$, and $\delta_1,\ldots,\delta_{n_3}$ are odd, $\delta_j = 2\tilde{\delta}_j + 1$, with $\tilde{\gamma}_j \ge 2$, $\tilde{\delta}_j > 1$. We thus have

$$\begin{aligned} n &= n_1 + n_2 + n_3, \\ r &= \sum_{j=1}^{n} k_j = 2n_1 + 2\sum_{j=1}^{n_2} \tilde{\gamma}_j + 2\sum_{j=1}^{n_3} \tilde{\delta}_j + n_3. \end{aligned}$$

(Thus, n_3 is necessarily of the same parity as r.) Moreover, by induction,

$$\frac{1}{\varrho}\sum_{j=1}^{n} \theta_{k_j} = n_1 \frac{\theta_2}{\varrho} + \sum_{j=1}^{n_2} \frac{\theta_{2\tilde{\gamma}_j}}{\varrho} + \sum_{j=1}^{n_3} \frac{\theta_{2\tilde{\delta}_j+1}}{\varrho} = n_1 + \sum_{j=1}^{n_2} (\tilde{\gamma}_j - 1) + \sum_{j=1}^{n_3} \tilde{\delta}_j$$

Now,

$$\sum_{j=1}^{n_2} \tilde{\gamma}_j + \sum_{j=1}^{n_3} \tilde{\delta}_j = \frac{r - n_3}{2} - n_1$$

and we deduce that

$$\frac{1}{\varrho}\sum_{j=1}^{n} \theta_{k_j} = \frac{r-n_3}{2} - n_2.$$

Now we need to maximise the quantity on the right hand side for $\boldsymbol{k} \in \mathbb{I}_{n,r}$ because of (4.2). Recalling that r and n_3 must have the same parity, we distinguish between even and odd r. If $r = 2\tilde{r}$ then $n_3 = 2\tilde{n}_3$, therefore

$$\frac{1}{\varrho}\sum_{j=1}^{n} \theta_{k_j} = \tilde{r} - \tilde{n}_3 - n_2 \le \tilde{r} = \left\lfloor \frac{r}{2} \right\rfloor .$$

Likewise, for $r = 2\tilde{r} + 1$ we have $n_3 = 2\tilde{n}_3 + 1$ and again

$$\frac{1}{\varrho}\sum_{j=1}^{n} \theta_{k_j} = \tilde{r} - \tilde{n}_3 - n_2 \le \tilde{r} = \left\lfloor \frac{r}{2} \right\rfloor .$$

Taking all this together completes the proof of the theorem. □

The implications for programming the method are clear, since it is possible to determine in advance how many terms $p_{r,m}(t)$ we need to compute for any given $r \ge 2$.

5. Examples

We consider first the Van der Pol oscillator. In this case we have $R(y) = \mu(1 - y^2)$ and $S(y) = y$, and it is not difficult to work out the ODEs for the first few $p_{r,0}(t)$ terms. The base equation is

$$p_{0,0}'' - \mu(1 - p_{0,0}^2)p_{0,0}' + p_{0,0} = \alpha_0, \qquad p_{0,0}(0) = y_0, \qquad p_{0,0}'(0) = y_0'.$$

Using the same notation as before, we have for $r \ge 1$

$$\begin{aligned}\mathcal{L}[p_{r,0}] &= p_{r,0}'' - \frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{r,0}] + S'(p_{0,0})p_{r,0} \\ &= p_{r,0}'' + 2\mu p_{0,0}p_{0,0}'p_{r,0} + \mu(1 - p_{0,0}^2)p_{r,0}' + p_{r,0},\end{aligned}$$

and

$$\begin{aligned}&\mathcal{L}[p_{1,0}] = 0, \\ &\mathcal{L}[p_{2,0}] = -\mu\left[p_{0,0}'p_{1,0}^2 + 2p_{0,0}p_{1,0}p_{1,0}'\right], \\ &\mathcal{L}[p_{3,0}] = -2\mu\left[p_{0,0}'p_{1,0}p_{2,0} + p_{0,0}p_{1,0}'p_{2,0} + p_{0,0}p_{1,0}p_{2,0}'\right] - \mu p_{1,0}^2 p_{1,0}'.\end{aligned}$$

We take the initial values $y(0) = 0$ and $y'(0) = 1$, and the forcing term $f_\omega(t) = 2\cos t \sin \omega t$. Thus we have $\alpha_1 = -\mathrm{i}\cos t$, $\alpha_{-1} = \mathrm{i}\cos t$ and $\alpha_m \equiv 0$ for $m \ne \pm 1$, and the initial values for the system of ODEs can be obtained from (3.2), (3.4) and (2.18),

$$\begin{aligned}p_{1,0}(0) &= 0, \qquad & p_{1,0}'(0) &= 2, \\ p_{2,0}(0) &= 0, \qquad & p_{2,0}'(0) &= 0, \\ p_{3,0}(0) &= 0, \qquad & p_{3,0}'(0) &= 4.\end{aligned}$$

Moreover, from (3.1) we have

$$p_{2,1}(t) = -\alpha_1(t) = \mathrm{i}\cos t, \qquad p_{2,-1}(t) = -\alpha_{-1}(t) = -\mathrm{i}\cos t,$$

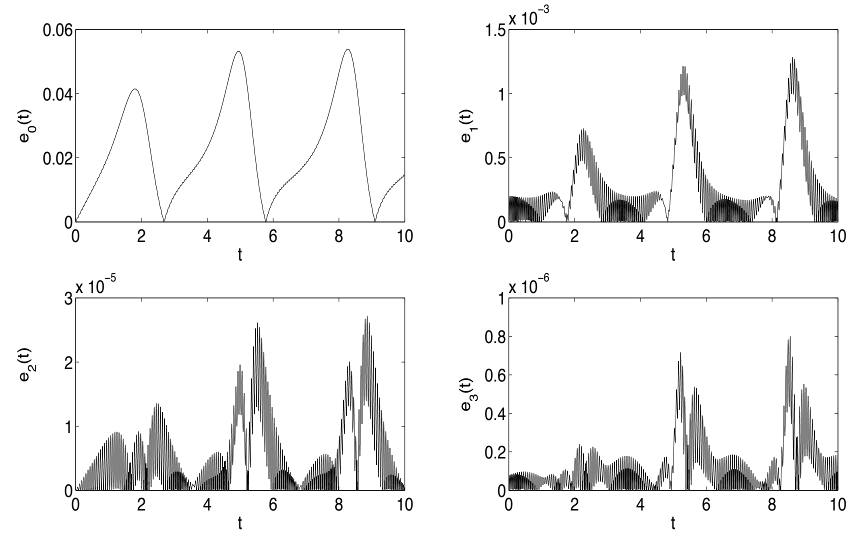

Figure 2. Errors in the approximation of the solution $y(t)$ of the forced Van der Pol oscillator with forcing term $f_\omega(t) = 2\cos t \sin\omega t$ and $\omega = 100$.

consequently

$$\psi_2(t) = p_{2,0}(t) + p_{2,1}(t)e^{i\omega t} + p_{2,-1}(t)e^{-i\omega t} = p_{2,0}(t) - 2\cos t \sin\omega t$$

Also, from (3.3),

$$p_{3,1}(t) = \mu(1 - p_{0,0}^2(t))\cos t + 2\sin t = p_{3,-1}(t),$$

so

$$\psi_3(t) = p_{3,0}(t) + 2\left[\mu(1 - p_{0,0}^2(t))\cos t + 2\sin t\right]\cos\omega t.$$

We will use all this information to assemble the numerical solver up to order 3. In Figures 2 and 3 we illustrate the errors in the approximation of the solution $y(t)$ and its derivative $y'(t)$, using different number of terms in the asymptotic expansion with $\omega = 100$. We compare the results with the solution of the original differential equation in MATLAB, using relative and absolute tolerance equal to 10^{-12}. The notation that has been used for the errors is

$$e_s(t) = \left| y(t) - \sum_{r=0}^{s} \frac{\psi_r(t)}{\omega^r} \right|, \qquad s \geq 0.$$

The next example illustrates the method applied to the Duffing equation with damping

$$y''(t) + ky'(t) + ay(t) + by(t)^3 = f_\omega(t), \qquad y(0) = 1, \quad y'(0) = 0.$$

We take $k = 1/2$, $a = 1$, $b = -1/3$, and a forcing term which is an AM modulated signal:

$$f_\omega(t) = c_1 \sin\omega_1 t + c_2 \sin\omega_1 t \, \sin\omega_2 t,$$

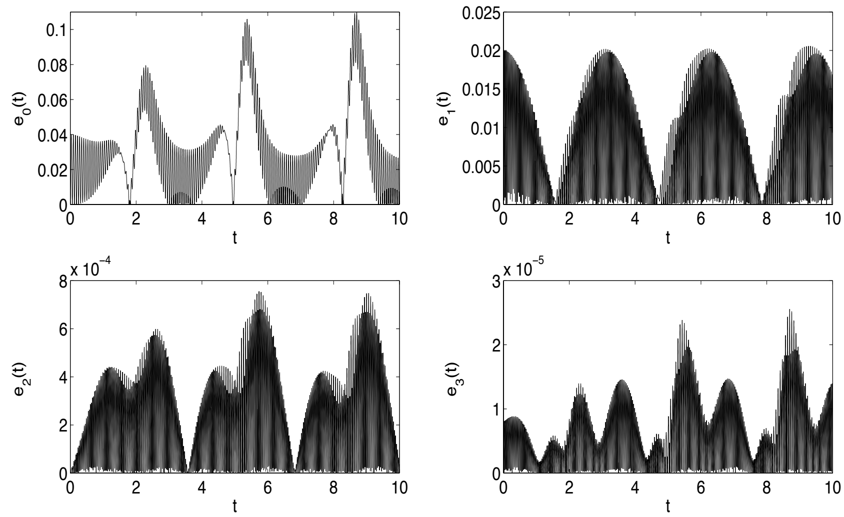

Figure 3. Errors in the approximation of the derivative of the solution $y'(t)$ of the forced Van der Pol oscillator with forcing term $f_\omega(t) = 2\cos t \sin \omega t$ and $\omega = 100$.

with $c_1 = 40$, $c_2 = 20$ and frequencies $\omega_1 = 1000$ and $\omega_2 = 100$. In order to construct the asymptotic expansion in a modulated Fourier series, we define $\omega := \gcd(\omega_1, \omega_2)$, that is the greatest common divisor of the two frequencies. Additionally, let $m_1 = \omega_1/\omega$ and $m_2 = \omega_2/\omega$.

In this case the base equation is

$$p''_{0,0} + kp'_{0,0} + ap_{0,0} + bp^3_{0,0} = \alpha_0,$$

and for $r \geq 1$:

$$\mathcal{L}[p_{r,0}] = p''_{r,0} - \frac{\mathrm{d}}{\mathrm{d}t}[R(p_{0,0})p_{r,0}] + S'(p_{0,0})p_{r,0} = p''_{r,0} + kp'_{r,0} + (a + 3bp^2_{0,0})p_{r,0}.$$

Moreover

$$\begin{aligned}
\mathcal{L}[p_{1,0}] &= 0 \\
\mathcal{L}[p_{2,0}] &= -3bp_{0,0}p^2_{1,0} \\
\mathcal{L}[p_{3,0}] &= -6bp_{0,0}p_{1,0}p_{2,0} - bp^3_{1,0}.
\end{aligned}$$

We can easily work out the initial values

$$\begin{aligned}
&p_{1,0}(0) = 0, \qquad p'_{1,0}(0) = \frac{c_1}{m_1}, \\
&p_{2,0}(0) = \frac{c_2}{2}\left[\frac{1}{(m_1 - m_2)^2} - \frac{1}{(m_1 + m_2)^2}\right], \qquad p'_{2,0}(0) = -kp_{2,0}(0),
\end{aligned}$$

as well as the other nonzero terms

$$\begin{aligned}
p_{2,m_1} &= \frac{\mathrm{i}c_1}{2m_1^2} = -p_{2,-m_1},\\
p_{2,m_1+m_2} &= \frac{c_2}{4(m_1+m_2)^2} = p_{2,-m_1-m_2},\\
p_{2,m_1-m_2} &= -\frac{c_2}{4(m_1-m_2)^2} = p_{2,-m_1+m_2},
\end{aligned}$$

so

$$\begin{aligned}
\psi_2(t) = {} & p_{2,0}(t) - \frac{c_1}{m_1^2}\sin m_1\omega t\\
& + \frac{c_2}{2(m_1+m_2)^2}\cos(m_1+m_2)\omega t - \frac{c_2}{2(m_1-m_2)^2}\cos(m_1-m_2)\omega t.
\end{aligned}$$

Similarly,

$$p_{3,0}(0) = \frac{kc_1}{m_1^3}, \qquad p'_{3,0}(0) = \frac{c_1}{m_1^3}\left[-k^2+a+3by_0^2\right],$$

and

$$\begin{aligned}
p_{3,m_1} &= \frac{-kc_1}{2m_1^3} = p_{3,-m_1},\\
p_{3,m_1+m_2} &= \frac{\mathrm{i}kc_2}{4(m_1+m_2)^3} = -p_{3,-m_1-m_2},\\
p_{3,m_1-m_2} &= -\frac{\mathrm{i}kc_2}{4(m_1-m_2)^3} = -p_{3,-m_1+m_2},
\end{aligned}$$

therefore

$$\begin{aligned}
\psi_3(t) = {} & p_{3,0}(t) - \frac{kc_1}{m_1^3}\cos m_1\omega t\\
& - \frac{kc_2}{2(m_1+m_2)^3}\sin(m_1+m_2)\omega t + \frac{kc_2}{2(m_1-m_2)^3}\sin(m_1-m_2)\omega t.
\end{aligned}$$

Figures 4 and 5 display the errors in the approximation of the solution $y(t)$ and its derivative $y'(t)$, using different number of terms in the asymptotic expansion in this example.

6. Conclusions and further research

We have presented a combined asymptotic-numerical method to solve efficiently second order differential equations with highly oscillatory forcing terms. The approach is based on using asymptotic expansions in inverse powers of the oscillatory parameter ω together with modulated Fourier expansions. With the aid of a computer algebra package such as MAPLE, it is possible to compute all the terms in this type of expansions to high accuracy.

A key feature of this approach is that, unlike classical numerical algorithms for ODEs, the performance of this method improves in the presence of high oscillation,

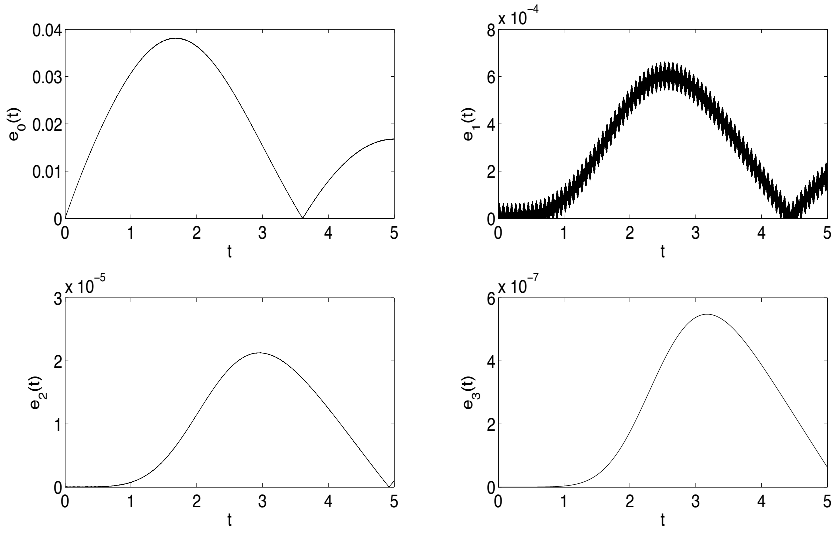

Figure 4. Errors in the approximation of the solution $y(t)$ of the forced Duffing oscillator with forcing term $f_\omega(t) = c_1 \sin \omega_1 t + c_2 \sin \omega_1 t \, \sin \omega_2 t$.

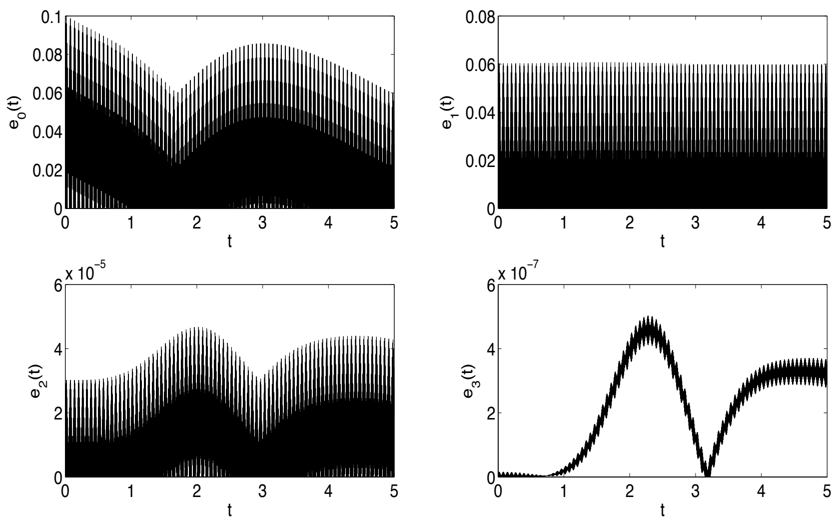

Figure 5. Errors in the approximation of the derivative of the solution $y'(t)$ of the forced Duffing oscillator with forcing term $f_\omega(t) = c_1 \sin \omega_1 t + c_2 \sin \omega_1 t \, \sin \omega_2 t$.

that is, when ω is large. This is a consequence of the asymptotic methodology that we have used, instead of the classical algorithms based on Taylor expansion of the solution.

We have presented numerical examples based on two equations which are very relevant in applications, the forced Van der Pol and Duffing oscillators. This is nothing but one possible application of this type of asymptotic-numerical solvers. See Condon *et al.* (2009b) for its use in solving systems of ODEs with a nonoscillatory linear part plus a highly oscillatory forcing term. Other scenarios that are currently being analysed are ODEs where the coefficients depend on ω (a situation which includes important examples such as the inverted pendulum) and differential-algebraic equations (DAEs), which are highly relevant in the modelling of electronic circuits.

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Appendix A. Two propositions in subsection 2.2

In this appendix we present the proofs of the two propositions that we used before.

Proposition 2.1. *For every $r \geq 1$ and $m \in \mathbb{Z}$ it is true that*

$$R(p_{0,0})p'_{r,m} + \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} p_{k_1,l_1}\cdots p_{k_n,l_n} p'_{r-s,l_{n+1}}$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\right].$$

Proof. Direct differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n}\right]$$
$$= \sum_{k=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_n,l_n} p'_{0,0}$$
$$+ \sum_{n=1}^{r} \frac{R^{(n-1)}(p_{0,0})}{n!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} [p'_{k_1,l_1}p_{k_2,l_2}\cdots p_{k_n,l_n} + p_{k_1,l_1}p'_{k_2,l_2}p_{k_3,l_3}\cdots p_{k_n,l_n}$$
$$+\cdots + p_{k_1,l_1}\cdots p_{k_{n-1},l_{n-1}}p'_{k_n,l_n}].$$

However, because of symmetry,

$$\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_{q-1},l_{q-1}} p'_{k_q,l_q} p_{k_{q+1},l_{q+1}}\cdots p_{k_n,l_n}$$
$$= \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1}\cdots p_{k_{n-1},l_{n-1}} p'_{k_n,l_n},$$

therefore

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r}\frac{R^{(n-1)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}\right]\\
=&\sum_{n=1}^{r}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}p'_{0,0}\\
+&\sum_{n=1}^{r}\frac{R^{(n-1)}(p_{0,0})}{(n-1)!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_{n-1},l_{n-1}}p'_{k_n,l_n}\\
=&\sum_{n=1}^{r}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}p'_{0,0}\\
+&R(p_{0,0})p'_{r,m}+\sum_{n=1}^{r-1}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n+1,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}p'_{k_{n+1},l_{n+1}}.
\end{aligned}
$$

In the last summation we let $s=k_1+\cdots+k_n$. Since $s+k_{n+1}=r$ and $k_j\geq 1$, we deduce that $s\in\{n,n+1,\ldots,r-1\}$. Moreover, $k_{n+1}=r-s$ and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{n=1}^{r}\frac{R^{(n-1)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}\right]\\
=&\sum_{n=1}^{r}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}p'_{0,0}\\
&+R(p_{0,0})p'_{r,m}+\sum_{n=1}^{r-1}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{s=n}^{r-1}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}p'_{r-s,l_{n+1}}\\
=&R(p_{0,0})p'_{r,m}+\sum_{n=1}^{r}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{s=n}^{r}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}p'_{r-s,l_{n+1}}.
\end{aligned}
$$

The last step follows because, letting $n\in\{1,2,\ldots,r\}$ and $s=r$ and noting that $p_{0,l_{n+1}}\neq 0$ only for $l_{n+1}=0$, we recover the first sum.

The proposition follows. □

Proposition 2.2. *For every $r\geq 1$ and $m\in\mathbb{Z}$ it is true that*

$$
\begin{aligned}
&R(p_{0,0})mp_{r+1,m}+\sum_{n=1}^{r}\frac{R^{(n)}(p_{0,0})}{n!}\sum_{s=n}^{r}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}}l_{n+1}p_{k_1,l_1}\cdots p_{k_n,l_n}p_{r-s+1,l_{n+1}}\\
&=m\sum_{n=1}^{r+1}\frac{R^{(n-1)}(p_{0,0})}{n!}\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}}\sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}}p_{k_1,l_1}\cdots p_{k_n,l_n}.
\end{aligned}
$$

Proof. Similar to the proof of the previous proposition. We let $\boldsymbol{k} \in \mathbb{I}_{n,r+1}$ and $|\boldsymbol{k}| = s$, hence $k_{n+1} = r - s + 1$, while $s \in \{n, n+1, \ldots, r\}$. In other words,

$$\sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}} = \sum_{\boldsymbol{k}\in\mathbb{I}_{n+1,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_{n+1},l_{n+1}}.$$

Therefore, shifting the index n,

$$R(p_{0,0}) m p_{r+1,m} + \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}}$$
$$= R(p_{0,0}) m p_{r+1,m} + \sum_{n=2}^{r+1} \frac{R^{(n-1)}(p_{0,0})}{(n-1)!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} l_n p_{k_1,l_1} \cdots p_{k_n,l_n}.$$

Finally, for $n = 1$ we have $\mathbb{I}_{1,r+1} = \{(r+1)\}$, $\mathbb{K}_{1,m} = \{(m)\}$, therefore

$$R(p_{0,0}) m p_{r+1,m} + \sum_{n=1}^{r} \frac{R^{(n)}(p_{0,0})}{n!} \sum_{s=n}^{r} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,s}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n+1,m}} l_{n+1} p_{k_1,l_1} \cdots p_{k_n,l_n} p_{r-s+1,l_{n+1}}$$
$$= \sum_{n=1}^{r+1} \frac{R^{(n-1)}(p_{0,0})}{(n-1)!} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} l_n p_{k_1,l_1} \cdots p_{k_n,l_n}.$$

Using the underlying symmetry, it is true for any $q \in \{1, 2, \ldots, n\}$ that

$$\sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} l_q p_{k_1,l_1} \cdots p_{k_n,l_n} = \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} l_n p_{k_1,l_1} \cdots p_{k_n,l_n}$$
$$= \frac{1}{n} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} (l_1 + \cdots + l_n) p_{k_1,l_1} \cdots p_{k_n,l_n} = \frac{m}{n} \sum_{\boldsymbol{k}\in\mathbb{I}_{n,r+1}} \sum_{\boldsymbol{l}\in\mathbb{K}_{n,m}} p_{k_1,l_1} \cdots p_{k_n,l_n}.$$

The lemma follows by straightforward substitution. □

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