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ON SECOND ORDER HYPERBOLIC EQUATION WITH TWO INDEPENDENT VARIABLES

BY

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In this paper Cauchy's problem for the equation

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

is considered in the class of Banach space-valued functions $z(x, y)$ having Bochner integrable partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x \partial y$. The reasonings are similar to those of [2], but results are slightly farther going. Theorem 2.5 shows that a condition introduced in [3], sufficient for the existence and uniqueness of solutions with continuous derivatives in the case of continuous f and continuous boundary data, is also sufficient for the existence and uniqueness of solutions with integrable derivatives if f and boundary data are suitably less regular.

Let us mention that by establishing theorem 2.5 for Banach space-valued functions a simple deduction of theorems on continuous dependence of solutions on f and boundary data is possible, similarly to [2], § 9, p. 102-106.

1. THE FUNCTION CLASS $W_1^{1,*}(\Delta_{a,b}; E)$

1.1. Assumptions. Let g be a function defined on $(-\infty, \infty)$, with values in $(-\infty, \infty]$, non-increasing, not equal identically to ∞, and such that $\lim_{x\to-\infty} g(x) = \infty$. For any $y \in (-\infty, \infty)$ put $h_+(y) = \sup\{x: g(x) > y\}$, $h_-(y) = \inf\{x: g(x) < y\}$, under the convention that $\inf \emptyset = +\infty$. Let h be a function defined on $(-\infty, \infty)$, with values in $(-\infty, \infty]$, and such that $h_+(y) \leqslant h(y) \leqslant h_-(y)$ for any $y \in (-\infty, \infty)$.

1.2. LEMMA. *Under the assumptions* 1.1, h_+ *and* h_- *are functions with values in* $(-\infty, \infty]$, *non-increasing on* $(-\infty, \infty)$, h_+ *is right-continuous, and* h_- *is left-continuous. The set*

$$D = \{y: -\infty < y < \infty,\ g^{-1}(\{y\}) \text{ is an interval of positive length}\}$$

is denumerable, $h_+(y) < h_-(y)$ *for* $y \in D$, $h_+(y) = h(y)$ *for* $y \in (-\infty, \infty)\setminus D$. *The function* h *is non-increasing,* D *is precisely the set of all points of discontinuity of* h, *and* $h_-(y) = h(y-0)$, $h_+(y) = h(y+0)$ *for any* $y \in (-\infty, \infty)$. *Furthermore, for any* $x \in (-\infty, \infty)$ *and* $y \in (-\infty, \infty)$ *the following equivalences hold*:

$$y > g(x-0) \Leftrightarrow x > h(y-0), \tag{1.2.1}$$

$$y < g(x+0) \Leftrightarrow x < h(y+0). \tag{1.2.2}$$

The proof is left to the reader.

1.3. LEMMA. *Under the assumptions* 1.1 *let* $a \in (-\infty, \infty)$ *and* $b \in \big(g(a-0), \infty\big)$ *be fixed and suppose that* g *is continuous in the interval* $\big(h(b-0), a\big)$ *and* h *is continuous in the interval* $\big(g(a-0), b\big)$. *Put*

$$a' = h\big(g(a-0)+0\big), \qquad b' = g\big(h(b-0)+0\big).$$

Then we have

$$h(b-0) < a' \leqslant a \quad \text{and} \quad g(a-0) < b' \leqslant b \tag{1.3.1}$$

or

$$h(b-0) = a' \quad \text{and} \quad g(a-0) = b'. \tag{1.3.2}$$

In each of these two cases if $x \in (a', a)$, *then* $g(x) = g(a-0)$, *and if* $y \in (b', b)$, *then* $h(y) = h(b-0)$. *Furthermore, in the case* (1.3.1), $g(x)$ *strictly decreases in the interval* $\big(h(b-0), a'\big)$ *from* b' *to* $g(a'-0) = g(a-0)$, *and the inverse function of* $g/\big(h(b-0), a'\big)$ *is* $h/\big(g(a-0), b'\big)$.

Proof. Putting $x = a$ and $y = g(a-0)$, we have $y \geqslant g(x+0)$, which by (1.2.2) implies that $a = x \geqslant h(y+0) = a'$. Since $g(a-0) < b$, we have $a' = h\big(g(a-0)+0\big) \geqslant h(b-0)$. Thus

$$h(b-0) \leqslant a' \leqslant a. \tag{1.3.3}$$

Similarly,

$$g(a-0) \leqslant b' \leqslant b. \tag{1.3.4}$$

Putting $x = a'$ and $y = g(a-0)$, we have $x = h(y+0)$ and so, by (1.2.2), $g(a-0) = y \geqslant g(x+0) = g(a'+0)$. Thus

$$g(a'+0) \leqslant g(a-0) \tag{1.3.5}$$

and so, since g is non-increasing, if $a' < a$, then

$$g(x) = g(a-0) \quad \text{for } x \in (a'\, a).$$

Similarly,

$$h(b'+0) \leqslant h(b-0),$$

and if $b' < b$, then

$$h(y) = h(b-0) \quad \text{for } y \in (b', b).$$

If $a' = h(b-0)$, then $b' = g(a'+0)$, and so, by (1.3.4) and (1.3.5), $b' = g(a-0)$. Similarly, $b' = g(a-0)$ implies $a' = h(b-0)$. This together with (1.3.3) and (1.3.4) shows that the alternative "(1.3.1) or (1.3.2)" is true.

Suppose now that $h(b-0) < a' \leqslant a$. In this case we have

$$g(a'-0) = g(a-0). \tag{1.3.6}$$

Indeed, if $a' = a$, then there is nothing to prove. If $h(b-0) < a' < a$, then g is continuous at $x = a'$, and so, by (1.3.5), $g(a'-0) = g(a'+0) \leqslant g(a-0)$ and, on the other hand, $g(a'-0) \geqslant g(a-0)$ since $a' < a$ and g is non-increasing. Furthermore, if $h(b-0) < a' \leqslant a$ and $h(b-0) < x < a'$, then, by (1.2.1), $h(b-0) < x$ implies $g(x-0) < b$, and $x < a' = h\big(g(a-0)+0\big)$ implies $g(a-0) < g(x+0)$, so that

$$g\big(h(b-0),\, a'\big) \subset \big(g(a-0),\, b\big). \tag{1.3.7}$$

According to (1.2.1) and (1.2.2), $g(x) \leqslant g(x-0)$ implies $x \leqslant h\big(g(x)-0\big)$ and $g(x) \geqslant g(x+0)$ implies $x \geqslant h\big(g(x)+0\big)$ so that

$$h\big(g(x)+0\big) \leqslant x \leqslant h\big(g(x)-0\big)$$

for every x. If $x \in \big(h(b-0),\, a'\big)$, then, by (1.3.7), $g(x) \in \big(g(a-0),\, b\big)$ and since h is continuous in $\big(g(a-0),\, b\big)$, we have $h\big(g(x)-0\big) = h\big(g(x)+0\big) = h\big(g(x)\big)$. Consequently,

$$h\big(g(x)\big) = x \quad \text{for } x \in \big(h(b-0),\, a'\big). \tag{1.3.8}$$

This implies that g strictly decreases in $\big(h(b-0),\, a'\big)$ and since g is continuous in this interval, we have by (1.3.6)

$$g\big(h(b-0),\, a'\big) = \Big(g(a'-0),\; g\big(h(b-0)+0\big)\Big) = \big(g(a-0),\, b'\big). \tag{1.3.9}$$

From (1.3.8) and (1.3.9) it follows that the inverse function of $g/\big(h(b-0),\, a'\big)$ is $h/\big(g(a-0),\, b'\big)$.

1.4. Definition. Under the assumptions 1.1 we put, for any $a \in (-\infty, \infty)$ and $b \in \big(g(a-0), \infty\big)$,

$$\Delta_{a,b} = \{(x, y)\colon -\infty < x < a,\; g(x-0) < y < b\},$$

$$\varphi_{a,b} = \{(x, y)\colon -\infty < x < a,\; g(x+0) \leqslant y \leqslant g(x-0),\; y < b\}.$$

According to (1.2.1) and (1.2.2), we then have

$$\Delta_{a,b} = \{(x, y)\colon -\infty < y < b,\; h(y-0) < x < a\},$$

$$\varphi_{a,b} = \{(x, y)\colon -\infty < y < b,\; h(y+0) \leqslant x \leqslant h(y-0),\; x < a\}.$$

1.5. Convention. Everywhere in the sequel, if the notation $\Delta_{a,b}$, or $\varphi_{a,b}$, will be used, then it will be supposed (without writing this explicitely) that the assumption 1.1 holds and that $b > g(a-0)$.

1.6. Definition. Let E be a Banach space. We denote by $W_1^{1,*}(\Delta_{a,b}; E)$ the class of all the E-valued distributions z on $\Delta_{a,b}$, which distributional partial derivatives $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x\partial y$ are represented by E-valued functions Bochner integrable on $\Delta_{a,b}$.

1.7. THEOREM. *Every distribution $z \in W_1^{1,*}(\Delta_{a,b}; E)$ is represented by an E-valued function strongly continuous on $\Delta_{a,b}$ and having strongly continuous extension onto $\Delta_{a,b} \cup \varphi_{a,b}$. For every $z \in W_1^{1,*}(\Delta_{a,b}; E)$ there are E-valued functions σ and τ of one real variable, strongly measurable on $\big(h(b-0), a\big)$ or $\big(g(a-0), b\big)$, respectively, and such that*

$$\int_{h(b-0)}^{a} \big(b-g(x)\big)\|\sigma(x)\|\,dx < \infty, \tag{1.7.1}$$

$$\int_{g(a-0)}^{b} \big(a-h(y)\big)\|\tau(y)\|\,dy < \infty, \tag{1.7.2}$$

and that for every $(x, y) \in \Delta_{a,b}$ and every $(x_0, y_0) \in \varphi_{a,b}$ we have

$$z(x, y) = z(x_0, y_0) + \int_{x_0}^{x} \sigma(u)\,du + \int_{y_0}^{y} \tau(v)\,dv + \iint_{\Delta_{x,y}} \frac{\partial^2 z(u, v)}{\partial u \partial v}\,du\,dv. \tag{1.7.3}$$

Here $z(x, y)$ and $z(x_0, y_0)$ denote the values at (x, y) and (x_0, y_0) of the function strongly continuous on $\Delta_{a,b} \cup \varphi_{a,b}$ and representing the given distribution z, and the integrals are taken in the Bochner sense.

Proof. Let $z \in W_1^{1,*}(\Delta_{a,b}; E)$ and let $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x\partial y$ be Bochner integrable on $\Delta_{a,b}$ representants of corresponding distributional derivatives of z. Assume $\varphi \in C_0^\infty(\Delta_{a,b})$ and consider the integral

$$I(\varphi) = -\iint_{\Delta_{a,b}} \left(\int_{g(x)}^{y} \frac{\partial^2 z(x, v)}{\partial x \partial v}\,dv \right) \frac{\partial \varphi}{\partial y}(x, y)\,dx\,dy.$$

By Fubini's theorem,

$$I(\varphi) = -\iiint_{D} \frac{\partial^2 z(x, v)}{\partial x \partial v} \frac{\partial \varphi(x, y)}{\partial y}\,dv\,dx\,dy,$$

where

$$D = \{(v, x, y) \colon g(x-0) < v < y,\ (x, y) \in \Delta_{a,b}\}.$$

But $\Delta_{a,b} = \{(x, y): -\infty < y < b,\ h(y-0) < x < a\}$ and the inequality $g(x-0) < y$ implies $h(y-0) < x$, so that

$$D = \{(v, x, y): x < a,\ g(x-0) < v < y < b\}$$
$$= \{(v, x, y): (x, v) \in \Delta_{a,b},\ v < y < b\}$$

and, by changing the roles of v and y,

$$I(\varphi) = -\iint_{\Delta_{a,b}} \frac{\partial^2 z(x, y)}{\partial z \partial y} \left(\int_y^b \frac{\partial \varphi(x, x)}{\partial v} dv\right) dx\, dy$$
$$= \iint_{\Delta_{a,b}} \frac{\partial^2 z(x, y)}{\partial x \partial y} \varphi(x, y)\, dx\, dy.$$

It means that the distributional derivative

$$\frac{\partial}{\partial y}\left(\int_{g(x)}^{y} \frac{\partial^2 z(x, v)}{\partial x \partial v} dv\right)$$

is represented by the Bochner integrable function $\partial^2 z(x, y)/\partial x \partial y$. But this implies that the distributional partial derivative with respect to y of the function

$$\frac{\partial z(x, y)}{\partial x} - \int_{g(x)}^{y} \frac{\partial^2 z(x, y)}{\partial x \partial v} dv$$

vanishes on $\Delta_{a,b}$ and so this function is equal almost everywhere on $\Delta_{a,b}$ to a function depending only on x. Denote the former function by σ. Then σ is strongly measurable on $(h(b-0), a)$ and

$$\int_{h(b-0)}^{a} (b - g(x)) \|\sigma(x)\| dx = \iint_{\Delta_{a,b}} \|\sigma(x)\| dx\, dy < \infty.$$

Similarly,

$$\frac{\partial z(x, y)}{\partial y} - \int_{h(y)}^{x} \frac{\partial^2 z(u, y)}{\partial u \partial y} du$$

is equal almost everywhere on $\Delta_{a,b}$ to a function τ depending only on y, strongly measurable on $(g(a-0), b)$, and satisfying (1.7.2). Now fix an arbitrary point $(x_0, y_0) \in \varphi_{a,b}$ and consider the E-valued function z_{x_0, y_0} strongly continuous on $\Delta_{a,b} \cup \varphi_{a,b}$ defined by the equality

$$z_{x_0, y_0}(x, y) = \int_{x_0}^{x} \sigma(u)\, du + \int_{y_0}^{y} \tau(v)\, dv + \iint_{\Delta_{x,y}} \frac{\partial^2 z(u, v)}{\partial u \partial v} du\, dv.$$

Let $\varphi \in C_0^\infty(\Delta_{a,b})$. By Fubini's theorem we have

$$-\iint_{\Delta_{a,b}} z_{x_0,y_0}(x,y)\frac{\partial\varphi(x,y)}{\partial y}\,dx\,dy$$

$$= -\iint_{\Delta_{a,b}}\left[\int_{y_0}^{y}\tau(v)\,dv+\int_{g(x)}^{y}\left(\int_{h(v)}^{x}\frac{\partial^2 z(u,v)}{\partial u\,\partial v}\,du\right)dv\right]\frac{\partial\varphi(x,y)}{\partial y}\,dx\,dy$$

$$= -\iint_{\Delta_{a,b}}\left[\tau(v)+\int_{h(v)}^{x}\frac{\partial^2 z(u,v)}{\partial u\,\partial v}\,du\right]\left[\int_{v}^{b}\frac{\partial\varphi(x,y)}{\partial y}\,dy\right]dx\,dv$$

$$= \iint_{\Delta_{a,b}}\left[\tau(y)+\int_{h(y)}^{x}\frac{\partial^2 z(u,y)}{\partial u\,\partial y}\,du\right]\varphi(x,y)\,dx\,dy = \iint_{\Delta_{a,b}}\frac{\partial z(x,y)}{\partial y}\varphi(x,y)\,dx\,dy.$$

It follows that the distributional derivative $\partial(z-z_{x_0,y_0})/\partial y$ vanishes on $\Delta_{a,b}$. Similarly, the distributional derivative $\partial(z-z_{x_0,y_0})/\partial x$ vanishes on $\Delta_{a,b}$. It follows that z is represented by a function strongly continuous on $\Delta_{a,b}\cup\varphi_{a,b}$, equal to z_{x_0,y_0} plus a constant. Since $z_{x_0,y_0}(x_0,y_0)=0$, this constant equals $z(x_0,y_0)$ and so (1.7.3) follows.

1.8. THEOREM. *An E-valued function z strongly continuous on $\Delta_{a,b}\cup\varphi_{a,b}$ belongs to $W_1^{1,*}(\Delta_{a,b};E)$ if and only if there are E-valued functions σ, τ and s with the following properties:*

1° *σ is strongly measurable on $\big(h(b-0),a\big)$ and satisfies* (1.7.1),

2° *τ is strongly measurable on $\big(g(a-0),b\big)$ and satisfies* (1.7.2),

3° *s is Bochner integrable on $\Delta_{a,b}$,*

4° *for every $(x,y)\in\Delta_{a,b}$ and $(x_0,y_0)\in\varphi_{a,b}$ we have*

$$z(x,y)=z(x_0,y_0)+\int_{x_0}^{x}\sigma(u)\,du+\int_{y_0}^{y}\tau(v)\,dv+\iint_{\Delta_{xy}}s(u,v)\,du\,dv. \tag{1.8.1}$$

If $z\in W_1^{1,*}(\Delta_{a,b};E)$ is given by formula (1.8.1) *with σ, τ and s satisfying* 1°-3°, *then the distributional derivatives $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial z\partial y$ are E-valued functions Bochner integrable on $\Delta_{a,b}$ defined by the equalities*

$$\frac{\partial z}{\partial x}(x,y)=\sigma(x)+\int_{g(x)}^{y}s(x,v)\,dv \tag{1.8.2}$$

and

$$\frac{\partial z}{\partial y}(x,y)=\tau(y)+\int_{h(y)}^{x}s(u,y)\,du, \tag{1.8.3}$$

and

$$\frac{\partial^2 z}{\partial x\,\partial y}=s(x,y) \tag{1.8.4}$$

almost everywhere on $\Delta_{a,b}$.

Proof. The "only if" part follows from theorem 1.7. The "if" part may be proved by arguments based on Fubini's theorem, similar to those used in the proof of theorem 1.7.

1.9. THEOREM. *If an E-valued function z strongly continuous on $\Delta_{a,b} \cup \varphi_{a,b}$ is defined by formula (1.8.1) with σ, τ and s satisfying* 1°-3°, *then*

$$(1.9.1) \qquad \lim_{\delta \downarrow 0} \int_{h(b-0)+\varepsilon}^{a-\varepsilon} \left\| \frac{z(x+\delta, g(x)) - z(x, g(x))}{\delta} - \delta(x) \right\| dx = 0$$

for every $\varepsilon \in \left(0, \dfrac{a-h(b-0)}{2}\right)$, *and*

$$(1.9.2) \qquad \lim_{\delta \downarrow 0} \int_{g(a-0)+\varepsilon}^{b-\varepsilon} \left\| \frac{z(h(y), y+\delta) - z(h(y), y)}{\delta} - \tau(y) \right\| dy = 0$$

for every $\varepsilon \in \left(0, \dfrac{b-g(a-0)}{2}\right)$.

Proof. Fixed $\varepsilon \in \left(0, \frac{1}{2}(a-h(b-0))\right)$, for $x \in \left(h(b-0)+\varepsilon, a-\varepsilon\right)$ and $\delta \in (0, \varepsilon)$ we have

$$\frac{1}{\delta}\left(z(x+\delta, g(x)) - z(x, g(x))\right) = \frac{1}{\delta}\int_x^{x+\delta} \sigma(u)\,du + \iint_{\Delta_{x+\delta, g(x)}} s(u, v)\,du\,dv.$$

Since

$$\lim_{\delta \to +0} \int_{h(b-0)+\varepsilon}^{a-\varepsilon} \left\| \frac{1}{\delta} \int_x^{x+\delta} \sigma(u)\,du - \sigma(x) \right\| dx = 0,$$

equality (1.9.1) will be proved if we show that

$$(1.9.3) \qquad \lim_{\delta \to +0} \int_{h(b-0)}^{a-\varepsilon} \left(\frac{1}{\delta} \iint_{\Delta_{x+\delta, g(x)}} \|s(u, v)\|\,du\,dv \right) dx = 0.$$

To prove (1.9.3) put

$$\lambda_\eta(u) = \int_{g(u)}^{\min(b, g(u)+\eta)} \|s(u, v)\|\,dv, \qquad N(u) = \int_{g(u)}^{b} \|s(u, v)\|\,dv$$

for almost every $u \in \left(h(b-0), a\right)$ and any $\eta > 0$, and

$$\lambda_{\eta,\delta}(x) = \frac{1}{\delta}\int_x^{x+\delta} \lambda_\eta(u)\,du, \qquad N_\delta(x) = \frac{1}{\delta}\int_x^{x+\delta} N(u)\,du$$

for $x \in (h(b-0), a-\varepsilon)$, $\delta \in (0, \varepsilon)$ and $\eta > 0$. Then

$$\lim_{\delta\to+0} \int_{h(b-0)}^{a-\varepsilon} |N_\delta(x) - N(x)|\, dx = 0 \tag{1.9.4}$$

and, for any $\delta \in (0, \varepsilon)$,

$$\begin{aligned}\int_{h(b-0)}^{a-\varepsilon} \lambda_{\eta,\delta}(x)\, dx &= \frac{1}{\delta} \int_{h(b-0)}^{a-\varepsilon} \left(\int_0^\delta \lambda_\eta(x+u)\, du \right) dx \\ &= \frac{1}{\delta} \int_0^\delta \left(\int_{h(b-0)}^{a-\varepsilon} \lambda_\eta(x+u)\, dx \right) du \leqslant \int_{h(b-0)}^{a} \lambda_\eta(x)\, dx \\ &= \iint_{\pi_\eta} \|s(u, v)\|\, dx\, dy,\end{aligned} \tag{1.9.5}$$

where

$$\pi_\eta = \{(x, y) : h(b-0) < x < a,\ g(x-0) < y < \min(b, g(x-0)+\eta)\}.$$

At last, for $\eta > 0$ and $\delta \in (0, \varepsilon)$, put

$$e_{\eta,\delta} = \{x : h(b-0) < x < a-\varepsilon,\ g(x+\delta) > g(x) - \eta\}.$$

Then for any $\eta > 0$,

$$\lim_{\delta\to+0} \operatorname{meas}\big((h(b-0), a-\varepsilon) \setminus e_{\eta,\delta}\big) = 0. \tag{1.9.6}$$

For any $\eta > 0$, $\delta \in (0, \varepsilon)$ and $x \in (h(b-0), a-\varepsilon)$ we have

$$\frac{1}{\delta} \iint_{\Delta_{x+\delta, g(x)}} \|s(u, v)\|\, du\, dv \leqslant \begin{cases} \lambda_{\eta,\delta}(x), & \text{if } x \in e_{\eta,\delta}, \\ N_\delta(x), & \text{if } x \notin e_{\eta,\delta}, \end{cases}$$

and so, by (1.9.5),

$$\begin{aligned}&\int_{h(b-0)}^{a-\varepsilon} \left(\frac{1}{\delta} \iint_{\Delta_{x+\delta, g(x)}} \|s(u, v)\|\, du\, dv \right) dx \\ &\leqslant \iint_{\pi_\eta} \|s(x, y)\|\, dx\, dy + \int_{h(b-0)}^{a-\varepsilon} |N_\delta(x) - N(x)|\, dx + \int_{(h(b-0), a-\varepsilon) \setminus e_{\eta,\delta}} N(x)\, dx,\end{aligned}$$

from where, by (1.9.4) and (1.9.6), equality (1.9.3) follows. Hence (1.9.1) is proved.

The proof of (1.9.2) is analogous.

2. *CAUCHY'S PROBLEM IN THE CLASS* $W_1^{1,*}(\Delta_{a,b}; E)$

2.1. Assumptions. Let E be a Banach space. Let $f(x, y, z, p, q)$ be an E-valued function defined for $(x, y) \in \Delta_{a,b}$ and $z, p, q \in E$, which for every fixed point $(x, y) \in \Delta_{a,b}$ is strongly continuous with respect to (z, p, q) on E^3 and for every fixed triple $(z, p, q) \in E^3$ is Bochner integrable with respect to (x, y) on $\Delta_{a,b}$. Let $\sigma(x)$ and $\tau(y)$ be E-valued functions Bochner integrable on $\big(h(b-0), a\big)$ or $\big(g(a-0), b\big)$, respectively. At last let $(x_0, y_0) \in \varphi_{a,b}$ and $z_0 \in E$ be given.

2.2. Definition. Under assumptions 2.1 we ask about a function, $z \in W_1^{1,*}(\Delta_{a,b}; E)$ satisfying the equation

$$\frac{\partial^2 z}{\partial x \partial y} = f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \tag{2.2.1}$$

almost everywhere in $\Delta_{a,b}$, such that

$$z(x_0, y_0) = z_0 \tag{2.2.2}$$

and that, furthermore,

$$\frac{\partial z(x, y)}{\partial x} = \sigma(x) + \int_{g(x)}^{y} \frac{\partial^2 z(x, v)}{\partial x \partial v} dv \tag{2.2.3}$$

and

$$\frac{\partial z(x, y)}{\partial y} = \tau(y) + \int_{h(y)}^{x} \frac{\partial^2 z(u, y)}{\partial u \partial y} du \tag{2.2.4}$$

almost everywhere in $\Delta_{a,b}$, the integrals taken in Bochner sense. Such a function z, if it exists, will be called a *solution of the class* $W_1^{1,*}(\Delta_{a,b}; E)$ of the Cauchy's problem for equation (2.2.1) under boundary conditions (2.2.2)-(2.2.4).

2.3. Connection with a problem considered in [2]. It follows from Lemma 1.3 and theorems 1.7-1.9 that under assumptions of continuity of $g(x)$ and $h(y)$ in open intervals $h(b-0) < x < a$ and $g(a-0) < y < b$ the Cauchy's problem defined above reduces to Cauchy's problem in the class $L_1^*(\Delta)$ considered in [2].

2.4. Assumptions. Let $t_0 = \inf\{x+y : (x, y) \in \Delta_{a,b}\}$ and let $\omega(t, r)$ be a function defined for $t \in [t_0, a+b)$ and $r \geqslant 0$, non-negative, for every fixed $t \in [t_0, a+b)$ continuous and non-decreasing in r on $[0, \infty)$, for every fixed $r \geqslant 0$ Lebesgue integrable in t on $[t_0, a+b)$, and such that

$$\omega(t, r) \leqslant L(t)(1+r) \quad \text{for } t \in [t_0, a+b) \text{ and } r \geqslant 0, \tag{2.4.1}$$

where $L(t)$ is a non-negative function Lebesgue integrable on $[t_0, a+b)$,

(2.4.2) $$\omega(t, 0) = 0 \quad \text{for a.e. } t \in [t_0, a+b),$$

and for every $\varepsilon \in (0, a+b-t_0]$ the unique non-negative function $R(t)$, continuous and satisfying the equation

$$R(t) = \int_{t_0}^{t} \omega\big(\tau, R(\tau)\big)\, d\tau \quad \text{on } [t_0, t_0+\varepsilon),$$

is $R(t) \equiv 0$ for $t \in [t_0, t_0+\varepsilon)$.

2.5. THEOREM. *Under assumptions* 2.1 *suppose that*

(2.5.1) $$\|f(x, y, z, p, q) - f(x, y, \tilde{z}, \tilde{p}, \tilde{q})\| \leqslant \omega\big(x+y, \max(K\|z-\tilde{z}\|, \|p-\tilde{p}\|, \|q-\tilde{q}\|)\big)$$

for every $(x, y) \in \Delta_{a,b}$ *and all* $z, p, q, \tilde{z}, \tilde{p}, \tilde{q} \in E$, *where* $K = \text{const} \geqslant 0$ *and the function* $\omega(t, r)$ *satisfies assumptions* 2.4. *Then the Cauchy's problem* (2.2.1)-(2.2.4) *has one and only one solution of the class* $W_1^{1,*}(\Delta_{a,b}; E)$.

The following Lemmas 2.6-2.8 are needed for the proof of this theorem.

2.6. LEMMA. *Let* $B(x, y)$ *be a non-negative function Lebesgue integrable on* $\Delta_{a,b}$ *and* $L(t)$ *a non-negative function Lebesgue integrable on* $[t_0, a+b)$, *where* $t_0 = \inf\{x+y: (x, y) \in \Delta_{a,b}\}$. *Let* K *be the linear operator of the space* $L_1(\Delta_{a,b})$ *of functions Lebesgue integrable on* $\Delta_{a,b}$ *into itself defined by the equality*

$$(\mathscr{K}s)(x, y) = B(x, y) \iint_{\Delta_{x,y}} s(u, v)\, du\, dv + L(x+y) \Big(\int_{g(x)}^{y} s(x, v)\, dv + \int_{h(y)}^{x} s(u, y)\, du \Big)$$

almost everywhere in $\Delta_{a,b}$ *for every* $s \in L_1(\Delta_{a,b})$. *Then the spectral radius of* $\mathscr{K}$ *equals zero.*

Proof. Assuming that $B(x, y) = 0$ for $(x, y) \notin \Delta_{a,b}$ put

$$\mathscr{L}(t) = 2L(t) + \frac{1}{2} \int_{-\infty}^{\infty} B\Big(\frac{t+\tau}{2}, \frac{t-\tau}{2}\Big)\, d\tau, \qquad t \in [t_0, a+b),$$

and for every $\lambda > 0$ define in $L_1(\Delta_{a,b})$ the norm $\| \ \|_\lambda$, equivalent to the usual one, putting

$$\|s\|_\lambda = \sup_{t \in [t_0, a+b]} e^{-\lambda \int_{t_0}^{t} \mathscr{L}(\omega)\, d\omega} \iint_{\substack{(x,y) \in \Delta_{a,b} \\ x+y \leqslant t}} |s(x, y)|\, dx\, dy.$$

We shall show that for every $\lambda > 0$

$$\|\mathcal{K}_s\|_\lambda \leqslant \frac{1}{\lambda}\|s\|_\lambda, \quad s \in L_1(\Delta_{a,b}), \tag{2.6.1}$$

whence the theorem follows immediately.

In order to prove (2.6.1) let $s \in L_1(\Delta_{a,b})$ and $\lambda > 0$ be arbitrarily fixed. Assuming that $s(x, y) = 0$ and $(\mathcal{K}_s)(x, y) = 0$ for $(x, y) \notin \Delta_{a,b}$, we have for every $T \geqslant t_0$

$$\iint_{x+y\leqslant T} |(\mathcal{K}_s)(x, y)|\,dx\,dy = \frac{1}{2}\int_{t_0}^{T}\int_{-\infty}^{\infty} (\mathcal{K}_s)\left(\frac{t+\tau}{2}, \frac{t-\tau}{2}\right) d\tau\, dt$$

$$= \frac{1}{2}\int_{t_0}^{T}\int_{-\infty}^{\infty} \left| B\left(\frac{t+\tau}{2}, \frac{t-\tau}{2}\right) \iint_{\Delta_{(t+\tau)/2,(t-\tau)/2}} s(u, v)\,du\,dv + L(t) \times\right.$$

$$\left.\times \int_{g((t+\tau)/2)}^{(t-\tau)/2} s\left(\frac{t+\tau}{2}, v\right) dv + L(t) \int_{h((t-\tau)/2)}^{(t+\tau)/2} s\left(u, \frac{t-\tau}{2}\right) du \right| d\tau\, dt$$

$$\leqslant \frac{1}{2}\int_{t_0}^{T}\left[\int_{-\infty}^{\infty} B\left(\frac{t+\tau}{2}, \frac{t-\tau}{2}\right) d\tau\, \|s\|_\lambda e^{\lambda\int_{t_0}^{t}\mathcal{L}(\sigma)d\sigma} + \right.$$

$$\left. + L(t)\int_{-\infty}^{\infty}\int_{-\infty}^{(t-\tau)/2} \left| s\left(\frac{t+\tau}{2}, v\right)\right| dv\,d\tau + L(t)\int_{-\infty}^{\infty}\int_{-\infty}^{(t+\tau)/2} \left| s\left(u, \frac{t-\tau}{2}\right)\right| du\,d\tau\right] dt.$$

We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{(t-\tau)/2} \left| s\left(\frac{t+\tau}{2}, v\right)\right| dv\,d\tau = 2\iint_{u+v\leqslant t/2} \left| s\left(\frac{t}{2} + u, v\right)\right| du\,dv$$

$$= 2\iint_{x+y\leqslant t} |s(x, y)|\,dx\,dy \leqslant 2\|s\|_\lambda e^{\lambda\int_{t_0}^{t}\mathcal{L}(\sigma)d\sigma}$$

and, similarly,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{(t+\tau)/2} \left| s\left(u, \frac{t-\tau}{2}\right)\right| du\,d\tau \leqslant 2\|s\|_\lambda e^{\lambda\int_{t_0}^{t}\mathcal{L}(\sigma)d\sigma}.$$

Consequently,

$$\iint_{x+y\leqslant T} |(\mathcal{K}s)(x, y)|\,dx\,dy \leqslant \|s\|_\lambda \int_{t_0}^{T} \mathcal{L}(t) e^{\lambda\int_{t_0}^{t}\mathcal{L}(\sigma)d\sigma} dt = \frac{1}{\lambda}\|s\|_\lambda\left(e^{\lambda\int_{t_0}^{T}\mathcal{L}(\sigma)d\sigma} - 1\right)$$

for every $T \geqslant t_0$, which implies (2.6.1).

2.7. Lemma. *Let* $t_0 = \inf\{x+y\colon (x,y)\in\Delta_{a,b}\}$, $k = \mathrm{const}\geqslant 0$, *and let* $L(t)$ *be a non-negative function Lebesgue integrable on* $[t_0, a+b)$.

Put

$$c = \exp\Big((a+b-t_0+2)\int_{t_0}^{a+b} L(t)\,dt\Big).$$

If a function $s(x,y)$ *Lebesgue integrable on* $\Delta_{a,b}$ *satisfies almost everywhere in* $\Delta_{a,b}$ *the inequality*

$$s(x,y)\leqslant L(x+y)\Big(k+\int_{\Delta_{x,y}} s(u,v)\,du\,dv+\int_{g(x)}^{y} s(x,v)\,dv+\int_{h(y)}^{x} s(u,y)\,du\Big),$$

then

$$s(x,y)\leqslant kcL(x+y)$$

almost everywhere in $\Delta_{a,b}$.

Proof. Put

$$r(t) = kL(t)\exp\Big((a+b-t_0+2)\int_{t_0}^{t} L(\tau)\,d\tau\Big);$$

$$d(x,y) = \max\big(0, s(x,y)-r(x+y)\big).$$

We need only to prove that $d(x,y) = 0$ almost everywhere in $\Delta_{a,b}$. In this order observe that

$$\begin{aligned}
&L(x+y)\Big(k+\iint_{\Delta_{x,y}} r(u+v)\,du\,dv+\int_{g(x)}^{y} r(x+v)\,dv+\int_{h(y)}^{x} r(u+y)\,du\Big)\\
&\quad\leqslant L(x+y)\Big(k+\int_{t_0-y}^{x}\int_{t_0-u}^{y} r(u+v)\,du\,dv+\int_{t_0-x}^{y} r(x+v)\,dv+\int_{t_0-y}^{y} r(u+y)\,du\Big)\\
&\quad= L(x+y)\Big(k+\int_{t_0}^{x+y}\int_{t_0}^{u} r(v)\,dv\,du+2\int_{t_0}^{x+y} r(v)\,dv\Big)\\
&\quad\leqslant L(x+y)\Big(k+(a+b-t_0+2)\int_{t_0}^{x+y} r(t)\,dt\Big) = r(x+y)
\end{aligned}$$

almost everywhere in $\Delta_{a,b}$, whence

$$0\leqslant d(x,y)\leqslant L(x+y)\Big(\iint_{\Delta_{x,y}} d(u,v)\,du\,dv+\int_{g(x)}^{y} d(x,v)\,dv+\int_{h(y)}^{x} d(u,y)\,du\Big)$$

almost everywhere in $\Delta_{a,b}$. The former inequality may be written as

(2.7.1) $$0\leqslant d(x,y)\leqslant (\mathscr{K}d)(x,y)$$

almost everywhere in $\Delta_{a,b}$, where $\mathscr{K}$ is the operator considered in Lemma 2.6 with $B(x,y) = L(x+y)$.

It follows from Lemma 2.6 that l.i.m. $\mathscr{K}^n d = 0$ on $\Delta_{a,b}$. On the other hand, it follows from (2.7.1) and from the monotonity of $\mathscr{K}$ that

$$0 \leqslant d(x, y) \leqslant (\mathscr{K}^n d)(x, y)$$

almost everywhere in $\Delta_{a,b}$ for $n = 1, 2, \ldots$ Consequently, $d(x, y) = 0$ almost everywhere in $\Delta_{a,b}$.

2.8. LEMMA. *Under assumptions* 2.4, *if a non-negative function* $d(x, y)$ *Lebesgue integrable on* $\Delta_{a,b}$ *satisfies almost everywhere in* $\Delta_{a,b}$ *the inequality*

(2.8.1)
$$d(x, y) \leqslant \omega\Big(x+y, \max\Big(K \iint_{\Delta_{x,y}} d(u, v)\,du\,dv, \int_{g(x)}^{y} d(x, v)\,dv, \int_{h(y)}^{x} d(u, y)\,du\Big)\Big),$$

where $K = \text{const} \geqslant 0$, *then* $d(x, y) = 0$ *almost everywhere in* $\Delta_{a,b}$.

Proof. Since by (2.4.1), (2.8.1) and Lemma 2.7 we have

$$d(x, y) \leqslant \text{const}\cdot L(x+y)$$

for almost every $(x, y) \in \Delta_{a,b}$, the set

$$Z = \{f: f \in L_1(t_0, a+b),\ f(x+y) \geqslant d(x, y) \text{ for almost every } (x, y) \in \Delta_{a,b}\}$$

is non-void. Define the function $r \in L_1(t_0, a+b)$ as the infimum of the set Z with respect to the relation of inequality almost everywhere in $(t_0, a+b)$. Since Z contains the infimum of every its countable subset, it follows that $r \in Z$ (namely, it is easy to see that if $f_n \in Z$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} \|f_n\| = \inf\{\|f\|: f \in Z\}$, then $r = \inf\{f_n: n = 1, 2 \ldots\}$). Thus r has following properties:

(2.8.2) $\quad r(x+y) \geqslant d(x, y)$ for almost every $(x, y) \in \Delta_{a,b}$,

(2.8.3) if $\varrho \in L_1(t_0, a+b)$ and $\varrho(x+y) \geqslant d(x, y)$ for almost every $(x, y) \in \Delta_{a,b}$, then $\varrho(t) \geqslant r(t)$ for almost every $t \in [t_0, a+b)$.

In view of (2.8.2), our lemma will be proved if we show that $r(t) = 0$ for almost every $t \in [t_0, a+b)$. By (2.8.1) and (2.8.2) we have

$$d(x, y) \leqslant \omega\Big(x+y, \max\Big(K \iint_{\Delta_{x,y}} r(u+v)\,du\,dv, \int_{g(x)}^{y} r(x+v)\,dv, \int_{h(y)}^{x} r(u+y)\,du\Big)\Big)$$
$$\leqslant \omega\Big(x+y, \max\Big(K \int_{t_0}^{x+y} \int_{t_0}^{u} r(v)\,dv\,du, \int_{t_0}^{x+y} r(v)\,dv\Big)\Big)$$

almost everywhere in $\Delta_{a,b}$, whence by (2.8.3) it follows that

(2.8.4)
$$0 \leqslant r(t) \leqslant \omega\Big(t, \max\Big(K \int_{t_0}^{t} \int_{t_0}^{u} r(\tau)\,d\tau\,du, \int_{t_0}^{t} r(\tau)\,d\tau\Big)\Big)$$

for almost every $t \in [t_0, a+b)$. Put

$$R(t) = \int_{t_0}^{t} r(\tau)\,d\tau,$$

$$t_1 = \max\{t\colon t \in [t_0, a+b],\ R(\tau) \equiv 0 \text{ for } \tau \in [t_0, t]\}.$$

The former definition is correct, since $R(t_0) = 0$. The proof will be complete if we show that $R(t) \equiv 0$ for $t \in [t_0, a+b]$, i.e. if we show that $t_1 = a+b$. Suppose that this is not true, so that $t_0 \leqslant t_1 < a+b$, and let $t_2 = \min(a+b, t_1 + 1/K)$. We then have

$$R(t_1) = 0$$

and, by (2.8.4),

$$\frac{dR(t)}{dt} \leqslant \omega\big(t, R(t)\big) \quad \text{for almost every } t \in [t_1, t_2].$$

By a theorem on differential inequalities [1] it follows that $R(t)$ is not greater than the maximal absolutely continuous solution $\bar{R}(t)$ of the Cauchy's problem

$$\begin{cases} \dfrac{d\bar{R}(t)}{dt} = \omega\big(t, R(t)\big) \quad \text{for almost every } t \in [t_1, t_1+\varepsilon], \\ \bar{R}(t_1) = 0 \end{cases}$$

in every interval $[t_0, t_1+\varepsilon]$, $0 < \varepsilon \leqslant t_2 - t_1$, in which $\bar{R}(t)$ exists.

Since, by (2.4.2), $\bar{R}(t)$ exists and equals zero in the whole $[t_1, t_2]$, we infer, that $R(t) \equiv 0$ for $t \in [t_1, t_2]$ in contradiction to the definition of t_1. The proof is completed.

2.9. Proof of theorem 2.5. It follows from Theorem 1.8 that a function $z \in W_1^{1,*}(\Delta_{a,b}; E)$ is a solution of the Cauchy's problem (2.2.1)-(2.2.4) if and only if it is given by formula (1.8.1), where s belongs to the space $L_1(\Delta_{a,b}; E)$ of E-valued functions Bochner integrable on $\Delta_{a,b}$ and satisfies almost everywhere in $\Delta_{a,b}$ the equality

$$(2.9.1) \quad s(x, y) = f\Big(x, y, z_0 + \int_{x_0}^{x} \sigma(u)\,du + \int_{y_0}^{y} \tau(v)\,dv + \iint_{\Delta_{x,y}} s(u, v)\,du\,dv,\ \sigma(x) + \int_{g(x)}^{y} s(x, v)\,dv,\ \tau(y) + \int_{h(y)}^{x} s(u, y)\,du\Big).$$

Hence we have to prove that there exists an $s \in L_1(\Delta_{a,b}; E)$ satisfying (2.9.1) almost everywhere in $\Delta_{a,b}$ and that it is determined uniquely up to the equality almost everywhere in $\Delta_{a,b}$.

The uniqueness of s follows at once from Lemma 2.8. Indeed, if s and $\tilde{s}$ belong to $L_1(\Delta_{a,b}; E)$ and satisfy (2.9.1), then $d(x, y) = \|s(x, y) - \tilde{s}(x, y)\|$ is a real non-negative function Lebesgue integrable on $\Delta_{a,b}$, which, by (2.5.1), satisfies inequality (2.8.1) almost everywhere in $\Delta_{a,b}$, and so, by Lemma 2.8, $d(x, y) = 0$ almost everywhere in $\Delta_{a,b}$.

For the proof of the existence of s let A be an arbitrary but fixed real non-negative function Lebesgue integrable on $\Delta_{a,b}$ satisfying almost everywhere in $\Delta_{a,b}$ the inequality

$$(2.9.2)\qquad A(x, y) \geqslant \|f(x, y, 0, 0, 0)\| + L(x+y)\Big(1 + K\|z_0 + \int_{x_0}^{x} \sigma(u)\,du + \int_{y_0}^{y} \tau(v)\,dv\| + \|\sigma(x)\| + \|\tau(y)\|\Big),$$

and let M be a real non-negative function Lebesgue integrable on $\Delta_{a,b}$ satisfying almost everywhere in $\Delta_{a,b}$ the equality

$$(2.9.3)\qquad M(x, y) = A(x, y) + KL(x+y) \iint_{\Delta_{x,y}} M(u, v)\,du\,dv + L(x+y)\Big(\int_{g(x)}^{y} M(x, v)\,dv + \int_{h(y)}^{x} M(u, y)\,du\Big).$$

The existence of M follows from Lemma 2.6. Namely, $M = \sum_{n=0}^{\infty} \mathscr{K}^n A$, where $\mathscr{K}$ is the operator considered in Lemma 2.6 with $B(x, y) = KL(x+y)$. For any $s \in L_1(\Delta_{a,b}; E)$ let F_s be the E-valued function defined by the equality

$$(2.9.4)\qquad (Fs)(x, y) = f\Big(x, y, z_0 + \int_{x_0}^{x} \sigma(u)\,du + \int_{y_0}^{y} \tau(v)\,dv + \iint_{\Delta_{x,y}} s(u, v)\,du\,dv,\ \sigma(x) + \int_{g(x)}^{y} s(x, v)\,dv,\ \tau(y) + \int_{h(y)}^{x} s(u, y)\,du\Big)$$

almost everywhere in $\Delta_{a,b}$. Then, as follows from assumption 2.1, Fs is strongly measurable on $\Delta_{a,b}$. Furthermore, it follows from (2.5.1), (2.4.1), (2.9.2) and (2.9.3) that if $\|s(x, y)\| \leqslant M(x, y)$ almost everywhere in $\Delta_{a,b}$, then $\|(Fs)(x, y)\| \leqslant M(x, y)$ almost everywhere in $\Delta_{a,b}$.

Let $s_0 \in L_1(\Delta_{a,b}; E)$ satisfy the inequality $\|s_0(x, y)\| \leqslant M(x, y)$ almost everywhere in $\Delta_{a,b}$. For every $n = 1, 2, \ldots$ put $s_n = Fs_{n-1}$. Then by the preceding remark we have

$$(2.9.5)\qquad \|s_n(x, y)\| \leqslant M(x, y)$$

almost everywhere in $\Delta_{a,b}$ for every $n = 1, 2, \ldots$ We shall show that the sequence $s_n(x, y)$, $n = 1, 2, \ldots$, converges almost everywhere in $\Delta_{a,b}$. Put

$$d(x, y) = \limsup_{n,m\to\infty} \|s_n(x, y) - s_m(x, y)\|.$$

Then, by (2.9.5), d is Lebesgue integrable on $\Delta_{a,b}$ and, by (2.9.5) and the Fatou Lemma,

$$\limsup_{n,m\to\infty} \iint_{\Delta_{x,y}} \|s_n(u, v) - s_m(u, v)\| du\, dv \leqslant \iint_{\Delta_{x,y}} d(u, v)\, du\, dv \tag{2.9.6}$$

for $(x, y) \in \Delta_{a,b}$ and, furthermore,

$$\limsup_{n,m\to\infty} \int_{g(x)}^{y} \|s_n(x, v) - s_m(x, v)\| dv \leqslant \int_{g(x)}^{y} d(x, v)\, dv \tag{2.9.7}$$

and

$$\limsup_{n,m\to\infty} \int_{h(y)}^{x} \|s_n(u, y) - s_m(u, y)\| du \leqslant \int_{h(y)}^{x} d(u, y)\, du \tag{2.9.8}$$

for almost every $(x, y) \in \Delta_{a,b}$. by (2.9.4) and (2.5.1) for every $n, m = 1, 2, \ldots$ we have

$$\|s_n(x, y) - s_m(x, y)\| = \|(Fs_{n-1})(x, y) - (Fs_{m-1})(x, y)\|$$
$$\leqslant \omega\Big(x+y, \max\Big(K \iint_{\Delta_{x,y}} \|s_{n-1}(u, v) - s_{m-1}(u, v)\| du\, dv,$$
$$\int_{g(x)}^{y} \|s_{n-1}(x, v) - s_{m-1}(x, v)\| dv, \int_{h(y)}^{x} \|s_{n-1}(u, y) - s_{m-1}(u, y)\| du$$

almost everywhere in $\Delta_{a,b}$. Since $\omega(t, r)$ is non-decreasing and continuous in r, it follows from (2.9.6)-(2.9.9) that d satisfies almost everywhere in $\Delta_{a,b}$ inequality (2.8.1), and thus, by Lemma 2.8, $d(x, y) = 0$ almost everywhere in $\Delta_{a,b}$. This shows that the sequence $s_n(x, y)$, $n = 1, 2, \ldots$, strongly converges almost everywhere in $\Delta_{a,b}$.

Put

$$s(x, y) = \lim_{n\to\infty} s_n(x, y). \tag{2.9.10}$$

Then, by (2.9.5), $s \in L_1(\Delta_{a,b}; E)$ and by the Lebesgue bounded convergence theorem we have

$$\lim_{n\to\infty} \iint_{\Delta_{x,y}} s_n(u, v)\, du\, dv = \iint_{\Delta_{x,y}} s(u, v)\, du\, dv \tag{2.9.11}$$

for $(x, y) \in \Delta_{a,b}$ and

$$(2.9.12)\qquad \begin{cases} \lim_{n\to\infty} \int\limits_{g(x)}^{y} s_n(x, v)\,dv = \int\limits_{g(x)}^{y} s(x, v)\,dv, \\ \lim_{n\to\infty} \int\limits_{h(y)}^{x} s_n(u, y)\,du = \int\limits_{h(y)}^{x} s(u, y)\,du \end{cases}$$

for almost every $(x, y) \in \Delta_{a,b}$. By (2.9.4) for every $n = 1, 2, \ldots$ we have

$$(2.9.13)\qquad s_n(x, y) = f\Big(x, y, z_0 + \int\limits_{x_0}^{x} \sigma(u)\,du + \int\limits_{y_0}^{y} \tau(v)\,dv +$$

$$+ \iint\limits_{\Delta_{x,y}} s_{n-1}(u, v)\,du\,dv,\ \sigma(x) + \int\limits_{g(x)}^{y} s_{n-1}(x, v)\,dv,\ \tau(y) + \int\limits_{h(y)}^{x} s_{n-1}(u, y)\,du\Big)$$

almost everywhere in $\Delta_{a,b}$. Since $f(x, y, z, p, q)$ is continuous with respect to (z, p, q), equalities (2.9.10)-(2.9.13) imply that s satisfies almost everywhere in $\Delta_{a,b}$ equality (2.9.1), which completes the proof.

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