ON SECOND ORDER SEMI-IMPLICIT FOURIER SPECTRAL METHODS FOR 2D CAHN-HILLIARD EQUATIONS

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ABSTRACT. We consider several seconder order in time stabilized semi-implicit Fourier spectral schemes for 2D Cahn-Hilliard equations. We introduce new stabilization techniques and prove unconditional energy stability for modified energy functionals. We also carry out a comparative study of several classical stabilization schemes and identify the corresponding stability regions. In several cases the energy stability is proved under relaxed constraints on the size of the time steps. We do not impose any Lipschitz assumption on the nonlinearity. The error analysis is obtained under almost optimal regularity assumptions.

1. INTRODUCTION

In this work we consider numerical schemes for solving the Cahn-Hilliard (CH) equation:

$$\begin{cases} \partial_t u = \Delta(-\nu\Delta u + f(u)), \quad (x,t) \in \Omega \times (0,\infty), \\ u\Big|_{t=0} = u_0. \end{cases} \tag{1.1}$$

The CH equation was originally introduced by Cahn and Hilliard in [4] to describe the phase separation and coarsening phenomena in non-uniform systems such as alloys, glasses and polymer mixtures. Here we consider dimension two and take the spatial domain Ω to be the usual 2π-periodic torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. The equation (1.1) in its present form is already non-dimensionalized. The function $u = u(x,t)$ is real-valued and typically represents the difference of the concentration of one of the phases. The term $f(u) = F'(u)$ with $F(u)$ being a given double-well potential as

$$F(u) = \frac{1}{4}(u^2-1)^2, \quad f(u) = u^3 - u.$$

The minima $u = \pm 1$ is connected with the formation of domains. In the fourth order dissipation term, the constant $\nu > 0$ is the diffusion coefficient. Since the equation is already non-dimensionalized, the size of ν controls the competition between the nonlinear coarsening effect and the linear smoothing effect. The CH equation can be alternatively viewed as the gradient flow of the Ginzburg-Landau type energy functional

$$E(u) := \int_\Omega (\frac{1}{2}\nu|\nabla u|^2 + F(u))dx \tag{1.2}$$

in H^{-1}. For smooth solutions the basic energy identity takes the form

$$\frac{d}{dt}E(u(t)) + \int_\Omega |\nabla(-\nu\Delta u + f(u))|^2 dx = 0.$$

Consequently $E(u(t)) \le E(u(s))$ for any $0 \le t < s < \infty$. By using this a priori H^1 bound and the fact that the critical space in 2D is L^2, one can deduce global wellposedness in H^s for any $s \ge 0$.

There is an extensive body of bibliography on the numerical simulation and analysis of the CH equation and related phase field models (cf. [3, 5, 6, 13, 20, 23, 25, 33, 8, 9, 14, 18, 19, 27, 32] and the references therein). In [10] Feng and Prohl first obtained the error analysis of a semi-discrete in time and fully discrete finite element method for CH. In [26], Sun derived a second order accurate finite difference scheme and obtained the error bound $O(\Delta x^2 + \Delta y^2 + \Delta t^2)$ in discrete L^2-norm. It is well known that explicit schemes usually suffer from very severe time step restrictions and do not obey energy conservation. Therefore in

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practice one usually resorts to semi-implicit or even fully implicit numerical schemes. In [5] Chen and Shen considered a semi-implicit Fourier spectral scheme for the CH equation (1.1). The numerical scheme is forward in time with the linear part treated implicitly and the nonlinear part evaluated explicitly, which is a typical feature of semi-implicit (linearly implicit) schemes. On the other hand, it is known that due to truncation errors semi-implicit schemes can lose energy stability for large time steps. As a result smaller time steps are usually enforced in practice. To resolve this problem, a new class of stabilized semi-implicit methods were introduced in [11, 18, 27, 31, 32]. A remarkable feature of these new schemes is that (in practice) larger time steps can be taken whilst not losing energy stability. Roughly speaking, for a p^{th} order (in time) method, the basic idea is to add an additional $O(\Delta t^p)$ well-chosen auxiliary term (henceforth called "stabilizing term") to the numerical scheme in order to alleviate the time step constraint. Formally speaking the $O(\Delta t^p)$ term vanishes as $\Delta t \to 0$ and numerical solution is expected to converge to the true PDE solution. In [32] the authors considered the modified Cahn-Hilliard-Cook equation of the form

$$\partial_t C = \nabla \cdot \big((1 - aC^2)\nabla(C^3 - C - \kappa\nabla^2 C)\big). \tag{1.3}$$

In the Fourier spectral approximation, they adopted a stabilization term as

$$-A\Delta^2(C^{n+1} - C^n),$$

where C^n is the numerical solution at time step t_n. Concerning the CH model (1.1), He, Liu and Tang in [18] introduced a semi-implicit Fourier spectral scheme with an $O(\Delta t)$ stabilization term

$$A\Delta(u^{n+1} - u^n).$$

To prove energy stability $E(u^{n+1}) \le E(u^n)$, they imposed a condition on the stabilization parameter A which reads as

$$A \ge \max_{x\in\Omega}\Big\{\frac{1}{2}|u^n(x)|^2 + \frac{1}{4}|u^{n+1}(x) + u^n(x)|^2\Big\} - \frac{1}{2}, \qquad \forall\, n \ge 0. \tag{1.4}$$

This bound is conditional since it depends on the numerical solution itself which implicitly could also depend on A.

In order to obtain energy stability without any a priori assumption on the numerical solution, Shen and Yang [27] developed a novel idea of using effectively Lipschitz nonlinearities. The remarkable observation is that in practical numerical simulations the numerical solutions always stay well bounded and the nonlinear term effectively coincides with a truncated nonlinearity. More precisely, by assuming

$$\max_{u\in\mathbb{R}} |\tilde{f}'(u)| \le L,$$

where $\tilde{f}(u)$ is a suitable "modification" of the original function $f(u)$, Shen and Yang proved unconditional energy stability for both Allen-Cahn and CH equations. This idea was followed up recently in [11] for the analysis of stabilized Crank-Nicolson or Adams-Bashforth scheme for Allen-Cahn and CH equations.

The main drawback of the aforementioned analytic developments is that to obtain energy stability, one either makes a Lipschitz assumption on the nonlinearity, or one assumes some additional L^∞ bounds on the numerical solution. An important problem is to remove these technical obstacles and prove energy stability for a large class of stabilized semi-implicit numerical schemes for general phase field models. In our recent work [22] by using harmonic analysis in borderline spaces (see for example [1, 2, 21]), we have obtained a result of this kind for 2D phase field models such as CH and thin film equations. We considered a first order in time stabilized semi-implicit Fourier spectral scheme and proved unconditional energy stability when the stabilization parameter is sufficiently large depending only on the diffusion coefficient and the initial data. In recent [24] we have settled the 3D CH case by a novel bootstrapping argument (to overcome the issue of uniform L^∞ bounds). Note that all these results are restricted to the first order in time methods where the energy can be shown to decrease monotonically in time. The situation with higher order in time methods are far more complex since it is known that energy is only approximately preserved over moderately long time intervals. This brings the question of how to design robust stabilized high order in time methods with good energy conservation. A further problem is to investigate the issue of conditional or unconditional energy stability, characterize the stabilization parameter and identify the stability region in various situations. The purpose of this work is to analyze a family of second order in time semi-implicit Fourier spectral schemes for the 2D CH equation (1.1). Perhaps a surprising result is that by choosing a good stabilization term we can prove unconditional energy stability albeit for a modified energy functional. Moreover for several classical second order schemes we have refined results which remove the Lipschitz assumption on the nonlinearity.

We now state the main results. Consider the following second order in time semi-implicit Fourier spectral scheme:

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau} = -\nu\Delta^2 u^{n+1} - A\tau(u^{n+1}-u^n) + \Delta\Pi_N(2f(u^n)-f(u^{n-1})), \quad n \ge 1, \tag{1.5}$$

where $\tau > 0$ denotes the time step. This scheme combines second-order backward differentiation (BD2) for the time derivative term with a second order extrapolation (EP2) for the nonlinear term. In order to start the iteration, we take $u^0 = \Pi_N u_0$, and compute u^1 according to the following first order in time scheme:

$$\begin{cases} \frac{u^1-u^0}{\tau_1} = -\nu\Delta^2 u^1 + \Delta\Pi_N f(u^0), \\ u^0 = \Pi_N u_0, \end{cases} \tag{1.6}$$

where $\tau_1 = \min\{\tau^{\frac{4}{3}}, 1\}$. The choice of τ_1 is such that the error after one iteration is $O(\tau^2)$ in accordance with the second order in time nature of the main scheme. We briefly explain the reason for this choice as follows. Roughly speaking, an error estimate on (1.6) (see Lemma 2.2) gives

$$\|u^1 - u(\tau_1)\|_2^2 \lesssim N^{-2s} + \tau_1 \cdot \tau_1^2,$$

where $u(\tau_1)$ denotes the true PDE solution at τ_1 and N^{-2s} is due to Fourier truncation of initial data. This then gives

$$\|u^1 - u(\tau_1)\|_2 \lesssim N^{-s} + \tau_1^{\frac{3}{2}}.$$

From this it is evident that we need $\tau_1 \lesssim \tau^{\frac{4}{3}}$.

For any $k = (k_1, k_2) \in \mathbb{Z}^2$, define

$$|k|_\infty = \max\{|k_1|, |k_2|\}.$$

For each integer $N \ge 2$, we introduce the space X_N as

$$X_N = \operatorname{span}\Big\{\cos(k\cdot x), \sin(k\cdot x) : \quad |k|_\infty \le N,\, k \in \mathbb{Z}^2\Big\}.$$

The L^2 projection operator $\Pi_N : L^2(\Omega) \to X_N$ is defined by the requirement

$$(\Pi_N u - u, \phi) = 0, \qquad \forall\, \phi \in X_N, \tag{1.7}$$

where $(\cdot,\cdot)$ denotes the usual L^2 inner product.

Note that in (1.5), the stabilization term is of the form $A\tau(u^{n+1}-u^n)$ which is formally of $O(\tau^2)$ thanks to the prefactor τ. The initial data u_0 will be assumed to have mean zero. It is easy to check that $u^0 = \Pi_N u_0$, and u^n, $n \ge 1$ all have mean zero. For L^2 functions with mean zero it is then possible to define $|\nabla|^s = (-\Delta)^{s/2}$, $s < 0$ as a Fourier multiplier $|k|^s$. The particular case we shall often need is $|\nabla|^{-1}$ which has smoothing effect of order 1. To this end, define for $n \ge 1$ a modified energy functional:

$$\tilde{E}(u^n) := E(u^n) + \frac{\nu}{4}\|u^n - u^{n-1}\|_2^2 + \frac{1}{4\tau}\||\nabla|^{-1}(u^n - u^{n-1})\|_2^2, \tag{1.8}$$

where

$$E(u) = \int_\Omega \Big(\frac{\nu}{2}|\nabla u|^2 + \frac{1}{4}(u^2-1)^2\Big)dx.$$

Theorem 1.1 (Unconditional stability for (1.5))**.** *Consider the scheme* (1.5)–(1.6) *with* $\nu > 0$*,* $\tau > 0$ *and* $N \ge 2$*. Assume* $u_0 \in H^4(\mathbb{T}^2)$ *with mean zero. Denote* $E_0 = E(u_0)$ *as the initial energy. There exists a constant* $\beta_c > 0$ *depending only on* E_0 *and* $\|u_0\|_{H^4}$*, such that if*

$$A \ge \beta(1 + \nu^{-18}|\log \nu|^{16}), \quad \beta \ge \beta_c,$$

then

$$\tilde{E}(u^{n+1}) \le \tilde{E}(u^n), \quad \forall\, n \ge 1,$$

where $\tilde{E}(\cdot)$ *is given by* (1.8).

Remark. It is expected that if the L^2 projection operator Π_N in (1.5) (similarly also in (1.9),(1.10) and (1.12)) is replaced by a (more computationally efficient) point-wise interpolation operator, one can also conduct a similar energy stability analysis. Some aliasing error control techniques have been developed for the Fourier pseudo-spectral method in recent years [15, 16]; similar ideas could also be applied to our proposed numerical scheme. Yet another issue is the generalization to 3D Cahn-Hilliard and similar phase-field models for which the main difficulty is the L^∞ control of the numerical solution. In [24] we have made some progress in the stability analysis of a class of first order numerical schemes by establishing some novel discrete smoothing estimates. We plan to address the more challenging 3D higher order cases elsewhere.

Remark. There have been many works on the second order accurate energy stable numerical schemes for the Chan-Hilliard equation, such as [12]. An alternate variable is used in the numerical design, denoted as a second order approximation to $v = \phi^2 - 1$. A linearized, second order accurate scheme is derived as the outcome of this idea, and an unconditional energy stability is established in a modified version. However, such an energy stability is applied to a pair of numerical variables (ϕ, v), and an H^1 stability for the original physical variable ϕ has not been justified. As a result, the convergence analysis is not available for this numerical approach.

In addition to [12], there have been a few other related works for the Cahn-Hilliard model, such as the Crank-Nicholson version, cf. [7, 17]. In these approaches, the energy stability for the original phase variable has been established at a theoretical level. As a result, this energy stability enables one to derive the H^1 numerical stability and the convergence analysis for the numerical schemes. Our proposed numerical scheme shares a similar merit as in [7, 17], so that the convergence analysis is available, which turns out to be a key advantage, in comparison with [12]. In addition, our scheme uses a purely explicit treatment of the nonlinear term, which makes the computational effort much simpler.

We have a slight variant of the scheme (1.5) which exhibits slightly better dependence on the parameter ν (for the stabilization parameter A). The scheme takes the form

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} = -\nu\Delta^2 u^{n+1} + A\tau\Delta(u^{n+1} - u^n) + \Delta\Pi_N(2f(u^n) - f(u^{n-1})), \quad n \ge 1, \tag{1.9}$$

where $u^0 = \Pi_N u_0$ and u^1 is computed via the same first order scheme (1.6). The main difference between (1.5) and (1.9) is that the stabilization term $-A\tau(u^{n+1} - u^n)$ is now replaced by a "higher order" analogue $A\tau\Delta(u^{n+1} - u^n)$.

Theorem 1.2 (Unconditional stability for (1.9))**.** *Consider the scheme* (1.9) *together with* (1.5) *with* $\nu > 0$, $\tau > 0$ *and* $N \ge 2$. *Assume* $u_0 \in H^4(\mathbb{T}^2)$ *with mean zero. Denote* $E_0 = E(u_0)$ *as the initial energy. There exists a parameter* $\beta_c > 0$ *depending only on* E_0 *and* $\|u_0\|_{H^4}$ *such that if*

$$A \ge \beta \cdot (1 + \nu^{-13}|\log \nu|^{12}), \quad \beta \ge \beta_c,$$

then

$$\tilde{E}(u^{n+1}) \le \tilde{E}(u^n), \quad \forall\, n \ge 1,$$

where $\tilde{E}(\cdot)$ *is given by* (1.8).

Our next two results are on the more "classical" second order schemes which have been often used in CH and Molecular Beam Epitaxy (MBE) models. Consider the scheme

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} + \nu\Delta^2 u^{n+1} - A\Delta(u^{n+1} - 2u^n + u^{n-1}) = \Delta\Pi_N(f(2u^n - u^{n-1})), \quad n \ge 1. \tag{1.10}$$

This scheme was first introduced by Xu and Tang in [31] for the 2D MBE equation. For the MBE model therein, the main unknown is the height function $h(x,t)$ and the scheme takes the form

$$\begin{aligned}&\frac{3h^{n+1}-4h^n+h^{n-1}}{2\Delta t}+\delta\Delta^2h^{n+1}-A\Delta h^{n+1}\\=&-2A\Delta h^n+A\Delta h^{n-1}-\nabla\cdot((1-|\nabla(2h^n-h^{n-1})|^2)\nabla(2h^n-h^{n-1})),\quad\forall\, n\geq 1.\end{aligned}$$

Denote

$$\tilde{E}^n=\frac{1}{\Delta t}\|h^n-h^{n-1}\|_2^2+\frac{1}{4}\||\nabla h^n|^2-1\|_2^2+\frac{\delta}{2}\|\Delta h^n\|_2^2+\frac{A}{2}\|\nabla(h^n-h^{n-1})\|_2^2.$$

Under the assumption that

$$A\geq\sup_{n\geq1}\||\nabla(2h^n-h^{n-1})|^2-1+\frac{1}{2}|\nabla(h^{n+1}+2h^n-h^{n-1})|^2\|_\infty,$$

Xu and Tang proved

$$\tilde{E}^{n+1}\leq\tilde{E}^n+O(\Delta t^2),$$

and

$$E(h^n)\leq E(h^1)+O(1)\Delta t,$$

where the $O(1)$ term is given by

$$O(1)=\left\|\frac{h^1-h^0}{\Delta t}\right\|_2^2+\frac{A}{2}\Delta t\left\|\frac{\nabla(h^1-h^0)}{\Delta t}\right\|_2^2+\sum_{i=0}^{n-1}\Delta t\left\|\frac{\nabla(h^i-h^{i-1})}{\Delta t}\right\|_2^2.$$

Our next result removes the a priori assumption on the numerical solution. Furthermore we have energy conservation for a modified energy functional for moderately small time steps. To state the result, we need to complement the scheme (1.10) with a carefully chosen first order scheme (to compute u^1) as follows

$$\begin{cases}\dfrac{u^1-u^0}{\tau_1}=-\nu\Delta^2u^1+\Delta\Pi_N(f(u^0)),\\ u^0=\Pi_Nu_0,\end{cases}\tag{1.11}$$

where

$$\tau_1=\min\{1,\tau^{\frac{4}{3}},\frac{1}{\sqrt{A+1}}\}.$$

The choice of such τ_1 is to guarantee the error estimate $O(\tau^2)$ (as explained before) and further to ensure the modified energy functional to be controlled by the initial data. The stability theorem below roughly states that we have energy decay under a very mild condition $\tau<8\nu$.

Theorem 1.3 (Case $A>0$). *For any $\theta_0>0$ the following holds: Consider the scheme* (1.10) *coupled with* (1.11). *Let $\nu>0$, $\tau>0$ satisfy*

$$\sqrt{\frac{2\nu}{\tau}}\geq\frac{1}{2}+\theta_0.$$

Let $u_0\in H^6(\mathbb{T}^2)$ with mean zero. There exists a constant $\beta_c>0$ depending only $(\theta_0,E(u_0),\|u_0\|_{H^6})$ such that if

$$A\geq\beta\cdot(1+\nu^{-4}(1+\nu)^6|\log\nu|^2),\quad\beta\geq\beta_c,$$

then

$$\begin{aligned}&E(u^{n+1})+\frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2+\frac{A+1}{2}\|u^{n+1}-u^n\|_2^2\\\leq\,&E(u^n)+\frac{1}{4\tau}\||\nabla|^{-1}(u^n-u^{n-1})\|_2^2+\frac{A+1}{2}\|u^n-u^{n-1}\|_2^2,\quad\forall\, n\geq1.\end{aligned}$$

Remark. Although the condition on A in Theorem 1.3 is very stiff especially when $0 < \nu \ll 1$, it is independent of the time step τ. Since the stabilization term is of the form $A\Delta(u^{n+1} - 2u^n + u^{n-1})$, the local truncation error is bounded by

$$C_\nu \cdot \tau^2,$$

where $C_\nu > 0$ is a constant depending on ν (and also on the Sobolev norms of the nearby exact PDE solution). If one insists on having ν-uniform error bounds, the aforementioned upper bound can be quite inferior as can be seen by taking $\tau \sim \nu$ and sending $\nu \to 0$. On the other hand it is a well-known open problem to extract ν-independent L^∞ upper bounds even for the PDE solution of CH for which maximum principle is no longer available. In general the constant C_ν depends on ν and we did not optimise this dependence here in order not to overburden the analysis. This issue of ν-dependent truncation error is also common for many other existing numerical schemes for phase field type models, since the rigorous analysis all involves bounding the various Sobolev norms of the solution which in turn implicitly depends on ν. We hope to investigate this important issue in the future.

For comparison we state the following theorem for the case $A = 0$, i.e. when the stabilization term is absent. In this case as expected the time step constraint is much more stringent than the case with stabilization.

Theorem 1.4 (Case $A = 0$)**.** *Consider the scheme* (1.10) *with* (1.11)*. Set $A = 0$. Let $u_0 \in H^6(\mathbb{T}^2)$ with mean zero. There exist constants $C_1 > 0$, $C_2 > 0$ depending only on $(E(u_0), \|u_0\|_{H^6})$ such that if*

$$\tau \le \begin{cases} C_1 \dfrac{\nu^9}{1+|\log \nu|^4}, & \text{when } 0 < \nu \le 1, \\ C_2 \dfrac{\nu^{-3}}{1+|\log \nu|^4}, & \text{when } \nu > 1, \end{cases}$$

then for all $n \ge 1$,

$$\begin{aligned} &E(u^{n+1}) + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1} - u^n)\|_2^2 + \frac{1}{2}\|u^{n+1} - u^n\|_2^2 \\ \le\ & E(u^n) + \frac{1}{4\tau}\||\nabla|^{-1}(u^n - u^{n-1})\|_2^2 + \frac{1}{2}\|u^n - u^{n-1}\|_2^2. \end{aligned}$$

Our next result is concerned with the scheme:

$$\frac{3u^{n+1} - 4u^n + u^{n-1}}{2\tau} + \nu\Delta^2 u^{n+1} - A\Delta(u^{n+1} - 2u^n + u^{n-1}) = \Delta\Pi_N(2f(u^n) - f(u^{n-1})), \quad n \ge 1. \tag{1.12}$$

Note the subtle difference between (1.10) and (1.12). Namely the nonlinear term is now replaced by $2f(u^n) - f(u^{n-1})$. One should note that from the computational efficiency point of view, the treatment $f(2u^n - u^{n-1})$ is better than $2f(u^n) - f(u^{n-1})$. Somewhat surprisingly, the latter also turns out to have inferior stability properties due to some spurious terms in the stability analysis.

In [27], Shen and Yang considered the scheme (1.12) for a suitably truncated nonlinearity satisfying

$$\max_{u \in \mathbb{R}} |f'(u)| \le L.$$

Under a condition on τ (which depends on L and ν) they proved energy stability. However the stability analysis therein is valid for all $A \ge 0$ and it was not known whether the scheme (1.12) with $A > 0$ has better stability than the case $A = 0$. See Remark 2.2 in [27] and the numerical experiments mentioned therein for more in-depth discussions.

The next theorem removes the Lipschitz assumption on the nonlinearity. Furthermore we discuss the case $A = 0$ versus the other case $A > 0$. Our analysis suggests that in the regime $\nu \to 0$ the scheme with $A > 0$ has better stability property than $A = 0$.

Theorem 1.5. *Consider the scheme* (1.12) *with* (1.11)*. Let $u_0 \in H^6(\mathbb{T}^2)$ with mean zero. There are constants $C_i > 0$, $i = 1, 2, 3, 4$ depending only on $(E(u_0), \|u_0\|_{H^6})$ such that the following holds:*

Case 1: $A = 0$. *If*

$$\tau \le \begin{cases} C_1 \dfrac{\nu^9}{1+|\log\nu|^4}, & \text{when } 0<\nu<1; \\ C_2 \dfrac{\nu^{-3}}{1+|\log\nu|^4}, & \text{when } \nu \ge 1, \end{cases}$$

then for any $n \ge 1$,

$$E(u^{n+1}) + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 + \frac{1}{2}\|u^{n+1}-u^n\|_2^2 \le E(u^n) + \frac{1}{4\tau}\||\nabla|^{-1}(u^n-u^{n-1})\|_2^2 + \frac{1}{2}\|u^n-u^{n-1}\|_2^2.$$

Case 2: $A = \mathrm{const}\cdot(\nu^{-4}+\nu^2)$. *If*

$$\tau \le \begin{cases} C_3 \dfrac{\nu^5}{1+|\log\nu|^2}, & \text{when } 0<\nu<1, \\ C_4 \dfrac{\nu^{-1}}{1+|\log\nu|^2}, & \text{when } \nu \ge 1, \end{cases}$$

then for any $n \ge 1$,

$$E(u^{n+1}) + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 + \frac{A+1}{2}\|u^{n+1}-u^n\|_2^2 \le E(u^n) + \frac{1}{4\tau}\||\nabla|^{-1}(u^n-u^{n-1})\|_2^2 + \frac{A+1}{2}\|u^n-u^{n-1}\|_2^2.$$

We now turn to the error analysis. We shall choose the representative case: the scheme (1.5) together with (1.6).

Theorem 1.6 (L^2 error estimate)**.** *Let* $\nu > 0$. *Let* $u_0 \in H^s$, $s \ge 8$ *with mean zero. Let* $u(t)$ *be the solution to the 2D CH equation* (1.1) *with initial data* u_0. *Let* u^1 *be defined according to* (1.6) *with initial data* $u^0 = \Pi_N u_0$. *Let* u^m, $m \ge 2$ *be defined according to* (1.5) *with initial data* u^0, u^1. *Assume A satisfies the same condition as in Theorem 1.2. Define* $t_0 = 0$, $t_1 = \tau_1$, $t_m = \tau_1 + (m-1)\tau$, $m \ge 2$. *Then for any* $m \ge 1$,

$$\|u(t_m) - u^m\|_2 \le C_1 \cdot e^{C_2 t_m} \cdot (N^{-s} + \tau^2),$$

where $C_1 > 0$, $C_2 > 0$ *are constants depending only on* (u_0, ν, s, A).

Remark 1.1. Similar error analysis results hold for the other schemes mentioned above. We omit such statements since the proofs are minor variations of the theme.

The rest of this paper is organized as follows:

§1. Introduction and preliminary results.
§2. Estimate of the first order scheme (1.6).
§3. Unconditional stability for the scheme (1.5) (Theorem 1.1).
§4. Unconditional stability for (1.9) (Theorem 1.2).
§5. Stability results for the scheme (1.11) (Theorem 1.3 for $A > 0$ and Theorem 1.4 for $A = 0$).
§6. Stability results for (1.12) (Theorem 1.5).
§7. Discretization lemma for the PDE solution.
§8. Error analysis for the scheme (1.5) (Theorem 1.6).
§9. Concluding remarks.

We end this introduction by collecting some notation and preliminaries used in this paper.

We denote by $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ the 2π-periodic torus. For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we use the Japanese bracket notation $\langle x\rangle = \sqrt{1 + x_1^2 + \cdots + x_d^2}$.

Let $\Omega = \mathbb{T}^d$. For any function $f : \Omega \to \mathbb{R}$, we use $\|f\|_{L^p} = \|f\|_{L^p(\Omega)}$ or sometimes $\|f\|_p$ to denote the usual Lebesgue L^p norm for $1 \le p \le \infty$.

For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We denote $X \lesssim_{Z_1,\cdots,Z_m} Y$ if $X \le CY$ where the constant C depends on the parameters $Z_1, \cdots, Z_m$.

We use the following convention for Fourier expansion on $\Omega = \mathbb{T}^d$:

$$(\mathcal{F}f)(k) = \hat{f}(k) = \int_\Omega f(x)e^{-ix\cdot k}dx, \quad f(x) = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}\hat{f}(k)e^{ik\cdot x}.$$

The usual Parseval takes the form

$$\int_\Omega |f(x)|^2 dx = \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}|\hat{f}(k)|^2.$$

For $f : \mathbb{T}^d \to \mathbb{R}$ and $s \ge 0$, we define the H^s-norm and $\dot{H}^s$-norm of f as

$$\begin{aligned}\|f\|_{H^s}^2 &= \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}(1+|k|^{2s})|\hat{f}(k)|^2,\\ \|f\|_{\dot{H}^s}^2 &= \frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d}|k|^{2s}|\hat{f}(k)|^2.\end{aligned} \tag{1.13}$$

provided the above series converge. In particular for $s = 1$

$$\|f\|_{\dot{H}^1} = \|\nabla f\|_2.$$

If f has mean zero, then $\hat{f}(0) = 0$ and clearly

$$\|f\|_{H^s} \sim \Big(\sum_{k\in\mathbb{Z}^d}|k|^{2s}|\hat{f}(k)|^2\Big)^{\frac{1}{2}}.$$

For f with mean zero, one can define the $\dot{H}^s$-norm for $s < 0$ by

$$\|f\|_{\dot{H}^s} = \Big(\frac{1}{(2\pi)^d}\sum_{0\ne k\in\mathbb{Z}^d}|k|^{2s}|\hat{f}(k)|^2\Big)^{\frac{1}{2}}$$

provided the series converges.

For mean zero functions, we can define the fractional Laplacian $|\nabla|^s = (-\Delta)^{s/2}$, $s \in \mathbb{R}$ by the relation

$$\widehat{|\nabla|^s f}(k) = |k|^s \hat{f}(k), \qquad 0 \ne k \in \mathbb{Z}^d.$$

The mean zero condition is only needed for $s < 0$. We shall also define $\langle\nabla\rangle^s$ for any $s \in \mathbb{R}$ as the Fourier multiplier $\langle k\rangle^s = (1+|k|^2)^{s/2}$.

In later sections, we will often use without explicit mentioning the following interpolation inequality on $\mathbb{T}^2$: for $s > 1$ and any $f \in H^s(\mathbb{T}^2)$, we have

$$\|f\|_{L^\infty(\mathbb{T}^2)} \lesssim_s 1 + \|f\|_{H^1(\mathbb{T}^2)}\sqrt{\log(3+\|f\|_{H^s(\mathbb{T}^2)})}. \tag{1.14}$$

We include an elementary proof of (1.14) for the sake of completeness. Let $R \ge 2$ be a number whose value will be chosen later. Then

$$\begin{aligned}\|f\|_\infty &\lesssim \sum_{k\in\mathbb{Z}^2}|\hat{f}(k)|\\ &\lesssim (\sum_{|k|\le R}|k|^2|\hat{f}(k)|^2)^{\frac{1}{2}}(\sum_{|k|\le R}|k|^{-2})^{1/2} + (\sum_{|k|>R}|k|^{-s})\|f\|_{H^s}\\ &\lesssim \|f\|_{H^1}\sqrt{\log R} + R^{-s}\|f\|_{H^s}.\end{aligned}$$

Choosing $R = 3 + \|f\|_{H^s}$ then yields the result.

□

Remark 1.2. An alternative version of (1.14) is

$$\|f\|_{L^\infty(\mathbb{T}^2)} \lesssim_s \|f\|_{H^1(\mathbb{T}^2)} \sqrt{\log\Big(3 + \frac{\|f\|_{H^s(\mathbb{T}^2)}}{\|f\|_{H^1}}\Big)}.$$

Note however that the constant term 1 needs to be present in (1.14). If this term is absent, one can take a sequence $f_n = \epsilon_n \phi$ with ϕ fixed and $\epsilon_n \to 0$ to arrive at a contradiction.

We shall need the following simple inequality when we extract the conditions on the stability parameter A. The typical condition on A takes the form:

$$A \ge \text{const} \cdot \nu^\alpha (|\log \nu|^\beta + |\log A|^\beta + 1).$$

It is then routine to derive the condition on A as

$$A \ge \text{const} \cdot (\nu^\alpha |\log \nu|^\beta + 1).$$

The following lemma clarifies this "routine" estimate.

Lemma 1.1. *Let $\alpha > 0$ and $C > 0$. There exists a constant $\epsilon_0 > 0$ depending only on α such that if $X > 0$, with*

$$X \le \epsilon_0 \cdot \frac{C}{\max\{|\log C|^\alpha, 1\}},$$

then

$$X |\log X|^\alpha \le C.$$

Similarly let $\alpha \in \mathbb{R}$, $\beta > 0$, $C > 0$, $\nu > 0$. There exists a constant $C_1 > 0$ depending on (C, α, β), such that if

$$X \ge C_1 (\nu^\alpha |\log \nu|^\beta + 1),$$

then

$$X \ge C \cdot \nu^\alpha \cdot (|\log \nu|^\beta + |\log X|^\beta + 1).$$

Proof. We shall only prove the first inequality. The argument for the second inequality is similar and therefore omitted. Consider first the case $|\log C| > 10$. Set

$$X = \frac{C}{|\log C|^\alpha} \cdot \eta,$$

where $\eta > 0$. Then

$$|\log X|^\alpha \lesssim_\alpha |\log C|^\alpha + |\log \eta|^\alpha.$$

Thus

$$\begin{aligned} X |\log X|^\alpha &\le C \cdot (\eta + \frac{1}{|\log C|^\alpha} \cdot \eta |\log \eta|^\alpha) \\ &\le C \cdot (\eta + \eta |\log \eta|^\alpha) \le C, \end{aligned}$$

if η is sufficiently small. On the other hand if $|\log C| \le 10$, then $C \sim 1$ and just need to take ϵ_0 to be a sufficiently small constant.

□

2. Estimate for the first order scheme (1.6)

In this section we gather estimates for the first iteration u^1 defined according to the scheme (1.6). More specifically Lemma 2.1 will be used in the stability proof later. Lemma 2.2 will be used for the error estimate later.

Lemma 2.1. *Consider (1.6). Assume $u_0 \in H^4(\mathbb{T}^2)$ with mean zero. Then*

$$\|u^1\|_\infty + \frac{\||\nabla|^{-1}(u^1 - u^0)\|_2^2}{\tau_1} + \frac{\nu}{2} \|\nabla u^1\|_2^2 \lesssim_{E(u_0), \|u_0\|_{H^4}} 1.$$

Proof. First we estimate $\|u^1\|_\infty$. Write

$$u^1 = \frac{1}{1+\tau_1\nu\Delta^2}u^0 + \frac{\tau_1\Delta\Pi_N}{1+\tau_1\nu\Delta^2}f(u^0).$$

Clearly

$$\begin{aligned}\|u^1\|_\infty &\lesssim \|\frac{1}{1+\tau_1\nu\Delta^2}u^0\|_{H^2} + \|\frac{\tau_1\Delta\Pi_N}{1+\tau_1\nu\Delta^2}f(u^0)\|_{H^2}\\ &\lesssim \|u^0\|_{H^2} + \|\Delta((u^0)^3-u^0)\|_{H^2} \qquad (\text{Note } \tau_1\le 1)\\ &\lesssim \|u_0\|_{H^4} + \|u_0\|_{H^4}^3.\end{aligned}$$

Taking L^2-inner product with $(-\Delta)^{-1}(u^1-u^0)$ on both sides of (1.6), we get

$$\begin{aligned}&\frac{1}{\tau_1}\||\nabla|^{-1}(u^1-u^0)\|_2^2 + \frac{\nu}{2}(\|\nabla u^1\|_2^2 - \|\nabla u^0\|_2^2 + \|\nabla u^1-\nabla u^0\|_2^2)\\ &\quad= -(f(u^0), u^1-u^0)\\ &\quad\lesssim \|f(u^0)\|_{\frac43}\|u^1-u^0\|_4 \lesssim_{\|u_0\|_{H^4}} 1.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{\tau_1}\||\nabla|^{-1}(u^1-u^0)\|_2^2 + \frac{\nu}{2}\|\nabla u^1\|_2^2 &\le \frac{\nu}{2}\|\nabla u^0\|_2^2 + C(\|u_0\|_{H^4})\\ &\lesssim_{E(u_0),\|u_0\|_{H^4}} 1,\end{aligned}$$

where in the above $C(\|u_0\|_{H^4})$ denotes a constant depending only on $\|u_0\|_{H^4}$. □

Lemma 2.2 (Error estimate for u^1)**.** *Consider*

$$\begin{cases}\frac{u^1-u^0}{\tau_1} = -\nu\Delta^2u^1 + \Delta\Pi_N f(u^0),\\ u^0 = \Pi_N u_0.\end{cases}$$

Compare with

$$\begin{cases}\partial_t u = -\nu\Delta^2 u + \Delta f(u), \quad 0<t\le\tau_1,\\ u(0)=u_0.\end{cases}$$

Let $u_0\in H^s$, $s\ge 6$ with mean zero. There exists a constant $D_1>0$ depending only on (u_0,ν,s), such that

$$\|u(\tau_1)-u^1\|_2 \le D_1\cdot(N^{-s}+\tau_1^{\frac32}).$$

Proof. We proceed in three steps.

Step 1: Rewrite the PDE into time-discretized form.

We first rewrite the PDE on the time interval $[0,\tau_1]$. Note that for a one-variable function $h=h(\tilde s)$, we have

$$\begin{aligned}h(0) &= h(\tau_1) + \int_{\tau_1}^0 h'(\tilde s)d\tilde s\\ &= h(\tau_1) - h'(\tau_1)\tau_1 + \int_0^{\tau_1} h''(\tilde s)\tilde s d\tilde s.\end{aligned}$$

Using this formula, we get

$$\begin{aligned}\frac{u(\tau_1)-u(0)}{\tau_1} &= (\partial_t u)(\tau_1) - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)(\tilde s)\tilde s d\tilde s\\ &= -\nu\Delta^2u(\tau_1) + \Delta f(u(\tau_1)) - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)(\tilde s)\tilde s d\tilde s\\ &= -\nu\Delta^2u(\tau_1) + \Delta\Pi_N f(u(0)) + \Delta\Pi_{>N}f(u(0)) + \Delta(f(u(\tau_1)) - f(u(0))) - \frac{1}{\tau_1}\int_0^{\tau_1}(\partial_{tt}u)(\tilde s)\tilde s d\tilde s,\end{aligned}$$

where $\Pi_{>N} = \mathrm{Id} - \Pi_N$. Now note that

$$\begin{aligned}\partial_{tt} u &= -\nu \partial_t \Delta^2 u + \Delta \partial_t (f(u)) \\ &= \Delta(-\nu \partial_t \Delta u + f'(u)\partial_t u).\end{aligned}$$

Thus

$$\frac{u(\tau_1) - u(0)}{\tau_1} = -\nu \Delta^2 u(\tau_1) + \Delta \Pi_N f(u(0)) + \Delta G^0,$$

where

$$G^0 = \Pi_{>N} f(u(0)) + f(u(\tau_1)) - f(u(0)) - \frac{1}{\tau_1}\int_0^{\tau_1} (-\nu \Delta \partial_t u + f'(u)\partial_t u)(\tilde{s})\tilde{s} d\tilde{s}.$$

Step 2: Estimate of G^0.

Clearly since $u_0 \in H^s$ and H^s is an algebra when $s > 1$ (in 2D), we have

$$\|\Pi_N f(u(0))\|_2 \lesssim N^{-s} \|f(u_0)\|_{H^s} \lesssim N^{-s}.$$

On the other hand,

$$f(u(\tau_1)) - f(u(0)) = f'(\xi)(u(\tau_1) - u(0)),$$

where ξ is a number between $u(0)$ and $u(\tau_1)$. Easy to check that $\|u(\tau_1)\|_\infty \lesssim_{u_0} 1$. Then

$$\|f(u(\tau_1)) - f(u(0))\|_2 \lesssim \|u(\tau_1) - u(0)\|_2 \lesssim \tau_1 \|\partial_t u\|_{L_t^\infty L_x^2([0,\tau_1])} \lesssim \tau_1.$$

Finally since $u \in L_t^\infty H^6$, we have

$$\begin{aligned}&\left\| \frac{1}{\tau_1}\int_0^{\tau_1} (-\nu \Delta \partial_t u + f'(u)\partial_t u)(\tilde{s})\tilde{s} d\tilde{s} \right\|_2 \\ \lesssim &\int_0^{\tau_1} \|\Delta \partial_t u\|_2 d\tilde{s} + \int_0^{\tau_1} \|f'(u)\partial_t u\|_2 d\tilde{s} \lesssim \tau_1.\end{aligned}$$

Thus

$$\|G^0\|_2 \lesssim N^{-s} + \tau_1.$$

Step 3: Estimate of $\|u(\tau) - u^1\|_2$.

Now we compare

$$\begin{cases} \dfrac{u(\tau_1) - u(0)}{\tau_1} = -\nu \Delta^2 u(\tau_1) + \Delta \Pi_N f(u(0)) + \Delta G^0, \\ \dfrac{u^1 - u^0}{\tau_1} = -\nu \Delta^2 u^1 + \Delta \Pi_N f(u^0), \\ u(0) = u_0, \quad u^0 = \Pi_N u_0. \end{cases}$$

Denote $e^1 = u(\tau_1) - u^1$ and $e^0 = u(0) - u^0$. Then we get

$$\frac{e^1 - e^0}{\tau_1} = -\nu \Delta^2 e^1 + \Delta \Pi_N (f(u(0)) - f(u^0)) + \Delta G^0.$$

Taking L^2-inner product with e^1 on both sides, we get

$$\begin{aligned}&\frac{1}{2\tau_1}(\|e^1\|_2^2 - \|e^0\|_2^2 + \|e^1 - e^0\|_2^2) + \nu \|\Delta e^1\|_2^2 \\ \le &\|f(u(0)) - f(u^0)\|_2 \cdot \|\Delta e^1\|_2 + \|G^0\|_2 \cdot \|\Delta e^1\|_2 \\ \lesssim &(\|e^0\|_2 + \|G^0\|_2) \cdot \|\Delta e^1\|_2 \\ \le &\frac{\text{const}}{\nu}\|e^0\|_2^2 + \frac{\text{const}}{\nu}\|G^0\|_2^2 + \frac{\nu}{2}\|\Delta e^1\|_2^2.\end{aligned}$$

It follows easily that

$$\begin{aligned}\|e^1\|_2^2 &\lesssim (1 + \tau_1)\|e^0\|_2^2 + \tau_1 \|G^0\|_2^2 \\ &\lesssim (1 + \tau_1)N^{-2s} + \tau_1(N^{-2s} + \tau_1^2) \lesssim N^{-2s} + \tau_1^3.\end{aligned}$$

Thus

$$\|e^1\|_2 \lesssim N^{-s} + \tau_1^{3/2}.$$

□

3. Proof of Theorem 1.1

In this section we establish the stability result Theorem 1.1. We begin with several lemmas.

Lemma 3.1. *Consider* (1.5) *with* $n \geq 1$. *Suppose* $E(u^n) \leq B$, $E(u^{n-1}) \leq B$ *for some* $B > 0$. *Then*

$$\|u^{n+1}\|_{H^1} \lesssim_B \ \nu^{-\frac{1}{2}} + \nu^{-1};$$
$$\|u^{n+1}\|_{H^2} \lesssim_B \ \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}};$$
$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}})}.$$

Proof. Since u^{n+1} has mean zero, it suffices to estimate $\|u^{n+1}\|_{\dot H^1}$ and $\|u^{n+1}\|_{\dot H^2}$ respectively. For simplicity of notation we shall write $\lesssim_B$ as $\lesssim$. Since $E(u^n) \leq B$, we have

$$\int_\Omega (\frac{1}{2}\nu|\nabla u^n|^2 + \frac{1}{4}((u^n)^2-1)^2) dx \leq B.$$

Clearly then

$$\|\nabla u^n\|_2 \lesssim \nu^{-1/2}, \quad \|u^n\|_4 \lesssim 1.$$

Similarly

$$\|\nabla u^{n-1}\|_2 \lesssim \nu^{-1/2}, \quad \|u^{n-1}\|_4 \lesssim 1.$$

These estimates will be used without mentioning below.

To bound $\|u^{n+1}\|_{\dot H^1}$, we first rewrite (1.5) as

$$u^{n+1} = \frac{4+2A\tau^2}{3+2\nu\tau\Delta^2+2A\tau^2} u^n + \frac{-1}{3+2\nu\tau\Delta^2+2A\tau^2} u^{n-1} + \frac{2\tau\Delta\Pi_N}{3+2\nu\tau\Delta^2+2A\tau^2}(2f(u^n)-f(u^{n-1})).$$

Observe that

$$\frac{\tau|k|^3}{3+2\tau\nu|k|^4+2A\tau^2} \lesssim \frac{\tau|k|^3}{\tau\nu|k|^4} \lesssim \frac{1}{\nu}\cdot|k|^{-1}, \quad \forall\, 0 \neq k \in \mathbb{Z}^2.$$

Then

$$\begin{aligned}\|u^{n+1}\|_{\dot H^1} &\lesssim \|u^n\|_{\dot H^1} + \|u^{n-1}\|_{\dot H^1} + \frac{1}{\nu}\|\langle\nabla\rangle^{-1}(2f(u^n)-f(u^{n-1}))\|_2 \\ &\lesssim \nu^{-\frac{1}{2}} + \nu^{-1}(\|(u^n)^3\|_{4/3} + \|(u^{n-1})^3\|_{4/3} + \|u^n\|_2 + \|u^{n-1}\|_2) \lesssim \nu^{-1/2} + \nu^{-1}.\end{aligned}$$

For $\|u^{n+1}\|_{\dot H^2}$, observe that for any $0 \neq k \in \mathbb{Z}^2$:

$$\frac{4+2A\tau^2}{3+2\tau\nu|k|^4+2A\tau^2} \lesssim (\frac{1}{\tau\nu} + \frac{A\tau}{\nu})\cdot|k|^{-4};$$
$$\frac{1}{3+2\tau\nu|k|^4+2A\tau^2} \lesssim \frac{1}{\tau\nu}\cdot|k|^{-4}.$$

Then

$$\begin{aligned}\|u^{n+1}\|_{\dot H^2} &\lesssim (\frac{1}{\tau\nu} + \frac{A\tau}{\nu})\|u^n\|_2 + \frac{1}{\tau\nu}\|u^{n-1}\|_2 + \frac{1}{\nu}\|2f(u^n)-f(u^{n-1})\|_2 \\ &\lesssim \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \frac{1}{\nu}(\|u^n\|_6^3 + \|u^n\|_2 + \|u^{n-1}\|_6^3 + \|u^{n-1}\|_2) \\ &\lesssim \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \frac{1}{\nu}(\nu^{-3/2}+1).\end{aligned}$$

Here in the estimate of $\|u^n\|_6$, we have used the crude bound $\|u^n\|_6 \lesssim \|u^n\|_{\dot{H}^1} \lesssim \nu^{-1/2}$. No interpolation is needed here since $\|u^{n+1}\|_{\dot{H}^2}$ only enters the logarithm part of the estimate of L^∞-norm of u^{n+1}, and any power gain is immaterial.

Finally for the estimate of $\|u^{n+1}\|_\infty$, we just use the interpolation inequality

$$\|f\|_\infty \lesssim 1 + \|f\|_{H^1}\sqrt{\log(3+\|f\|_{H^2})} \lesssim (1+\|f\|_{H^1})\sqrt{\log(3+\|f\|_{H^2})},$$

and the fact that $1+\nu^{-\frac{1}{2}}+\nu^{-1} \lesssim 1+\nu^{-1}$.

Since the bound on $\|u^{n+1}\|_{H^2}$ is of the form

$$\|u^{n+1}\|_{H^2} \le C_B \cdot M,$$

where C_B depends on B, we have

$$\log(3+C_B M) \le \tilde{C}_B \log(3+M),$$

where $\tilde{C}_B$ is another constant depending only on C_B. The desired inequality then follows.

□

Lemma 3.2. *Consider the scheme* (1.5). *For any $n \ge 1$, we have*

$$\begin{aligned}
&E(u^{n+1}) - E(u^n) + \frac{\nu}{2}\|\nabla(u^{n+1}-u^n)\|_2^2 + (A\tau + \frac{1}{\tau})\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 + \frac{1}{2}\|u^{n+1}-u^n\|_2^2 \\
&\quad + \frac{1}{4\tau}(\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2 - \||\nabla|^{-1}(u^n-u^{n-1})\|_2^2) + \frac{1}{4\tau}\||\nabla|^{-1}u^{n+1} - 2|\nabla|^{-1}u^n + |\nabla|^{-1}u^{n-1}\|_2^2 \\
\le &(\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1+3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)}{\nu})\|u^{n+1}-u^n\|_2^2 + \frac{\nu}{4}\|u^n - u^{n-1}\|_2^2.
\end{aligned}$$

In particular, if

$$\sqrt{\nu(A\tau+\frac{1}{\tau})} + \frac{1}{2} \ge \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1+3\|u^n\|_\infty^2+3\|u^{n-1}\|_\infty^2)^2}{\nu},$$

then

$$\tilde{E}(u^{n+1}) \le \tilde{E}(u^n), \quad n \ge 1,$$

where

$$\tilde{E}(u^{n+1}) = E(u^{n+1}) + \frac{\nu}{4}\|u^{n+1}-u^n\|_2^2 + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2.$$

Proof. We first rewrite (1.5) as

$$\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau} + \nu\Delta^2 u^{n+1} + A\tau(u^{n+1}-u^n) = \Delta\Pi_N(2f(u^n) - f(u^{n-1})).$$

We take L^2 inner product with $(-\Delta)^{-1}(u^{n+1}-u^n)$ on both sides and estimate each term separately.

For simplicity of notation we denote $\delta u^{n+1} := u^{n+1}-u^n$. Note that $\delta^2 u^{n+1} = u^{n+1} - 2u^n + u^{n-1}$. Clearly

$$3u^{n+1} - 4u^n + u^{n-1} = 2\delta u^{n+1} + \delta^2 u^{n+1}.$$

Then

$$\begin{aligned}
&(3u^{n+1}-4u^n+u^{n-1}, (-\Delta)^{-1}(u^{n+1}-u^n)) \\
=&2\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + (\delta^2|\nabla|^{-1}u^{n+1}, \delta|\nabla|^{-1}u^{n+1}) \\
=&2\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{1}{2}(\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 - \|\delta|\nabla|^{-1}u^n\|_2^2 + \|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2).
\end{aligned}$$

Here we used the simple identity

$$\begin{aligned}
(\delta a^n)a^n &= (a^n - a^{n-1})a^n \\
&= \frac{1}{2}(|a^n|^2 - |a^{n-1}|^2 + |a^n - a^{n-1}|^2) \\
&= \frac{1}{2}(|a^n|^2 - |a^{n-1}|^2 + |\delta a^n|^2).
\end{aligned}$$

Similarly

$$(\nu\Delta^2 u^{n+1}, (-\Delta)^{-1}\delta u^{n+1}) = \nu(\nabla u^{n+1}, \delta\nabla u^{n+1})$$
$$= \frac{\nu}{2}(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2 + \|\delta\nabla u^{n+1}\|_2^2).$$

Also

$$(A\tau\delta u^{n+1}, (-\Delta)^{-1}\delta u^{n+1}) = A\tau\|\delta|\nabla|^{-1}u^{n+1}\|_2^2.$$

Collecting the estimates, we get

$$\text{LHS} = \frac{\nu}{2}\|\nabla u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 - (\frac{\nu}{2}\|\nabla u^n\|_2^2 + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2)$$
$$+ \frac{\nu}{2}\|\delta\nabla u^{n+1}\|_2^2 + (A\tau + \frac{1}{\tau})\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2.$$

For the nonlinear term, note that

$$(\Delta\Pi_N(2f(u^n) - f(u^{n-1})), (-\Delta)^{-1}\delta u^{n+1}) = -(2f(u^n) - f(u^{n-1}), \delta u^{n+1}).$$

Now split

$$2f(u^n) - f(u^{n-1}) = f(u^n) + (f(u^n) - f(u^{n-1})).$$

Observe (recall $F(z) = (z^2-1)^2/4$)

$$F(u^{n+1}) - F(u^n) = f(u^n)\delta u^{n+1} + \int_0^1 \tilde{f}(u^n + s\delta u^{n+1})(1-s)ds(\delta u^{n+1})^2 - \frac{1}{2}(\delta u^{n+1})^2,$$

where $\tilde{f}(z) = 3z^2$.

Therefore

$$f(u^n)\delta u^{n+1} \geq F(u^{n+1}) - F(u^n) + \frac{1}{2}(\delta u^{n+1})^2 - \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2)\cdot(\delta u^{n+1})^2.$$

On the other hand, noting that $f'(z) = 3z^2 - 1$, we get

$$(f(u^n) - f(u^{n-1}))\cdot\delta u^{n+1} \geq -(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)\cdot|\delta u^n|\cdot|\delta u^{n+1}|.$$

Thus

$$\text{RHS} \leq -\int_\Omega F(u^{n+1})dx + \int_\Omega F(u^n)dx - \frac{1}{2}\|\delta u^{n+1}\|_2^2 + \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2)\cdot\|\delta u^{n+1}\|_2^2$$
$$+ (1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)\cdot\|\delta u^{n+1}\|_2\cdot\|\delta u^n\|_2.$$

By using the inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$, we get

$$(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)\cdot\|\delta u^{n+1}\|_2\cdot\|\delta u^n\|_2$$
$$\leq \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)^2}{\nu}\|\delta u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^n\|_2^2.$$

Collecting the estimates, we then obtain

$$E(u^{n+1}) - E(u^n) + \frac{\nu}{2}\|\delta\nabla u^{n+1}\|_2^2 + (A\tau + \frac{1}{\tau})\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{1}{2}\|\delta u^{n+1}\|_2^2$$
$$+ \frac{1}{4\tau}(\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 - \|\delta|\nabla|^{-1}u^n\|_2^2) + \frac{1}{4\tau}\|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2$$
$$\leq (\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)}{\nu})\|\delta u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^n\|_2^2.$$

Now

$$\frac{\nu}{4}\|\delta\nabla u^{n+1}\|_2^2 + (A\tau + \frac{1}{\tau})\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 \geq \sqrt{\nu(A\tau + \frac{1}{\tau})}\|\delta\nabla u^{n+1}\|_2\cdot\|\delta|\nabla|^{-1}u^{n+1}\|_2$$
$$\geq \sqrt{\nu(A\tau + \frac{1}{\tau})}\|\delta u^{n+1}\|_2^2.$$

Clearly if

$$\sqrt{\nu(A\tau+\frac{1}{\tau})}+\frac{1}{2}\geq\frac{3}{2}(\|u^n\|_\infty^2+\|u^{n+1}\|_\infty^2)+\frac{(1+3\|u^n\|_\infty^2+3\|u^{n-1}\|_\infty^2)^2}{\nu},$$

then

$$\begin{aligned}&E(u^{n+1})+\frac{\nu}{4}\|\delta\nabla u^{n+1}\|_2^2+\frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2\\ &\leq\ E(u^n)+\frac{\nu}{4}\|\delta u^n\|_2^2+\frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2.\end{aligned}$$

Since $\|\delta\nabla u^{n+1}\|_2\geq\|\delta u^{n+1}\|_2$, we clearly get

$$\tilde{E}(u^{n+1})\leq\tilde{E}(u^n).$$

□

Proof of Theorem 1.1. In this proof we shall denote by C a generic constant which depends only on $E(u_0)$ and $\|u_0\|_{H^4}$. The value of C may vary from line to line.

Set

$$B=\max\{\tilde{E}(u^1),\,E(u^0)\}.$$

By Lemma 2.1, we have

$$B\lesssim_{\|u_0\|_{H^4},E(u_0)}\ 1.$$

We shall inductively prove for every $m\geq 2$:

$$\begin{aligned}&\tilde{E}(u^m)\leq B,\quad \tilde{E}(u^m)\leq\tilde{E}(u^{m-1}),\\ &\|u^m\|_\infty\leq\alpha_B\cdot(1+\nu^{-1})\cdot\sqrt{\log(3+\frac{A\tau}{\nu}+\frac{1}{\tau\nu}+\nu^{-1}+\nu^{-\frac{5}{2}})},\end{aligned}$$

where $\alpha_B>0$ is the same constant as in Lemma 3.1. We shall specify the choice of the parameter A during the course of the proof.

We first check the case $m=2$. Note that $E(u^1)\leq\tilde{E}(u^1)\leq B$, $E(u^0)\leq B$, therefore we can apply Lemma 3.1 and obtain

$$\|u^2\|_\infty\leq\alpha_B\cdot(1+\nu^{-1})\cdot\sqrt{\log(3+\frac{A\tau}{\nu}+\frac{1}{\tau\nu}+\nu^{-1}+\nu^{-\frac{5}{2}})}.$$

We only need to check $\tilde{E}(u^2)\leq\tilde{E}(u^1)$. By Lemma 3.2, this amounts to checking the inequality

$$\sqrt{\nu(A\tau+\frac{1}{\tau})}+\frac{1}{2}\geq\frac{3}{2}(\|u^1\|_\infty^2+\|u^2\|_\infty^2)+\frac{(1+3\|u^1\|_\infty^2+3\|u^0\|_\infty^2)^2}{\nu}.$$

By Lemma 2.1 (for $\|u^1\|_\infty$) and using the bound on $\|u^2\|_\infty$, we then only need to choose A such that

$$\sqrt{\nu(A\tau+\frac{1}{\tau})}+\frac{1}{2}\geq C\cdot(1+\nu^{-2})\log(3+\frac{1}{\nu}(A\tau+\frac{1}{\tau})+\nu^{-1}+\nu^{-\frac{5}{2}})+C\nu^{-1}.$$

Denote $X=A\tau+\frac{1}{\tau}$. Note that $X\geq 2\sqrt{A}$ which can be made large by taking A large. In terms of X, we need

$$\sqrt{\nu X}+\frac{1}{2}\geq C\cdot(1+\nu^{-2})\cdot\log(3+\frac{1}{\nu}X+\nu^{-1}+\nu^{-\frac{5}{2}})+C\cdot\nu^{-1}.$$

Now discuss two cases.

Case 1: $0<\nu\leq 1/2$. In this case we only need

$$\sqrt{X}\geq C\cdot\nu^{-5/2}(|\log\nu|+|\log X|).$$

Easy to see that we need

$$X\geq C\cdot\nu^{-5}|\log\nu|^2.$$

Case 2: $\nu>1/2$. Then we need to fulfil

$$\sqrt{\nu X}\geq C\cdot(|\log X|+1).$$

It suffices to take

$$X \geq C.$$

Concluding from both cases, we obtain the condition on X as

$$X \geq C \cdot (1 + \nu^{-5} |\log \nu|^2).$$

Recalling $X = A\tau + \frac{1}{\tau}$, the condition on A then takes the form

$$A \geq C \cdot (1 + \nu^{-10} |\log \nu|^4).$$

We now check the induction step. Assume the induction hypothesis hold for all $2 \leq m \leq n$ ($n \geq 2$). Then for $m = n+1$, we can use Lemma 3.1 to get

$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot (1 + \nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}})}.$$

We then only need to check the inequality $\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n)$. By Lemma 3.2, we only need to prove

$$\sqrt{\nu(A\tau + \frac{1}{\tau})} + \frac{1}{2} \geq \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)^2}{\nu},$$

Note that for $n \geq 3$, by using the induction bounds, we have

$$\max\{\|u^n\|_\infty, \|u^{n-1}\|_\infty\} \leq \alpha_B \cdot (1 + \nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}})}.$$

If $n = 2$, then by Lemma 2.1,

$$\|u^{n-1}\|_\infty = \|u^1\|_\infty \leq C.$$

By the above L^∞ bounds on (u^{n-1}, u^n, u^{n+1}), we then obtain the inequality on A as

$$\begin{aligned}\sqrt{\nu(A\tau + \frac{1}{\tau})} + \frac{1}{2} \geq \frac{C}{\nu} &+ C \cdot (1 + \nu^{-4}) \cdot (\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}}))^4 \\ &+ C \cdot (1 + \nu^{-2}) \cdot (\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}}))^2.\end{aligned}$$

In terms of $X = A\tau + \frac{1}{\tau}$, we have

$$\begin{aligned}\sqrt{\nu X} + \frac{1}{2} \geq \frac{C}{\nu} &+ C \cdot (1 + \nu^{-4}) \cdot (\log(3 + \frac{1}{\nu}X + \nu^{-1} + \nu^{-\frac{5}{2}}))^4 \\ &+ C \cdot (1 + \nu^{-2}) \cdot (\log(3 + \frac{1}{\nu}X + \nu^{-1} + \nu^{-\frac{5}{2}}))^2.\end{aligned}$$

Now consider two cases.

Case 1: $0 < \nu \leq 1/2$. Then we need

$$\sqrt{\nu X} \geq C \cdot \nu^{-4}(|\log \nu| + |\log X|)^4.$$

Thus need

$$X \geq C \cdot \nu^{-9} |\log \nu|^8.$$

Case 2: $\nu > 1/2$. Then we need

$$\sqrt{\nu X} \geq C + C(|\log X| + 1)^4.$$

It suffices to take

$$X \geq C.$$

Concluding from both cases, we get the condition on X as

$$X \geq C \cdot (\nu^{-9} |\log \nu|^8 + 1).$$

Recalling $X = A\tau + \frac{1}{\tau}$, the condition on A then takes the form

$$A \geq C \cdot (1 + \nu^{-18} |\log \nu|^{16}).$$

We have completed the induction step. The theorem is proved.

4. Proof of Theorem 1.2

In this section we give the proof of stability for the scheme (1.9), i.e. Theorem 1.2.

Lemma 4.1. *Consider the scheme* (1.9) *with* $n \geq 1$. *Suppose* $E(u^n) \leq B$, $E(u^{n-1}) \leq B$ *for some* $B > 0$, *then*

$$\|u^{n+1}\|_{H^1} \lesssim_B \nu^{-1/2} + \nu^{-1};$$
$$\|u^{n+1}\|_{H^2} \lesssim_B \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-5/2};$$
$$\|u^{n+1}\|_\infty \leq \alpha_B \cdot (1+\nu^{-1}) \cdot \sqrt{\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-5/2})},$$

where $\alpha_B > 0$ *is a constant depending only on* B.

Proof. Rewrite

$$u^{n+1} = \frac{4 - 2A\tau^2\Delta}{3 + 2\nu\tau\Delta^2 - 2A\tau^2\Delta} u^n - \frac{1}{3 + 2\nu\tau\Delta^2 - 2A\tau^2\Delta} u^{n-1} + \frac{2\tau\Delta\Pi_N}{3 + 2\nu\tau\Delta^2 - 2A\tau^2\Delta}(2f(u^n) - f(u^{n-1})).$$

We shall write $\lesssim_B$ as $\lesssim$. Since $E(u^n) \leq B$ and $E(u^{n-1}) \leq B$, we have

$$\|\nabla u^n\|_2 + \|\nabla u^{n-1}\|_2 \lesssim \nu^{-1/2}, \quad \|u^n\|_4 + \|u^{n-1}\|_4 \lesssim 1.$$

Then since u^{n+1} has mean zero, we have

$$\begin{aligned}\|u^{n+1}\|_{H^1} &\lesssim \|u^{n+1}\|_{\dot H^1} \\ &\lesssim \|u^n\|_{\dot H^1} + \|u^{n-1}\|_{\dot H^1} + \frac{1}{\nu}\|\langle\nabla\rangle^{-1}(2f(u^n) - f(u^{n-1}))\|_2 \\ &\lesssim \nu^{-1/2} + \nu^{-1}.\end{aligned}$$

Next observe for any $0 \neq k \in \mathbb{Z}^2$:

$$\frac{4 + 2A\tau^2|k|^2}{3 + 2\nu\tau|k|^4 + 2A\tau^2|k|^2} \lesssim (\frac{1}{\nu\tau} + \frac{A\tau}{\nu})|k|^{-2},$$
$$\frac{1}{3 + 2\nu\tau|k|^4 + 2A\tau^2|k|^2} \lesssim \frac{1}{\tau\nu} \cdot |k|^{-4}.$$

Then

$$\begin{aligned}\|u^{n+1}\|_{\dot H^2} &\lesssim (\frac{1}{\nu\tau} + \frac{A\tau}{\nu})\|u^n\|_2 + \frac{1}{\tau\nu}\|u^{n-1}\|_2 + \frac{1}{\nu}\|2f(u^n) - f(u^{n-1})\|_2 \\ &\lesssim \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \frac{1}{\nu}(\|u^n\|_6^3 + \|u^n\|_2 + \|u^{n-1}\|_6^3 + \|u^{n-1}\|_2) \\ &\lesssim 1 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \frac{1}{\nu}(\nu^{-3/2} + 1).\end{aligned}$$

Finally for the estimate of $\|u^{n+1}\|_\infty$, we just appeal to the logarithm interpolation inequality. ∎

Lemma 4.2. *Consider the scheme* (1.9) *with* $n \geq 1$. *If*

$$A\tau + \frac{1}{2} + \sqrt{\frac{\nu}{\tau}} \geq \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)^2}{\nu},$$

then

$$\tilde E(u^{n+1}) \leq \tilde E(u^n), \quad n \geq 1,$$

where

$$\tilde E(u^{n+1}) = E(u^{n+1}) + \frac{\nu}{4}\|u^{n+1} - u^n\|_2^2 + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1} - u^n)\|_2^2.$$

Proof. Taking L^2-inner product with $(-\Delta)^{-1}\delta u^{n+1} = (-\Delta)^{-1}(u^{n+1}-u^n)$ on both sides of (1.9), we get

$$\frac{1}{\tau}\||\nabla|^{-1}\delta u^{n+1}\|_2^2 + \frac{1}{4\tau}(\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 - \|\delta|\nabla|^{-1}u^n\|_2^2 + \|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2) + \frac{\nu}{2}(\|\nabla u^{n+1}\|_2^2 - \|\nabla u^n\|_2^2$$
$$+ \|\delta\nabla u^{n+1}\|_2^2) + A\tau\|\delta u^{n+1}\|_2^2 = -(2f(u^n) - f(u^{n-1}), \delta u^{n+1})$$
$$\leq -\int_\Omega F(u^{n+1})dx + \int_\Omega F(u^n)dx - \frac{1}{2}\|\delta u^{n+1}\|_2^2 + \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2)\|\delta u^{n+1}\|_2^2$$
$$+ (1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)\cdot\|\delta u^{n+1}\|_2\cdot\|\delta u^n\|_2.$$

Clearly then

$$E(u^{n+1}) + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 - (E(u^n) + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2) + \frac{1}{4\tau}\|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2$$
$$+ \frac{1}{\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{\nu}{2}\|\delta\nabla u^{n+1}\|_2^2 + (A\tau + \frac{1}{2})\|\delta u^{n+1}\|_2^2$$
$$\leq \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2)\|\delta u^{n+1}\|_2^2 + (1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)\|\delta u^{n+1}\|_2\cdot\|\delta u^n\|_2$$
$$\leq \big(\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)^2}{\nu}\big)\|\delta u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^n\|_2^2.$$

Note that

$$\frac{1}{\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{\nu}{2}\|\delta\nabla u^{n+1}\|_2^2$$
$$= \frac{1}{\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta\nabla u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta\nabla u^{n+1}\|_2^2$$
$$\geq \sqrt{\frac{\nu}{\tau}}\|\delta u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^{n+1}\|_2^2.$$

We then obtain

$$E(u^{n+1}) + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{\nu}{4}\|\delta u^{n+1}\|_2^2 - (E(u^n) + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2 + \frac{\nu}{4}\|\delta u^n\|_2^2)$$
$$+ (A\tau + \frac{1}{2} + \sqrt{\frac{\nu}{\tau}})\|\delta u^{n+1}\|_2^2 \leq (\frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)^2}{\nu})\|\delta u^{n+1}\|_2^2.$$

Clearly if

$$A\tau + \frac{1}{2} + \sqrt{\frac{\nu}{\tau}} \geq \frac{3}{2}(\|u^n\|_\infty^2 + \|u^{n+1}\|_\infty^2) + \frac{(1 + 3\|u^n\|_\infty^2 + 3\|u^{n-1}\|_\infty^2)^2}{\nu},$$

then

$$\tilde{E}(u^{n+1}) \leq \tilde{E}(u^n), \quad n \geq 1,$$

where

$$\tilde{E}(u^{n+1}) = E(u^{n+1}) + \frac{\nu}{4}\|u^{n+1} - u^n\|_2^2 + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1} - u^n)\|_2^2.$$

□

4.1. **Proof of Theorem 1.2.** The main argument is similar to that in Theorem 1.1. Therefore we just point out the needed changes. By Lemma 4.1 and Lemma 4.2, we only need to check the inequality

$$A\tau + \frac{1}{2} + \sqrt{\frac{\nu}{\tau}} \geq C\cdot\nu^{-1} + C\cdot(1 + \nu^{-4})\cdot(\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}}))^4$$
$$+ C\cdot(1 + \nu^{-2})\cdot(\log(3 + \frac{A\tau}{\nu} + \frac{1}{\tau\nu} + \nu^{-1} + \nu^{-\frac{5}{2}}))^2,$$

where $C > 0$ is a constant depending only on $(E(u_0), \|u_0\|_{H^4})$.

We shall only consider the case $0 < \tau < 1$. The case $\tau \geq 1$ is simpler and therefore omitted.

Now note that

$$|\log(3+\frac{A\tau}{\nu}+\frac{1}{\tau\nu}+\nu^{-1}+\nu^{-\frac{5}{2}})|\lesssim 1+|\log A|+|\log\tau|+|\log\langle\frac{1}{\nu}\rangle|.$$

Consider two cases:
Case 1: $0<\nu\le 1/2$. Then we only need to check

$$A\tau+\sqrt{\frac{\nu}{\tau}}\ge C\cdot\nu^{-1}+\nu^{-4}(1+|\log A|^4+|\log\tau|^4+|\log\nu|^4).$$

Or more simply

$$A\tau+\sqrt{\frac{\nu}{\tau}}\ge C\cdot\nu^{-4}|\log\nu|^4+C\cdot\nu^{-4}|\log A|^4+C\cdot\nu^{-4}\cdot|\log\tau|^4.$$

If $\tau<\eta\cdot\nu^9/|\log\nu|^8$, then one can take $A=1$ and the desired inequality follows if η is sufficiently small. On the other hand if $\tau\ge\eta\nu^9/|\log\nu|^8$, then we can take

$$A>\beta\cdot\nu^{-13}|\log\nu|^{12},$$

where β is sufficiently large.

Case 2: $\nu>1/2$. Then we only need to check

$$A\tau+\sqrt{\frac{\nu}{\tau}}\ge C\cdot\nu^{-1}+C(1+|\log A|^4+|\log\tau|^4)$$
$$+C(1+|\log A|^2+|\log\tau|^2).$$

Clearly if $0<\tau\ll 1$, then we take $A=1$. On the other hand if $\tau\sim 1$, then it suffices to take $A\gg 1$.

Concluding from both cases, we found that it suffices to take

$$A\ge\beta\cdot(1+\nu^{-13}|\log\nu|^{12}),$$

where β is sufficiently large (depending only on $(E(u_0),\|u_0\|_{H^4})$). ∎

5. Proofs of Theorem 1.4 and Theorem 1.4

Lemma 5.1. *Consider the scheme* (1.10) *with* $n\ge 1$. *Suppose* $E(u^n)\le B\cdot(1+\nu)^2$, $E(u^{n-1})\le B\cdot(1+\nu)^2$ *for some* $B>0$. *Then*

$$\|u^{n+1}\|_{H^1}\lesssim_B\ \nu^{-1}+\nu^{\frac{1}{2}};$$
$$\|u^{n+1}\|_{H^2}\lesssim_B\ \frac{1}{\nu\tau}+\frac{A+1}{\nu^{\frac{1}{2}}}+\frac{1}{\nu^{\frac{1}{2}}\tau}+\frac{A}{\nu}+\nu^{-5/2};$$
$$\|u^{n+1}\|_\infty\le\alpha_B\cdot(\nu^{-1}+\nu^{1/2})\cdot\sqrt{1+|\log(A+1)|+|\log\nu|+|\log\langle\frac{1}{\tau}\rangle|},$$

where $\alpha_B>0$ *is a constant depending only on* B.

Proof. We shall write $\lesssim_B$ as $\lesssim$. Note that

$$\|\nabla u^{n-1}\|_2+\|\nabla u^n\|_2\lesssim\nu^{-1/2}(1+\nu),\quad \|u^{n-1}\|_4+\|u^n\|_4\lesssim(1+\nu)^{1/2}.$$

Write

$$u^{n+1}=\frac{4-2A\tau\Delta}{3-2A\tau\Delta+2\nu\tau\Delta^2}u^n+\frac{-1+2A\tau\Delta}{3-2A\tau\Delta+2\nu\tau\Delta^2}u^{n-1}+\frac{2\tau\Delta\Pi_N}{3-2A\tau\Delta+2\nu\tau\Delta^2}f(2u^n-u^{n-1}).$$

Clearly

$$\|u^{n+1}\|_{H^1}\lesssim\|u^{n+1}\|_{\dot H^1}\lesssim\|u^n\|_{\dot H^1}+\|u^{n-1}\|_{\dot H^1}+\frac{1}{\nu}\|\langle\nabla\rangle^{-1}(f(2u^n-u^{n-1}))\|_2$$
$$\lesssim\nu^{-\frac{1}{2}}(1+\nu)+\frac{1}{\nu}(\|(u^n)^3\|_{4/3}+\|(u^{n-1})^3\|_{4/3}+\|u^n\|_2+\|u^{n-1}\|_2)$$
$$\lesssim\nu^{-\frac{1}{2}}(1+\nu)+\nu^{-1}(1+\nu)^{\frac{3}{2}}\lesssim\nu^{-1}+\nu^{1/2}.$$

On the other hand,

$$\begin{aligned}\|u^{n+1}\|_{H^2} &\lesssim \|u^{n+1}\|_{\dot{H}^2}\\ &\lesssim (\frac{1}{\nu\tau}+\frac{A}{\nu})(\|u^n\|_2+\|u^{n-1}\|_2)+\frac{1}{\nu}\|f(2u^n-u^{n-1})\|_2\\ &\lesssim (\frac{1}{\nu\tau}+\frac{A}{\nu})(1+\nu)^{\frac12}+\frac{1}{\nu}\cdot(\|u^n\|_6^3+\|u^{n-1}\|_6^3+\|u^n\|_2+\|u^{n-1}\|_2)\\ &\lesssim (\frac{1}{\nu\tau}+\frac{A}{\nu})\cdot(1+\nu)^{\frac12}+\frac{1}{\nu}\cdot(\nu^{-\frac32}(1+\nu)^3+(1+\nu)^{\frac12})\\ &\lesssim \frac{1}{\nu\tau}+\frac{A}{\nu}+\frac{1}{\nu^{\frac12}\tau}+\frac{A+1}{\nu^{\frac12}}+\nu^{-5/2}.\end{aligned}$$

Finally the L^∞ bound just follows from the logarithm interpolation inequality. Note that we always have $\nu^{-1}+\nu^{1/2}\gtrsim 1$.

□

Lemma 5.2. *Consider the scheme* (1.10) *with* $n\ge 1$. *If* $\nu>0$, $\tau>0$ *satisfy*

$$\sqrt{\frac{2\nu}{\tau}}>\frac12,$$

and

$$2(A+1)(\sqrt{\frac{2\nu}{\tau}}-\frac12)\ge 729\cdot\max\{\|u^{n-1}\|_\infty^4,\|u^n\|_\infty^4,\|u^{n+1}\|_\infty^4\},$$

then

$$\begin{aligned}&E(u^{n+1})+\frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1}-u^n)\|_2^2+\frac{A+1}{2}\|u^{n+1}-u^n\|_2^2\\ \le\ &E(u^n)+\frac{1}{4\tau}\||\nabla|^{-1}(u^n-u^{n-1})\|_2^2+\frac{A+1}{2}\|u^n-u^{n-1}\|_2^2.\end{aligned}$$

Proof. Taking L^2-inner product with $(-\Delta)^{-1}(u^{n+1}-u^n)$ on both sides of (1.10), we get

$$\begin{aligned}(\frac{3u^{n+1}-4u^n+u^{n-1}}{2\tau},(-\Delta)^{-1}\delta u^{n+1}) &= \frac{1}{2\tau}(2\delta u^{n+1}+\delta^2u^{n+1},(-\Delta)^{-1}\delta u^{n+1})\\ &=\frac{1}{2\tau}(2\|\delta|\nabla|^{-1}u^{n+1}\|_2^2+\frac12(\|\delta|\nabla|^{-1}u^{n+1}\|_2^2-\|\delta|\nabla|^{-1}u^n\|_2^2+\|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2));\\ (\nu\Delta^2u^{n+1},(-\Delta)^{-1}\delta u^{n+1}) &= \frac{\nu}{2}(\|\nabla u^{n+1}\|_2^2-\|\nabla u^n\|_2^2+\|\delta\nabla u^{n+1}\|_2^2);\\ (-A\Delta(\delta^2u^{n+1}),(-\Delta)^{-1}\delta u^{n+1}) &= \frac{A}{2}(\|\delta u^{n+1}\|_2^2-\|\delta u^n\|_2^2+\|\delta^2u^{n+1}\|_2^2).\end{aligned}$$

For the (nonlinear) RHS, we have

$$\begin{aligned}&(\Delta\Pi_N f(2u^n-u^{n-1}),(-\Delta)^{-1}\delta u^{n+1})\\ =&-(f(2u^n-u^{n-1}),\delta u^{n+1})\\ =&(2u^n-u^{n-1},\delta u^{n+1})+((u^{n+1})^3-(2u^n-u^{n-1})^3,\delta u^{n+1})-((u^{n+1})^3,\delta u^{n+1}).\end{aligned}$$

Now

$$\begin{aligned}&(2u^n-u^{n-1},\delta u^{n+1})\\ =&-(\delta^2u^{n+1},\delta u^{n+1})+(u^{n+1},\delta u^{n+1})\\ =&-\frac12(\|\delta u^{n+1}\|_2^2-\|\delta u^n\|_2^2+\|\delta^2u^{n+1}\|_2^2)+\frac12(\|u^{n+1}\|_2^2-\|u^n\|_2^2+\|\delta u^{n+1}\|_2^2).\end{aligned}$$

By convexity, we have

$$-((u^{n+1})^3,\delta u^{n+1})\le-(\frac14(u^{n+1})^4,1)+(\frac14(u^n)^4,1).$$

On the other hand,

$$\begin{aligned}&|((u^{n+1})^3-(2u^n-u^{n-1})^3,\delta u^{n+1})|\\ =&|(u^{n+1})^3-(u^{n+1}-\delta^2u^{n+1})^3,\delta u^{n+1})|\\ \le&\ \|\delta u^{n+1}\|_2\cdot\|\delta^2u^{n+1}\|_2\cdot 3\max\{\|u^{n+1}\|_\infty^2,\|2u^n-u^{n-1}\|_\infty^2\}\\ \le&\ \|\delta u^{n+1}\|_2\cdot\|\delta^2u^{n+1}\|_2\cdot 27\max\{\|u^n\|_\infty^2,\|u^{n-1}\|_\infty^2,\|u^{n+1}\|_\infty^2\}.\end{aligned}$$

Therefore

$$\begin{aligned}\text{RHS}\le\ &-(\int_\Omega F(u^{n+1})dx+\frac12\|\delta u^{n+1}\|_2^2)+(\int_\Omega F(u^n)dx+\frac12\|\delta u^n\|_2^2)\\ &-\frac12\|\delta^2u^{n+1}\|_2^2+\frac12\|\delta u^{n+1}\|_2^2\\ &+\|\delta u^{n+1}\|_2\cdot\|\delta^2u^{n+1}\|_2\cdot 27\cdot\max\{\|u^n\|_\infty^2,\|u^{n-1}\|_\infty^2,\|u^{n+1}\|_\infty^2\}.\end{aligned}$$

Clearly

$$\frac1\tau\|\delta|\nabla|^{-1}u^{n+1}\|_2^2+\frac\nu2\|\delta\nabla u^{n+1}\|_2^2\ge\sqrt{\frac{2\nu}{\tau}}\|\delta u^{n+1}\|_2^2.$$

Collecting all the estimates, we obtain

$$\begin{aligned}&E(u^{n+1})+\frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2+\frac{A+1}{2}\|\delta u^{n+1}\|_2^2\\ \le&\ E(u^n)+\frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2+\frac{A+1}{2}\|\delta u^n\|_2^2\\ &-\Big((\sqrt{\frac{2\nu}{\tau}}-\frac12)\|\delta u^{n+1}\|_2^2-\|\delta u^{n+1}\|_2\cdot\|\delta^2u^{n+1}\|_2\cdot 27\max\{\|u^n\|_\infty^2,\|u^{n-1}\|_\infty^2,\|u^{n+1}\|_\infty^2\}+\frac{A+1}{2}\|\delta^2u^{n+1}\|_2^2\Big).\end{aligned}$$

Thus if

$$2(A+1)(\sqrt{\frac{2\nu}{\tau}}-\frac12)\ge 729\cdot\max\{\|u^{n-1}\|_\infty^4,\|u^n\|_\infty^4,\|u^{n+1}\|_\infty^4\},$$

then the desired inequality follows. □

Lemma 5.3 (Estimate on u^1)**.** *Consider the scheme*

$$\begin{cases}\dfrac{u^1-u^0}{\tau_1}=-\nu\Delta^2u^1+\Delta\Pi_N(f(u^0)),\\ u^0=\Pi_Nu_0,\end{cases}\tag{5.1}$$

where

$$\tau_1=\min\{1,\,\tau^{4/3},\,\frac{1}{\sqrt{A+1}}\}.$$

Suppose $u_0\in H^6$*. Then*

$$\begin{aligned}&\|u^1\|_{H^4}\lesssim_{\|u_0\|_{H^6}}1,\\ &(A+1)\|u^1-u^0\|_2^2\lesssim_{\|u_0\|_{H^6}}(1+\nu)^2,\\ &E(u^1)+\frac{1}{\tau_1}\ ||\nabla|^{-1}(u^1-u^0)\|_2^2\lesssim_{E(u_0),\|u_0\|_{H^6}}1.\end{aligned}$$

Proof. First note that

$$u^1=\frac{1}{1+\nu\tau_1\Delta^2}u^0+\frac{\tau_1\Delta}{1+\nu\tau_1\Delta^2}\Pi_N(f(u^0)).$$

Then

$$\|u^1\|_{H^4}\lesssim\|u^0\|_{H^4}+\|f(u^0)\|_{H^6}\lesssim_{\|u_0\|_{H^6}}1.$$

On the other hand, by (5.1), we get

$$\frac{1}{\tau_1}\|u^1-u^0\|_2 \le \nu\|u^1\|_{H^4}+\|\Delta\Pi_N(f(u^0))\|_2 \lesssim_{\|u_0\|_{H^6}} 1+\nu.$$

This implies

$$(A+1)\|u^1-u^0\|_2^2 \le \frac{1}{\tau_1^2}\|u^1-u^0\|_2^2 \lesssim_{\|u_0\|_{H^6}} (1+\nu)^2.$$

Taking L^2-inner product with $(-\Delta)^{-1}(u^1-u^0)$ on both sides of (5.1), we get

$$\begin{aligned}&\frac{1}{\tau_1}\||\nabla|^{-1}(u^1-u^0)\|_2^2+\frac{\nu}{2}(\|\nabla u^1\|_2^2-\|\nabla u^0\|_2^2+\|\nabla u^1-\nabla u^0\|_2^2)\\ =&-(f(u^0),u^1-u^0)\lesssim_{\|u_0\|_{H^6}} 1.\end{aligned}$$

Thus

$$E(u^1)+\frac{1}{\tau_1}\||\nabla|^{-1}(u^1-u^0)\|_2^2 \lesssim_{E(u_0),\|u_0\|_{H^6}} 1.$$

□

5.1. **Proof of Theorem 1.3.** The main argument is similar to Theorem 1.1. Thus we only sketch the needed changes. Recall

$$\tilde{E}(u^n)=E(u^n)+\frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2+\frac{A+1}{2}\|\delta u^n\|_2^2,\quad n\ge 1.$$

By Lemma 5.3, we have

$$\tilde{E}(u^1)\le C_1\cdot(1+\nu)^2,$$

where $C_1>0$ depends only on $(E(u_0),\|u_0\|_{H^6})$. Note that

$$E(u^0)\le \tilde{C}_1<\infty,$$

where $\tilde{C}_1$ depends only on $(E(u_0),\|u_0\|_{H^1})$. Then we set $B=\max\{C_1,\tilde{C}_1\}$ and inductively prove for every $m\ge 2$,

$$\begin{aligned}&\tilde{E}(u^m)\le B\cdot(1+\nu)^2,\quad \tilde{E}(u^m)\le\tilde{E}(u^{m-1}),\\ &\|u^m\|_\infty\le\alpha_B\cdot(\nu^{-1}+\nu^{\frac12})\cdot\sqrt{1+|\log(A+1)|+|\log\nu|+|\log\langle\frac{1}{\tau}\rangle|},\end{aligned}$$

where $\alpha_B>0$ is the same constant as in Lemma 5.1.

By Lemma 5.2 and Lemma 5.1, we then only need to verify the main inequality

$$2(A+1)(\sqrt{\frac{2\nu}{\tau}}-\frac12)\ge C_2\cdot(\nu^{-4}+\nu^2)\cdot(1+|\log(A+1)|^2+|\log\nu|^2+|\log\langle\frac{1}{\tau}\rangle|^2),$$

where $C_2>0$ depends only on $(E(u_0),\|u_0\|_{H^6})$.

We shall only discuss the case $0<\tau<1/2$. In this case $|\log\langle\frac{1}{\tau}\rangle|\sim|\log\tau|$. The other case $\tau\ge 1/2$ is much simpler and therefore omitted.

Note that $A+1\ge 1$. To remove the dependence of the above inequality on $\log\tau$, we first consider the inequality

$$\sqrt{\frac{2\nu}{\tau}}-\frac12\ge C_2\cdot(\nu^{-4}+\nu^2)\cdot|\log\langle\frac{1}{\tau}\rangle|^2.$$

It is clear that if $|\log\tau|\gg|\log\nu|+1$ (i.e., τ is sufficiently small), then the inequality holds.

Then it suffices for us to verify the inequality

$$(A+1)(\sqrt{\frac{2\nu}{\tau}}-\frac12)\ge C_3\cdot(\nu^{-4}+\nu^2)\cdot(1+|\log A|^2+|\log\nu|^2),$$

where $C_3>0$ depends only on $(E(u_0),\|u_0\|_{H^6})$.

By our assumption on (ν,τ), there exists a constant $\theta_1>0$ such that

$$\sqrt{\frac{2\nu}{\tau}}-\frac{1}{2}\geq\theta_1>0.$$

Then we only need to check

$$A\geq C_4\cdot(\nu^{-4}+\nu^2)\cdot(1+|\log A|^2+|\log\nu|^2),$$

where $C_4>0$ depends only on $(\theta_1,E(u_0),\|u_0\|_{H^6})$.

Now consider two cases.

Case 1: $0<\nu\leq 1/2$. Then we need

$$A\gg\nu^{-4}(1+|\log A|^2+|\log\nu|^2).$$

Thus it suffices to take

$$A\gg\nu^{-4}|\log\nu|^2+1.$$

Case 2: $\nu>1/2$. Then

$$A\gg\nu^2(1+|\log A|^2+|\log\nu|^2).$$

It is enough to take

$$A\gg\nu^2|\log\nu|^2+1.$$

Concluding from both cases, we obtain

$$A\gg\nu^{-4}(1+\nu)^6|\log\nu|^2+1.$$

□

5.2. **Proof of Theorem 1.4.** We only need to check the inequality

$$\sqrt{\frac{2\nu}{\tau}}-\frac{1}{2}\gg(\nu^{-4}+\nu^2)(1+|\log\nu|^2+|\log\langle\frac{1}{\tau}\rangle|^2).$$

If $0<\nu<1$, then we need

$$\sqrt{\frac{\nu}{\tau}}\gg\nu^{-4}(1+|\log\nu|^2+|\log\tau|^2).$$

It suffices to take

$$\tau\ll\frac{\nu^9}{1+|\log\nu|^4}.$$

If $\nu\geq 1$, then need

$$\sqrt{\frac{\nu}{\tau}}\gg\nu^2(1+|\log\nu|^2+|\log\tau|^2).$$

It suffices to take

$$\tau\ll\frac{\nu^{-3}}{1+|\log\nu|^4}.$$

□

6. Proof of Theorem 1.5

Lemma 6.1. *Consider the scheme* (1.12) *with* $n \geq 1$. *Suppose* $E(u^n) \leq B\cdot(1+\nu)^2$, $E(u^{n-1}) \leq B\cdot(1+\nu)^2$ *for some* $B > 0$. *Then*

$$\begin{aligned}
&\|u^{n+1}\|_{H^1} \lesssim_B \ \nu^{-1} + \nu^{\frac{1}{2}};\\
&\|u^{n+1}\|_{H^2} \lesssim_B \ \frac{1}{\nu\tau} + \frac{A+1}{\nu^{\frac{1}{2}}} + \frac{1}{\nu^{\frac{1}{2}}\tau} + \frac{A}{\nu} + \nu^{-5/2};\\
&\|u^{n+1}\|_\infty \leq \alpha_B \cdot (\nu^{-1} + \nu^{1/2}) \cdot \sqrt{1 + |\log(A+1)| + |\log \nu| + |\log\langle \frac{1}{\tau}\rangle|},
\end{aligned}$$

where $\alpha_B > 0$ *is a constant depending only on* B.

Proof. This is essentially the same as the proof of Lemma 5.1. We omit details. $\square$

Lemma 6.2. *Consider the scheme* (1.12) *with* $n \geq 1$. *If* $\nu > 0$, $\tau > 0$, $A > 0$ *satisfy*

$$2(A+1)\Big(\sqrt{\frac{2\nu}{\tau}} - \frac{1}{2} - 0.46\|u^{n+1}\|_\infty^2\Big) \geq 529 \cdot \max\{\|u^{n-1}\|_\infty^4, \|u^n\|_\infty^4, \|u^{n+1}\|_\infty^4\},$$

then

$$\begin{aligned}
&E(u^{n+1}) + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1} - u^n)\|_2^2 + \frac{A+1}{2}\|u^{n+1} - u^n\|_2^2\\
\leq\, &E(u^n) + \frac{1}{4\tau}\||\nabla|^{-1}(u^n - u^{n-1})\|_2^2 + \frac{A+1}{2}\|u^n - u^{n-1}\|_2^2.
\end{aligned}$$

Proof. Taking L^2-inner product with $(-\Delta)^{-1}\delta u^{n+1}$ on both sides, we get

$$\begin{aligned}
\text{LHS} =\, &\frac{1}{\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{\nu}{2}\|\delta\nabla u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta^2|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{A}{2}\|\delta^2 u^{n+1}\|_2^2\\
&+ \frac{\nu}{2}\|\nabla u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{A}{2}\|\delta u^{n+1}\|_2^2 - \Big(\frac{\nu}{2}\|\nabla u^n\|_2^2 + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2 + \frac{A}{2}\|\delta u^n\|_2^2\Big)\\
\geq &\frac{\nu}{2}\|\nabla u^{n+1}\|_2^2 + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^{n+1}\|_2^2 + \frac{A}{2}\|\delta u^{n+1}\|_2^2 - \Big(\frac{\nu}{2}\|\nabla u^n\|_2^2 + \frac{1}{4\tau}\|\delta|\nabla|^{-1}u^n\|_2^2 + \frac{A}{2}\|\delta u^n\|_2^2\Big)\\
&+ \sqrt{\frac{2\nu}{\tau}}\|\delta u^{n+1}\|_2^2 + \frac{A}{2}\|\delta^2 u^{n+1}\|_2^2.
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= -(2f(u^n) - f(u^{n-1}), \delta u^{n+1})\\
&= \underbrace{(2u^n - u^{n-1}, \delta u^{n+1})}_{(a)} + \underbrace{((u^{n-1})^3 - 2(u^n)^3, \delta u^{n+1})}_{(b)}.
\end{aligned}$$

For (a) we have

$$\begin{aligned}
(a) &= (-\delta^2 u^{n+1}, \delta u^{n+1}) + (u^{n+1}, \delta u^{n+1})\\
&= -\frac{1}{2}(\|\delta u^{n+1}\|_2^2 - \|\delta u^n\|_2^2 + \|\delta^2 u^{n+1}\|_2^2) + \frac{1}{2}(\|u^{n+1}\|_2^2 - \|u^n\|_2^2 + \|\delta u^{n+1}\|_2^2).
\end{aligned}$$

For (b) we use $u^{n-1} = \delta^2 u^{n+1} + 2u^n - u^{n+1}$ and write

$$\begin{aligned}
&(u^{n-1})^3 - 2(u^n)^3\\
&= (\delta^2 u^{n+1} + 2u^n - u^{n+1})^3 - 2(u^n)^3\\
&= (\delta^2 u^{n+1})^3 + 3(\delta^2 u^{n+1})^2(2u^n - u^{n+1}) + 3\delta^2 u^{n+1}(2u^n - u^{n+1})^2 + (2u^n - u^{n+1})^3 - 2(u^n)^3.
\end{aligned}$$

Clearly

$$\begin{aligned}
3\delta^2 u^{n+1}(u^{n+1} - 2u^n)^2 &= 3\delta^2 u^{n+1}(\delta^2 u^{n+1} - u^{n-1})^2\\
&= 3(\delta^2 u^{n+1})^3 - 6(\delta^2 u^{n+1})^2 u^{n-1} + 3\delta^2 u^{n+1} \cdot (u^{n-1})^2.
\end{aligned}$$

Then

$$
\begin{aligned}
&(u^{n-1})^3 - 2(u^n)^3 \\
=&(\delta^2 u^{n+1})^2 (u^{n+1} - 2u^n - 2u^{n-1}) + 3\delta^2 u^{n+1} \cdot (u^{n-1})^2 + 6(u^n)^3 - 12(u^n)^2 u^{n+1} + 6u^n (u^{n+1})^2 - (u^{n+1})^3 \\
=&(\delta^2 u^{n+1})^2 \cdot (u^{n+1} - 2u^n - 2u^{n-1}) + 3\delta^2 u^{n+1} \cdot (u^{n-1})^2 + 6u^n (u^{n+1} - u^n)^2 - (u^{n+1})^3.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(b) \le & \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot \|\delta^2 u^{n+1}\|_\infty \cdot (\|u^{n+1}\|_\infty + 2\|u^n\|_\infty + 2\|u^{n-1}\|_\infty) + \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 3 \cdot \|u^{n-1}\|_\infty^2 \\
& + ((\delta u^{n+1})^2, 6u^n u^{n+1} - 6(u^n)^2) - ((u^{n+1})^3, u^{n+1} - u^n).
\end{aligned}
$$

It is easy to check that

$$
\frac{(u^n)^4}{4} = \frac{(u^{n+1})^4}{4} + (u^{n+1})^3 (u^n - u^{n+1}) + (\frac{1}{4}(u^n)^2 + \frac{1}{2} u^n u^{n+1} + \frac{3}{4}(u^{n+1})^2) \cdot (\delta u^{n+1})^2.
$$

Then

$$
\begin{aligned}
&((\delta u^{n+1})^2, 6u^n u^{n+1} - 6(u^n)^2) - ((u^{n+1})^3, u^{n+1} - u^n) \\
=&(\frac{1}{4}(u^n)^4, 1) - (\frac{1}{4}(u^{n+1})^4, 1) - ((\delta u^{n+1})^2, \frac{25}{4}(u^n)^2 - \frac{11}{2} u^n u^{n+1} + \frac{3}{4}(u^{n+1})^2) \\
=&(\frac{1}{4}(u^n)^4, 1) - (\frac{1}{4}(u^{n+1})^4, 1) - ((\delta u^{n+1})^2, \frac{25}{4}(u^n - \frac{11}{25} u^{n+1})^2 - 0.46 (u^{n+1})^2).
\end{aligned}
$$

Collecting the estimates, we obtain

$$
\begin{aligned}
\text{RHS} \le & -(\int_\Omega F(u^{n+1}) dx + \frac{1}{2}\|\delta u^{n+1}\|_2^2) + (\int_\Omega F(u^n) dx + \frac{1}{2}\|\delta u^n\|_2^2) \\
& - \frac{1}{2}\|\delta^2 u^{n+1}\|_2^2 + \|\delta u^{n+1}\|_2^2 \cdot (\frac{1}{2} + 0.46\|u^{n+1}\|_\infty^2) \\
& + \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 23 \cdot \max\{\|u^{n-1}\|_\infty^2, \|u^n\|_\infty^2, \|u^{n+1}\|_\infty^2\}.
\end{aligned}
$$

Comparing LHS with RHS, we get

$$
\begin{aligned}
& E(u^{n+1}) + \frac{1}{4\tau}\||\nabla|^{-1}(u^{n+1} - u^n)\|_2^2 + \frac{A+1}{2}\|u^{n+1} - u^n\|_2^2 \\
\le & E(u^n) + \frac{1}{4\tau}\||\nabla|^{-1}(u^n - u^{n-1})\|_2^2 + \frac{A+1}{2}\|u^n - u^{n-1}\|_2^2 \\
& - \Big(\frac{A+1}{2}\|\delta^2 u^{n+1}\|_2^2 + (\sqrt{\frac{2\nu}{\tau}} - \frac{1}{2} - 0.46\|u^{n+1}\|_\infty^2) \cdot \|\delta u^{n+1}\|_\infty^2 \\
& \quad - \|\delta^2 u^{n+1}\|_2 \cdot \|\delta u^{n+1}\|_2 \cdot 23 \max\{\|u^{n-1}\|_\infty^2, \|u^n\|_\infty^2, \|u^{n+1}\|_\infty^2\} \Big).
\end{aligned}
$$

The desired inequality then follows if

$$
2(A+1)(\sqrt{\frac{2\nu}{\tau}} - \frac{1}{2} - 0.46\|u^{n+1}\|_\infty^2) \ge 529 \cdot \max\{\|u^{n-1}\|_\infty^4, \|u^n\|_\infty^4, \|u^{n+1}\|_\infty^4\}.
$$

□

6.1. **Proof of Theorem 1.5.** This is similar to the proof of Theorem 1.3. By an induction argument together with Lemma 6.1 and Lemma 6.2, we only need to check the inequality

$$
\begin{aligned}
& 2(A+1)\Big(\sqrt{\frac{2\nu}{\tau}} - \frac{1}{2} - C_1 \cdot (\nu^{-2} + \nu) \cdot (1 + |\log(A+1)| + |\log \nu| + |\log\langle \frac{1}{\tau}\rangle|)\Big) \\
> & C_2 \cdot (\nu^{-4} + \nu^2)(1 + |\log(A+1)|^2 + |\log \nu|^2 + |\log\langle \frac{1}{\tau}\rangle|^2),
\end{aligned}
$$

where $C_1 > 0$, $C_2 > 0$ are constants depending only on $(E(u_0), \|u_0\|_{H^6})$.

Now consider two cases:

Case 1: $A = 0$. Then we need

$$
\sqrt{\frac{2\nu}{\tau}} \gg (\nu^{-4} + \nu^2) \cdot (1 + |\log \nu|^2 + |\log \tau|^2).
$$

Easy to check that we need

$$\tau \ll \frac{\nu^9}{1+|\log \nu|^4}, \quad \text{if } 0<\nu<1;$$

and

$$\tau \ll \frac{\nu^{-3}}{1+|\log \nu|^4}, \quad \text{if } \nu \ge 1.$$

Case 2: $A = \text{const} \cdot (\nu^{-4}+\nu^2)$. Then we need

$$\sqrt{\frac{2\nu}{\tau}} \gg (\nu^{-2}+\nu)(|\log\nu|+|\log\tau|)+(|\log\nu|^2+|\log\tau|^2).$$

If $0<\nu<1$, then need

$$\tau \ll \frac{\nu^5}{1+|\log\nu|^2}.$$

If $\nu \ge 1$, then need

$$\tau \ll \frac{\nu^{-1}}{1+|\log\nu|^2}.$$

□

7. Time discretization of the PDE

In this section we prove a useful lemma for the error estimate later. It basically says the PDE solution can be rewritten as the solution to an iterative system mimicking (1.5) with controllable error terms.

Lemma 7.1 (PDE in time-discretized form)**.** *Let $u=u(t)$ be the solution to* (1.1) *with initial data $u_0 \in H^s$, $s \ge 8$ (u_0 has mean zero). Define $t_0=0$, $t_1=\tau_1$, $t_m=\tau_1+(m-1)\tau$ for $m\ge 2$. Then for any $n \ge 1$,*

$$\begin{aligned}&\frac{3u(t_{n+1})-4u(t_n)+u(t_{n-1})}{2\tau}\\ =&-\nu\Delta^2 u(t_{n+1})-A\tau(u(t_{n+1})-u(t_n))+\Delta\Pi_N(2f(u(t_n))-f(u(t_{n-1})))+\Delta G^n,\end{aligned}$$

where

$$\|G^n\|_2^2 \lesssim_{A,\nu,u_0} \tau^3 \int_{t_{n-1}}^{t_{n+1}} (\|\partial_t u(\tilde{s})\|_2^2+\|\Delta\partial_{tt}u(\tilde{s})\|_2^2)d\tilde{s}+N^{-2s}.$$

For any $m\ge 2$,

$$\tau\sum_{n=1}^{m-1}\|G^n\|_2^2 \lesssim (1+t_m)\cdot(\tau^4+N^{-2s}).$$

Proof. We proceed in several steps.

Step 1: Rewriting the PDE.

Recall

$$\partial_t u = -\nu\Delta^2 u+\Delta(f(u)).$$

Note that for a one-variable function $h=h(t)$, we have

$$h(t)=h(t_0)+h'(t_0)(t-t_0)+\frac{1}{2}h''(t_0)(t-t_0)^2+\frac{1}{2}\int_{t_0}^t h'''(\tilde{s})(\tilde{s}-t)^2 d\tilde{s}.$$

We then write

$$u(t_n)=u(t_{n+1})-(\partial_t u)(t_{n+1})\tau+\frac{1}{2}(\partial_{tt}u)(t_{n+1})\tau^2+\frac{1}{2}\int_{t_{n+1}}^{t_n}(\partial_{ttt}u)(\tilde{s})(\tilde{s}-t_n)^2 d\tilde{s},$$

$$u(t_{n-1})=u(t_{n+1})-(\partial_t u)(t_{n+1})\cdot 2\tau+2(\partial_{tt}u)(t_{n+1})\tau^2+\frac{1}{2}\int_{t_{n+1}}^{t_{n-1}}(\partial_{ttt}u)(\tilde{s})(\tilde{s}-t_{n-1})^2 d\tilde{s}.$$

This gives

$$
\begin{aligned}
&\frac{3u(t_{n+1})-4u(t_n)+u(t_{n-1})}{2\tau}\\
=&(\partial_t u)(t_{n+1})+\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(\partial_{ttt}u)(\tilde{s})(\tilde{s}-t_n)^2d\tilde{s}-\frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}}(\partial_{ttt}u)(\tilde{s})(\tilde{s}-t_{n-1})^2d\tilde{s}\\
=&-\nu\Delta^2u(t_{n+1})-A\tau(u(t_{n+1})-u(t_n))+\Delta\Pi_N(2f(u(t_n))-f(u(t_{n-1})))\\
&+A\tau(u(t_{n+1})-u(t_n))+\Delta\Pi_{>N}(2f(u(t_n))-f(u(t_{n-1})))\\
&+\Delta(f(u(t_{n+1}))-(2f(u(t_n))-f(u(t_{n-1}))))\\
&+\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(\partial_{ttt}u)(\tilde{s})(\tilde{s}-t_n)^2d\tilde{s}-\frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}}(\partial_{ttt}u)(\tilde{s})(\tilde{s}-t_{n-1})^2d\tilde{s}.
\end{aligned}
$$

Note that

$$\partial_{ttt}u=\Delta(-\nu\Delta\partial_{tt}u+\partial_{tt}(f(u))).$$

Therefore

$$
\begin{aligned}
G^n=&A\tau\Delta^{-1}(u(t_{n+1})-u(t_n))+\Pi_{>N}(2f(u(t_n))-f(u(t_{n-1})))\\
&+(f(u(t_{n+1}))-2f(u(t_n))+f(u(t_{n-1})))\\
&+\frac{1}{\tau}\int_{t_n}^{t_{n+1}}(-\nu\Delta\partial_{tt}u+\partial_{tt}(f(u)))(\tilde{s})(\tilde{s}-t_n)^2d\tilde{s}\\
&-\frac{1}{4\tau}\int_{t_{n-1}}^{t_{n+1}}(-\nu\Delta\partial_{tt}u+\partial_{tt}(f(u)))(\tilde{s})(\tilde{s}-t_{n-1})^2d\tilde{s}\\
=:&I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

We shall estimate each term separately. To simply the notation, we will write $\lesssim_{A,\nu,u_0}$ simply as $\lesssim$.
Step 2: Estimates of $I_1,\cdots,I_5$.
Estimate of I_1:
Since

$$u(t_{n+1})-u(t_n)=\int_{t_n}^{t_{n+1}}\partial_t u(\tilde{s})d\tilde{s},$$

we have

$$
\begin{aligned}
\|I_1\|_2^2&\lesssim\tau^2\int_{t_n}^{t_{n+1}}\|\Delta^{-1}\partial_t u(\tilde{s})\|_2^2d\tilde{s}\cdot\tau\\
&\lesssim\tau^3\int_{t_n}^{t_{n+1}}\|\partial_t u(\tilde{s})\|_2^2d\tilde{s},
\end{aligned}
$$

where in the second inequality we have used the fact that $\partial_t u$ has mean zero.
Estimate of I_2: Since $u\in L_t^\infty H^s$, we have

$$\|I_2\|_2\lesssim N^{-s}(\|f(u(t_n))\|_{H^s}+\|f(u(t_{n-1}))\|_{H^s})\lesssim N^{-s}.$$

Estimate of I_3: For a one-variable function $h=h(t)$, we have the formula:

$$h(t)=h(t_0)+h'(t_0)(t-t_0)-\int_{t_0}^{t}h''(\tilde{s})\cdot(\tilde{s}-t)d\tilde{s}.$$

Then

$$
\begin{aligned}
f(u(t_n))&=f(u(t_{n+1}))-(\partial_t f(u))(t_{n+1})\cdot\tau+\int_{t_n}^{t_{n+1}}\partial_{tt}(f(u))(\tilde{s})(\tilde{s}-t_n)d\tilde{s};\\
f(u(t_{n-1}))&=f(u(t_{n+1}))-(\partial_t f(u))(t_{n+1})\cdot 2\tau+\int_{t_{n-1}}^{t_{n+1}}\partial_{tt}(f(u))(\tilde{s})(\tilde{s}-t_{n-1})d\tilde{s}.
\end{aligned}
$$

So

$$\begin{aligned}&f(u(t_{n+1}))-2f(u(t_n))+f(u(t_{n-1}))\\ =&-2\int_{t_n}^{t_{n+1}}\partial_{tt}(f(u))(\tilde{s})\cdot(\tilde{s}-t_n)d\tilde{s}+\int_{t_{n-1}}^{t_{n+1}}\partial_{tt}(f(u))(\tilde{s})\cdot(\tilde{s}-t_{n-1})d\tilde{s}.\end{aligned}$$

We then obtain

$$\|f(u(t_{n+1}))-2f(u(t_n))+f(u(t_{n-1}))\|_2^2\lesssim\int_{t_{n-1}}^{t_{n+1}}\|\partial_{tt}(f(u))(\tilde{s})\|_2^2d\tilde{s}\cdot\tau^3.$$

We shall estimate $\int\|\partial_{tt}f(u)\|_2^2d\tilde{s}$ in the estimate of I_4 and I_5 below.

Estimate of I_4 and I_5: Noting that $\partial_t(f(u))=f'(u)\partial_t u$, we have

$$\partial_{tt}(f(u))=f'(u)\partial_{tt}u+f''(u)(\partial_t u)^2.$$

Since $\partial_{tt}u$ has mean zero, we have

$$\|\partial_{tt}u\|_2\lesssim\|\Delta\partial_{tt}u\|_2.$$

Easy to check that $\|\partial_t u\|_{L_t^\infty L_x^\infty}\lesssim 1$. Then

$$\begin{aligned}\|I_4+I_5\|_2^2&\lesssim\int_{t_{n-1}}^{t_{n+1}}(\|\Delta\partial_{tt}u\|_2^2+\|\partial_{tt}(f(u))\|_2^2)d\tilde{s}\cdot\tau^3\\&\lesssim\int_{t_{n-1}}^{t_{n+1}}(\|\Delta\partial_{tt}u\|_2^2+\|\partial_t u\|_2^2)d\tilde{s}\cdot\tau^3.\end{aligned}$$

Step 3: Estimate of $\tau\sum_{n=1}^{m-1}\|G^n\|_2^2$.

It suffices to prove for any $T>0$

$$\int_0^T(\|\partial_t u(\tilde{s})\|_2^2+\|\Delta\partial_{tt}u(\tilde{s})\|_2^2)d\tilde{s}\lesssim 1+T.$$

By using the smoothing effect, we have

$$\sup_{t\ge1}\|u\|_{H^{100}}\lesssim 1.$$

Thus it follows easily that

$$\int_1^T(\|\partial_t u(\tilde{s})\|_2^2+\|\Delta\partial_{tt}u(\tilde{s})\|_2^2)d\tilde{s}\lesssim 1+T.$$

It remains to prove

$$\int_0^1(\|\partial_t u(\tilde{s})\|_2^2+\|\Delta\partial_{tt}u(\tilde{s})\|_2^2)d\tilde{s}\lesssim 1.$$

The estimate of $\|\partial_t u\|_2$ is trivial since $u\in L_t^\infty H^4$. We shall focus on the estimate of $\|\Delta\partial_{tt}u\|_{L_t^2L_x^2([0,1])}$. To do this we shall first estimate $\|\partial_t\Delta^2u\|_{L_t^2L_x^2([0,1])}$.

Estimate of $\|\partial_t\Delta^2u\|_{L_t^2L_x^2([0,1])}$:

Clearly

$$\partial_t\Delta^2u=-\nu\Delta^4u+\Delta^2(f(u)).$$

Multiplying both sides by $\partial_t\Delta^2u$ and integrating by parts, we get

$$\|\partial_t\Delta^2u\|_2^2=-\frac{\nu}{2}\frac{d}{dt}\big(\|\Delta^3u\|_2^2\big)+\int\Delta^2(f(u))\cdot\partial_t\Delta^2u\,dx.$$

Then

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\big(\|\Delta^3u\|_2^2\big)&\le-\frac12\|\partial_t\Delta^2u\|_2^2+\text{const}\cdot\|\Delta^2(f(u))\|_2^2\\&\le-\frac12\|\partial_t\Delta^2u\|_2^2+\text{const}\cdot(\|u\|_{H^4}^6+\|u\|_{H^4}^2).\end{aligned}$$

For $u_0 \in H^6$, integrating in time then gives

$$\|\partial_t \Delta^2 u\|_{L_t^2 L_x^2([0,1])} \lesssim 1.$$

Estimate of $\|\partial_{tt}\Delta u\|_{L_t^2 L_x^2([0,1])}$:
Write $v = \partial_t u$. Then for v we have the equation

$$\partial_t v = -\nu \Delta^2 v + \Delta(f'(u)v).$$

We need to estimate $\|\Delta \partial_t v\|_2$. Multiplying both sides of the equation by $\Delta^2 \partial_t v$ and integrating by parts, we get

$$\|\partial_t \Delta v\|_2^2 = -\frac{\nu}{2}\frac{d}{dt}(\|\Delta^2 v\|_2^2) + \int \Delta^2(f'(u)v)\Delta\partial_t v dx.$$

Thus

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}(\|\Delta^2 v\|_2^2) &\le -\frac{1}{2}\|\partial_t \Delta v\|_2^2 + \frac{1}{2}\|\Delta^2(f'(u)v)\|_2^2 \\ &\le -\frac{1}{2}\|\partial_t \Delta v\|_2^2 + \text{const}\cdot(\|f'(u)\|_\infty \|v\|_{H^4} + \|f'(u)\|_{H^4}\|v\|_\infty)^2.\end{aligned}$$

Since $v = \partial_t u$ and $\|\Delta^2 v\|_{L_t^2 L_x^2([0,1])} = \|\partial_t \Delta^2 u\|_{L_t^2 L_x^2([0,1])} \lesssim 1$, integrating in time then gives

$$\|\partial_{tt}\Delta u\|_{L_t^2 L_x^2([0,1])} \lesssim 1.$$

Here we used the fact that $v(0) = (\partial_t u)(0) \in H^8$.

□

8. Error estimate for CH

In this section we carry out the error estimate for CH in L^2 and complete the proof of Theorem 1.6.

8.1. **Auxiliary L^2 error estimate for near solutions.** Consider

$$\begin{cases}\frac{3v^{n+1}-4v^n+v^{n-1}}{2\tau} = -\nu\Delta^2 v^{n+1} - A\tau(v^{n+1}-v^n) + \Delta\Pi_N(2f(v^n)-f(v^{n-1})) + \Delta G^n, & n \ge 1,\\ \frac{3\tilde{v}^{n+1}-4\tilde{v}^n+\tilde{v}^{n-1}}{2\tau} = -\nu\Delta^2 \tilde{v}^{n+1} - A\tau(\tilde{v}^{n+1}-\tilde{v}^n) + \Delta\Pi_N(2f(\tilde{v}^n)-f(\tilde{v}^{n-1})), & n \ge 1,\end{cases} \tag{8.1}$$

where $(v^1, \tilde{v}^1, v^0, \tilde{v}^0)$ are given.

We first recall a simple lemma.

Lemma 8.1 (Discrete Gronwall inequality)**.** *Let $\tau > 0$ and $y_n \ge 0$, $\alpha_n \ge 0$, $\beta_n \ge 0$ for $n = 0, 1, 2, \cdots$. Suppose*

$$\frac{y_{n+1}-y_n}{\tau} \le \alpha_n y_n + \beta_n, \quad \forall\, n \ge 0.$$

Then for any $m \ge 1$, we have

$$y_m \le \exp\Big(\tau\sum_{n=0}^{m-1}\alpha_n\Big)(y_0 + \tau\sum_{n=0}^{m-1}\beta_n).$$

Proof. See Lemma 4.1 in [22]. □

Proposition 8.1. *For solutions of (8.1), assume for some $N_1 > 0$,*

$$\sup_{n\ge 0}\|v^n\|_\infty + \sup_{n\ge 0}\|\nabla v^n\|_2 + \sup_{n\ge 0}\|\nabla \tilde{v}^n\|_2 \le N_1.$$

Then for any $m \ge 2$,

$$\begin{aligned}&\|v^m - \tilde{v}^m\|_2^2 \\ &\le \exp\Big((m-1)\tau\cdot\frac{C(1+N_1^4)}{\nu}\Big)\cdot\Big((7+2A\tau^2)\|v^1-\tilde{v}^1\|_2^2 + 3\|v^0-\tilde{v}^0\|_2^2 + \frac{2\tau}{\nu}\sum_{n=1}^{m-1}\|G^n\|_2^2\Big),\end{aligned}$$

where $C > 0$ is an absolute constant.

Proof of Proposition 8.1. We shall denote by the letter C an absolute constant whose value may vary from line to line. Denote $e^n = v^n - \tilde{v}^n$. Then

$$\frac{3e^{n+1} - 4e^n + e^{n-1}}{2\tau} + \nu\Delta^2 e^{n+1} + A\tau(e^{n+1} - e^n) = \Delta\Pi_N(2f(v^n) - 2f(\tilde{v}^n)) - \Delta\Pi_N(f(v^{n-1}) - f(\tilde{v}^{n-1})) + \Delta G^n.$$

Taking L^2-inner product with e^{n+1} on both sides, we get

$$\begin{aligned}
&\frac{1}{2\tau}(3e^{n+1} - 4e^n + e^{n-1}, e^{n+1}) + \nu\|\Delta e^{n+1}\|_2^2 + \frac{A\tau}{2}(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 + \|e^{n+1} - e^n\|_2^2)\\
&= 2(f(v^n) - f(\tilde{v}^n), \Delta e^{n+1}) - (f(v^{n-1}) - f(\tilde{v}^{n-1}), \Delta e^{n+1}) + (G^n, \Delta e^{n+1})\\
&\le \frac{C}{\nu}\|f(v^n) - f(\tilde{v}^n)\|_2^2 + \frac{C}{\nu}\|f(v^{n-1}) - f(\tilde{v}^{n-1})\|_2^2 + \frac{1}{2\nu}\|G^n\|_2^2 + \frac{2\nu}{3}\|\Delta e^{n+1}\|_2^2.
\end{aligned}$$

Now write

$$\begin{aligned}
&(3e^{n+1} - 4e^n + e^{n-1}, e^{n+1} - e^n)\\
&\quad = 2\|e^{n+1} - e^n\|_2^2 + \frac{1}{2}(\|e^{n+1} - e^n\|_2^2 - \|e^n - e^{n-1}\|_2^2 + \|e^{n+1} - 2e^n + e^{n-1}\|_2^2);\\
&(3e^{n+1} - 4e^n + e^{n-1}, e^n) = 3(e^{n+1} - e^n, e^n) - (e^n - e^{n-1}, e^n)\\
&\quad = \frac{3}{2}(\|e^{n+1}\|_2^2 - \|e^n\|_2^2 - \|e^{n+1} - e^n\|_2^2)\\
&\quad\quad - \frac{1}{2}(\|e^n\|_2^2 - \|e^{n-1}\|_2^2 + \|e^n - e^{n-1}\|_2^2).
\end{aligned}$$

Thus

$$\begin{aligned}
&(3e^{n+1} - 4e^n + e^{n-1}, e^{n+1})\\
&= \frac{3}{2}(\|e^{n+1}\|_2^2 - \|e^n\|_2^2) - \frac{1}{2}(\|e^n\|_2^2 - \|e^{n-1}\|_2^2) + \|e^{n+1} - e^n\|_2^2 - \|e^n - e^{n-1}\|_2^2\\
&\quad + \frac{1}{2}\|e^{n+1} - 2e^n + e^{n-1}\|_2^2.
\end{aligned}$$

To estimate $\|f(v^n) - f(\tilde{v}^n)\|_2$, note that $\tilde{v}^n = v^n - e^n$, and

$$\begin{aligned}
f(v^n) - f(v^n - e^n) &= (v^n)^3 - (v^n - e^n)^3 - e^n\\
&= -(e^n)^3 - e^n - 3v^n(e^n)^2 + 3(v^n)^2 e^n.
\end{aligned}$$

We have

$$\begin{aligned}
\|f(v^n) - f(\tilde{v}^n)\|_2^2 &\lesssim \|e^n\|_6^6 + (1 + \|v^n\|_\infty^4)\|e^n\|_2^2 + \|v^n\|_\infty^2\|e^n\|_4^4\\
&\lesssim \|e^n\|_2^2(\|\nabla e^n\|_2^4 + 1 + \|v^n\|_\infty^4 + \|v^n\|_\infty^2\|\nabla e^n\|_2^2)\\
&\lesssim (1 + N_1^4)\|e^n\|_2^2.
\end{aligned}$$

Similarly

$$\|f(v^{n-1}) - f(\tilde{v}^{n-1})\|_2^2 \lesssim (1 + N_1^4)\|e^{n-1}\|_2^2.$$

Collecting the estimates, we get

$$\begin{aligned}
&\frac{\frac{3}{2}\|e^{n+1}\|_2^2 - \frac{1}{2}\|e^n\|_2^2 + \|e^{n+1} - e^n\|_2^2}{2\tau} + \frac{A\tau}{2}\|e^{n+1}\|_2^2\\
&\le \frac{\frac{3}{2}\|e^n\|_2^2 - \frac{1}{2}\|e^{n-1}\|_2^2 + \|e^n - e^{n-1}\|_2^2}{2\tau} + \frac{A\tau}{2}\|e^n\|_2^2\\
&\quad + C\cdot\frac{(1 + N_1^4)}{\nu}(\|e^n\|_2^2 + \|e^{n-1}\|_2^2) + \frac{1}{2\nu}\|G^n\|_2^2.
\end{aligned}$$

Denote

$$X^{n+1} = \frac{3}{2}\|e^{n+1}\|_2^2 - \frac{1}{2}\|e^n\|_2^2 + \|e^{n+1} - e^n\|_2^2.$$

Note that

$$\begin{aligned} X^{n+1} &= \frac{5}{2}\|e^{n+1}\|_2^2 + \frac{1}{2}\|e^n\|_2^2 - 2(e^{n+1}, e^n) \\ &= \frac{1}{2}\|e^{n+1}\|_2^2 + \frac{1}{2}\|2e^{n+1} - e^n\|_2^2 \\ &= \frac{5}{2}\|e^{n+1} - \frac{2}{5}e^n\|_2^2 + \frac{1}{10}\|e^n\|_2^2. \end{aligned}$$

Therefore

$$X^{n+1} \geq \frac{1}{10}\max\{\|e^{n+1}\|_2^2,\ \|e^n\|_2^2\}.$$

We then obtain

$$\begin{aligned} &\frac{X^{n+1} + A\tau^2\|e^{n+1}\|_2^2 - (X^n + A\tau^2\|e^n\|_2^2)}{2\tau} \\ &\leq \frac{C\cdot(1+N_1^4)}{\nu}\cdot(X^n + A\tau^2\|e^n\|_2^2) + \frac{1}{2\nu}\|G^n\|_2^2. \end{aligned}$$

By Lemma 8.1, we get for any $m \geq 2$,

$$X^m \leq e^{(m-1)\tau\cdot\frac{C(1+N_1^4)}{\nu}}(X^1 + A\tau^2\|e^1\|_2^2 + \frac{\tau}{\nu}\sum_{n=1}^{m-1}\|G^n\|_2^2).$$

Recall $X^m = \frac{1}{2}\|e^m\|_2^2 + \frac{1}{2}\|2e^m - e^{m-1}\|_2^2 \geq \frac{1}{2}\|e^m\|_2^2$. This implies for any $m \geq 2$:

$$\begin{aligned} &\|v^m - \tilde{v}^m\|_2^2 \\ \leq& 2\exp((m-1)\tau\cdot\frac{C(1+N_1^4)}{\nu})\cdot(\frac{3}{2}\|e^1\|_2^2 - \frac{1}{2}\|e^0\|_2^2 + \|e^1 - e^0\|_2^2 + A\tau^2\|e^1\|_2^2 + \frac{\tau}{\nu}\sum_{n=1}^{m-1}\|G^n\|_2^2) \\ \leq& \exp((m-1)\tau\cdot\frac{C(1+N_1^4)}{\nu})\cdot(3\|v^1 - \tilde{v}^1\|_2^2 - \|v^0 - \tilde{v}^0\|_2^2 + 4\|v^1 - \tilde{v}^1\|_2^2 + 4\|v^0 - \tilde{v}^0\|_2^2 \\ &+ 2A\tau^2\|v^1 - \tilde{v}^1\|_2^2 + \frac{2\tau}{\nu}\sum_{n=1}^{m-1}\|G^n\|_2^2) \\ \leq& \exp((m-1)\tau\cdot\frac{C(1+N_1^4)}{\nu})\cdot((7 + 2A\tau^2)\|v^1 - \tilde{v}^1\|_2^2 + 3\|v^0 - \tilde{v}^0\|_2^2 + \frac{2\tau}{\nu}\sum_{n=1}^{m-1}\|G^n\|_2^2). \end{aligned}$$

□

8.2. **Proof of Theorem 1.6.** We shall denote by C a constant depending only on (ν, u_0, s, A). The value of C may vary from line to line. For simplicity we will denote $\lesssim_{\nu,A,u_0,s}$ as $\lesssim$.

Define $t_0 = 0$, $t_1 = \tau_1$, $t_m = \tau_1 + (m-1)\tau$ for any $m \geq 2$.

By Lemma 7.1, we have for $n \geq 1$,

$$\begin{aligned} &\frac{3u(t_{n+1}) - 4u(t_n) + u(t_{n-1})}{2\tau} \\ =& -\nu\Delta^2 u(t_{n+1}) - A(u(t_{n+1}) - u(t_n)) + \Delta\Pi_N(2f(u(t_n)) - f(u(t_{n-1}))) + \Delta G^n, \end{aligned}$$

where for any $m \geq 2$,

$$\tau\sum_{n=1}^{m-1}\|G^m\|_2^2 \lesssim (1 + m\tau)\cdot(\tau^4 + N^{-2s}).$$

Easy to check that

$$\sup_{n\geq 0}\|u(t_n)\|_\infty + \sup_{n\geq 0}\|\nabla u(t_n)\|_2 + \sup_{n\geq 0}\|\nabla u^n\|_2 \lesssim 1.$$

Then by Proposition 8.1 and Lemma 2.2, we have for any $m \geq 2$,

$$\begin{aligned}&\|u(t_m)-u^m\|_2^2\\ \leq&\exp(C\cdot(m-1)\tau)\Big((7+2A\tau^2)\|u(t_1)-u^1\|_2^2+3\|u_0-\Pi_N u_0\|_2^2+C\cdot(1+m\tau)\cdot(\tau^4+N^{-2s})\Big)\\ &\leq \exp(C\cdot t_m)\cdot C\cdot(1+\tau^2)(N^{-2s}+\tau^4).\end{aligned}$$

Thus for any $m \geq 2$:

$$\|u^m-u(t_m)\|_2 \lesssim e^{Ct_m}(N^{-s}+\tau^2).$$

□

9. Concluding remarks

In this work we considered several second order in time stabilized semi-implicit Fourier spectral methods for the 2D Cahn-Hilliard equation with double well potential. We proposed two novel stabilization schemes and proved unconditional energy stability independent of the time step τ. The stabilization parameter is taken to be sufficiently large, depending only on initial data and the diffusion coefficient. The corresponding error analysis is carried out in full detail. As a comparative study we also revisited two classical second order in time semi-implicit Fourier spectral schemes. We prove energy stability for moderately small time steps and identified the corresponding stability region. Our analysis suggests that in general the stabilization term can indeed relieve the size constraint on the time steps. On the other hand the form of the stabilization term is crucial for conditional or unconditional energy stability.

It is expected that our analysis can be generalized to other phase field models such as the thin film equations, the molecular beam epitaxy (MBE) equations, the Allen-Cahn equation and the like. An intriguing issue is to investigate general stabilization techniques such as biharmonic stabilization $-\Delta^2(u^{n+1}-u^n)$, fractional Laplacian stabilization $-(\Delta)^s(u^{n+1}-u^n)$ $(s>0)$ or more general pseudodifferential operators. Other interesting topics include phase field models with higher order dissipations ([8]), nonlinear diffusion models ([28]), and decoupled energy stable numerical schemes ([29], [30]). Another issue is to lower the dependence on ν in stability and error estimates by using more refined resolvent bounds of the linearized Cahn-Hilliard operator. We plan to address some of these issues in future publications.

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