J. Math.
Vol. 13,

Soc. Japan
No. 2, 1961

On semi-hereditary rings

By Shizuo ENDO

(Received June 18, 1960)
(Revised Nov. 14, 1960)

§ 1. Introduction.

A ring R with unit element is called " left (right) semi-hereditary " according
to [21 if any finitely generated left (right) ideal of R is projective.
The purpose of this paper is to determine completely the structure of commutative semi-hereditary rings. A. Hattori has recently given in [6] a homological characterization of Prefer rings, i. e., semi-hereditary integral domains. This was generalized by M. Harada [51 to commutative rings whose total quotient rings are regular. The results of this paper will include those results of [5] and [61.
In § 3 we shall give a necessary and sufficient condition for a ring to be regular by using the quotient rings. Also we shall introduce a notion of quasi-regular rings and show some properties of them.
In § 4 we shall characterize semi-hereditary rings by using the quotient rings as follows : A ring R is semi-hereditary if and only if the total quotient ring K of R is regular and the quotient ring Rm of R with respect to any maximal ideal m of R is a valuation ring. Furthermore we shall introduce a notion of algebraic extensions of regular rings and show that the integral closure R' of a semi-hereditary ring R in any algebraic extension K' of the total quotient ring K of R is also semi-hereditary.
In § 5, we shall first prove that a local ring R is a valuation ring if and only if w. gl. dim R <_ 1. Secondly we shall show, as a generalization of [6], Theorem 2, that a ring R with the total quotient ring K is semi-hereditary if and only if w. gl. dim R <_ 1 and w. gl. dim K= 0, or if and only if any torsion-free R-module is flat.

§ 2. Notations and terminologies.

Throughout this paper a ring will mean a commutative ring with unit element 1. Our notations and terminologies are, in general, the same as in [2] but we shall make the following modifications.
A local ring will mean a (not always Noetherian) ring with only one

110 S. ENDO

maximal ideal and a regular ring R will mean a ring such that for any a R there is an element b of R with aba = a (cf. von Neumann [10]).
Let R be a ring, M an R-module and S a multiplicatively closed subset of
R. Then the quotient ring and module of R, M with respect to S are defined as in [2] and denoted by RS, MS respectively. If S is the complementary set of a prime ideal p in R, then we shall use R~, M~ instead of RS, MS.
Let R be a ring and T be the set of all non zero divisors in R. Then the quotient ring K of R with respect to T will be called the " total quotient ring" of R. An element u of an R-module M will be called a " torsion element" if

to = 0 for some t T. If we denote by t(M) the set of all torsion elements in M, then t(M) becomes an R-module and will be called a " torsion submodule " of M. If t(M) = M, M will be called a " torsion module ", and on the other hand, if t(M) = 0, it will be called a " torsion-free " module. Furthermore an R-module M will be called a "divisible" module if for any t E T, u M there is an element v of M with u = tv.

3. Regular rings and quasi-regular rings.

First we shall prove the following1
THEOREM 1. A ring R is regular if and only if the quotient ring Rm of R with respect to any maximal ideal m of R is a field.
PROOF. The only if part : If R is regular, then any Rm is obviously reg-ular, hence we have only to show that if a local ring R is regular, it is a field. Let m be a maximal ideal of a local ring R. If there is a non-unit a in R, then a is contained in m. Since R is regular, we have alb = a for a sui-table element b of R, hence (1-ab)a = 0. Since ab E m, 1-ab is a unit of R. Therefore a =0. Thus R must be a field.
The if part : Let a be an element of R and set Ii = {b; ba = 0, b E R}. Since any Rm is a field, b is not contained in any maximal ideal m containing a. Setting c = (a, b), c is not contained in any maximal ideal of R and so we have R = (a, b). Since (a)b = 0, (a) is a direct summand of R. Accordingly we have (a) = (e) for a suitable idempotent e of R and also have b = (1-e). Fur-thermore, if we set d =1-era, then d is clearly a unit of R and we have de = ae = a. So we obtain ad-1a = a. This proves that R is regular. COROLLARY 1. A ring R is regular if and only if any element of R is expres-sible as a product of a unit and an idempotent in R.
COROLLARY 2. Let R be a regular ring and a be a finitely generated ideal
of R. Then a is generated by a single idempotent.

1) The contents of Theorem 1 and its corollary 1 were published in author's paper [3].

Semi-hereditary rings 111

PROOF. By Corollary 1 we may assume that a is generated by a finite number of idempotents of R. It suffices to show this in case a = (e1, e2), where :e1, e2 are idempotents of R. Setting e = e1+e2-ele2, we obtain easily e2 = e and eei = ei for i =1, 2. Therefore a = (e). Thus our proof is completed. COROLLARY 3. Any regular ring is semi-hereditary.
A ring R is called a " quasi-regular" ring if the total quotient ring K of
R is regular.
PROPOSITION 1. Let R be a quasi-regular ring and K be the total quotient ring of R. Let a be an ideal of R such that aKn R = a. Then R/a is also a quasi-regular ring and K/aK can be regarded as the total quotient ring of R/a. PROOF. R/a can be regarded as the subring of K/aK by identifying R+aK/aK to R/a. Then K/aK is obviously contained in the total quotient ring of R/a. Since the homomorphic image of a regular ring is also regular, K/aK is regular. Then the total quotient ring of R/a must coincide with K/aK, for the total quotient ring of a regular ring is itself.
PROPOSITION 2. Let R be a quasi-regular ring with the total quotient ring K
and S be a multiplicatively closed subset of R. Then the quotient ring RS of R with respect to S is also a quasi-regular ring and the quotient ring KS of K with respect to S is the total quotient ring of Rs.
PROOF. Set as = {a; as = 0 for some s S, a E R}. Let a be an element
of a5Kn R. Then we have a = alas+a2a2+ ... +atat, ai E as, cai E K. Let si
t be an element of S for each i such that siai = 0, and set s = II si. Then we
i=1

obtain sa = 0, hence a E as. This shows as = a5Kn R. So, by Proposition 1, R/as is a quasi-regular ring with the total quotient ring K/asK. Since KS = K/asK and Ks Rs J R/as, Rs is a quasi-regular ring with the total quotient ring KS.
PROPOSITION 3. Let R be a quasi regular ring and a be a finitely generated ideal of R. Then the following statements are equivalent:
1) a is projective.
2) For any maximal ideal m aRm is zero or generated by a single non zero divisor of Rm.
3) a-la is a direct summand of R.
4) a is a direct summand of an invertible ideal of R.
PROOF. The implications 1)-* 2) and 4)- ~ 1) are obvious.
The implication 2)-* 3) : If we set b = {b; ba =0, b R}, then we have
in for any maximal ideal m of R such that aRm =0. On the other hand,
in case aRm 0, aRm is invertible in Rm by our assumption. Setting a = (al,

n

an) and a = aRm, we have =1, ai a-1, where 1, ai are the residues

1=1

of 1, ai in Rm respectively. Set am ={a; as = 0 for some s R-m, a E R}.

112 S. ENDO

Since ai ar', we have aiaj = sij R/ant-m/am, tij E R/am for any Z and j. If we set s" = we have saia j R/am. By Proposition 2 we can
i,.7

now choose a representative ai of ai in K for any i. Furthermore, by choos-ing suitably an element s' of R-m, we obtain s'saiaj E R for any i and j, where s is a representative of s in R• So s'sai E a-1 for any i. Now we have ss's" = (ss's"ai)ai for s" R-m. The left hand side of this formula is not contained in m but the right hand side is contained in a-la. This shows that if aRm 0, then a-la m. Hence, if we set c = (b, a-la), c is not contained in any maximal ideal of R, and so we have R = (b, a-1 a). Since Iia -1 a = 0, a-1 a must be a direct summand of R.
The implication 3) -+ 4). Suppose that a-la is a direct summand of R. Then there is an idempotent e of R such that a-la = (e). If we set b = (1-e, a), then b is invertible as b-' = (1-e, a-1e). Since a is a direct summand of b, this proves our assertion.
PROPOSITION 4. Let R be an integrally closed quasi-regular ring with the total quotient ring K and a be an ideal of R such that a = aKn R. Then R/a is also integrally closed.
PROOF. By Proposition 1 K/aK is the total quotient ring of R. Now let
a be an element of K/aK integral over R/a. Then we have &+a1an-I+ • • • + an = 0, di R/a. Denote by a, ai representatives of a, ai in K, R, respectively. Then & +alan-1+ +an = aK. Since K is regular, we have ,0 = re for a unit r and an idempotent e of K, by Corollary 1 to Theorem 1. Then we have e R, for R is integrally closed. So e E a = aKn R. From (1-e)/9 = 0 we obtain ((1-e)a)n+a1((1-e)a)n-1+ ... +a(1-e) = 0. Since R is integrally closed, we have (1--e)a E R. As (1-e)a = a, a must be in R/a.
PROPOSITION 5. Let R be a quasi-regular ring. Then R is integrally closed
if and only if the quotient ring Rm of R with respect to any maximal ideal m of R is integrally closed. In general, if R is integrally closed, the quotient ring RS of R with respect to any multiplicatively closed subset S of R is integrally closed. PROOF. The only if part is contained in the second part and also the second part is easily obtained from Propositions 2 and 4. Hence we have only to show the if part. Let K be the total quotient ring of R and a be an element of K integral over R. If we set S = R-m for any maximal ideal m of R, then KS can be regarded as the total quotient ring of Rm according to Proposition
2. The residue a of a in KS is, then, integral over R,n, so, by our assumption, we have a Rt,t. Hence we have sa R for some s E S. If we set c = {c; ca R, c E R}, then we must have c = R, i. e., a E R.
PROPOSITION 6. Let R be a local quasi-regular ring. If R is integrally closed, then it is an integral domain.
PROOF. Let K be the total quotient ring of R. If K is not a field, then

Semi-hereditary rings 113

there exists an idempotent e of K which is not a unit element by Corollary 1 to Theorem 1. Since R is integrally closed, e is contained in R. Then R is expressible as a direct sum of Re and R(1-e). Since R is local, this is obvi-ously a contradiction. Consequently K is a field. Thus R is an integral domain. PROPOSITION 7. Let R be a quasi-regular ring. Then an R-module M is a torsion free module if and only if the quotient module Mm with respect to any maximal ideal m of R is a torsion free R,,,-module.
PROOF. The if part is evident, hence we have only to show the only if part. Suppose that M is a torsion-free R-module and that au = 0 for a non zero divisor a of Rm and an element u of Mm. Set a,n = {a ; as = 0 for some s E R-m, a R} and M' ={ u; su = 0 for some s R-m, u E M}. Then we may assume a R/am and u M/M' by multiplying suitably elements of R/a,,, -m/a,,, to a, u. Denote by a a representative of a in R and by u a representative of u in M. Since a is a non zero divisor of R/arn„ an ideal c = (a, am) of R contains a non zero divisor b of R for R is quasi-regular. Setting b = raga', r R, a' am, we have bu = rau = 0. Hence, for a suitable element s of R-m, we have sbu = 0. As b is a non zero divisor, we obtain su = 0. This shows u = 0. Thus M,, is a torsion-free Rm-module.

§ 4. Characterization by quotient rings and algebraic extension.

Here we shall prove our main theorem.2~
THEOREM 2. A ring R is semi-hereditary if and only if the total quotient ring K of R is regular and the quotient ring R,n of R with respect to any maximal ideal m of R is a valuation ring.
PROOF. The only if part : Assume that R is semi-hereditary. Then any Rm is obviously semi-hereditary, hence it is a valuation ring as any finitely generated projective ideal of a local ring is a principal ideal generated by a single non zero divisor. Similarly K is also semi-hereditary. Now suppose that K is not regular. Then, by Theorem 1, there exists a maximal ideal m' of K such that Km' is not a field but a valuation ring. If we set ~' = {a'; a's' = 0 for some s' E K-m', a' E K}, ' is a prime ideal of K strictly contained in m'. Let a' be an element of m' not contained in '. Since K is semi-heredi-

tary, a principal ideal (a') is projective over K. If we set b = {b'; b'a' = 0, b' K}, then b' is a direct summand of K and therefore we have Ii' = (e') for a suitable idempotent e' of K Further set c' = e'ra'. Then c' is contained in m' since a' ~', hence c' is a non unit. On the other hand, if c'd' = 0,

2) Mr. M. Nagata reported to the author that there exists a ring R such that K
is not regular but any Ris a valuation ring. So we can not omit the condition that K is regular from the condition in our theorem.

114 S. ENDO

d' E K, then d'e' = d'a' = 0. Since d' E (e') n (1-e'), we obtain d' = 0. Thus c'" is a non zero divisor. Consequently c' is a non unit and a non zero divisor of K. This contradicts the fact that K is the total quotient ring of R.
The if part : Let a be a finitely generated ideal of R. Since R is quasi-regular and any Rm is a valuation ring, a satisfies the condition 2) in Proposi-tion 3. Hence a is projective. This shows that R is semi-hereditary. COROLLARY 1. A ring R is semi-hereditary if and only if any finitely genera-
ted ideal of R is a direct summand of an invertible ideal of R.

PROOF. It is obvious by Proposition 3 and Theorem 2.
COROLLARY 2. A semi-hereditary ring is integrally closed.
PROOF. This follows from Proposition 5 immediately.
Let R be a valuation ring and K be the quotient field of R. Let K' be
an algebraic extension of K and R' be the integral closure of R in K'. It is well known that the quotient ring R'm- of R' with respect to any maximall ideal m' of R' is a valuation ring (cf. [9]). This fact shows, according to Theorem 2, that R' is a Pruf er ring. We shall give a generalization of this to general semi-hereditary rings.
PROPOSITION 8. Let R be a regular ring and R' be a subring of R such that
R is integral over R'. Then R' is also a regular ring.
PROOF. By Theorem 1 it suffices to prove that for any maximal ideal m'
of R' R'm' is a field. Now put S' = R-m'. Then RS is integral over Therefore any maximal ideal m of R5 contains m'RS,. If we set n = (1 m where m runs over all maximal ideals of RS,, n contains m'RS,. Since RS5 is regular, we have n = 0, so m'RS, = 0. Consequently m'R'm, = 0. This shows that
is a field.

Let R, R' be regular rings with the common unit element such that
R C R'. Then R' is called an "algebraic extension" of R if R' is integral over R.

THEOREM 3. Let R be a semi-hereditary ring and K be the total quotient
ring of R. Let K' be an algebraic extension of K and R" be any intermediate ring between R and K'. Then the integral closure R" of R" in its total quotient ring is also a semi-hereditary ring.
PROOF. Let K" be the total quotient ring of R". Then, by Proposition $,. K" is regular. Hence we may assume K' = K". Now let R' be the integral closure of R in K'. Then we have obviously R' C R". First we shall prove that R' is semi-hereditary. By the definition of algebraic extension K' is the total quotient ring of R'. Then, by Theorem 2, it suffices to show that the quotient ring R'm, of R' with respect to any maximal ideal m' of R' is a valu-ation ring. Set m = m' n R and S = R-m. Then R'S is a quasi-regular ringg with the total quotient ring K'S by Proposition 2. Since R' is integrally

Semi-hereditary rings 115

closed in K', R'S is integrally closed in K'S by Proposition 5. Also it is obvious that R'S is integral over Rm. Hence R'S is the integral closure of Rm in K'S. Since we have R'm, _ (R's)mfR's, we can suppose that R is a valuation ring. If we set :;3' = {a' ; a's' = 0, for some s' R'-m', a' R'}, then ~' is a prime ideal of R' by Proposition 6. As is easily seen we can regard K'/p'K' C R'/p' R. Since R' is integrally closed, R'/p' is also integrally closed in K'/'K' by Proposition 4. Since K'/p'K' is an algebraic extension (in the ordinary sense) of the quotient field K of R, R'/p' is a Prefer ring, as is well-known. Now we have R'm, _ (R'/Y')m'ip'. Hence R'm, must be a valuation ring. Thus R' is semi-hereditary. From this we may assume R = R', K=K' and R C R" C K. Let m" be a maximal ideal of R" and set m = m" n R. Then m is a prime ideal of R. If we set S = R-m, we have KS C R"s Rm. Since Rm is a valua-tion ring, R"8 is also a valuation ring. Accordingly R"" = RS, Again, by Theorem 2, R" must be semi-hereditary.

§ 5. Homological characterization.

Now we refer to some well-known facts (cf. [2]).
(I) Let R be a ring and M be an R-module. Then M is R-flat, i. e.A
w. dim RM= 0 if and only if for each relation atiui = 0, ai E R, zc1 M, there

2 exist elements r,j R, vj E M, finite in number, such that u.1= rz,jvj,] r1ja1= & (cf. [2, VI, Ex. 6]).
(II) Let R be a ring, M be an R-module and S be a multiplicatively closed. subset of R. Then from (I) it follows immediately that RS is R-flat as an R-module and we have MS N RS ® M as RS-modules. For any R-modules M, N and any integer n > 0 we have (Tor R (M, N))8 Tor RS(M8, NS). If M is an R8-module, then if we regard M as an R-module, we have MS = M, w. dimRM=
w. dimRSM. From these we obtain easily that for any R-module M we have w. dimRM= sup. w. dimRmM and w. gl. dim R = sup. w. gl. dim Rm, where m runs

m m

over all maximal ideals of R (cf. [2, VII, Ex. 9, 10, 11]).
(III) Let R be a ring with the total quotient ring K and M be an R-module. Then, by (II) we have w. dimRK= 0. If we set K= K/R, then we have an exact sequence

0 - TorR(M, K) -> M-~ MK-

- R

and t(M) TorR(M, ,K). Therefore M is torsion-free if and only if Tor' (Mr K) = 0 and is a torsion module if and only if M~ K= 0. Again, by (II), for R any torsion-free divisible R-module M, we have w. dimRM= w. dimRM since M can be regarded as a K -module. Conversely any K -module can be regarded as a torsion-free divisible R-module (cf. [2, VII]).

116 S. ENDO

We shall begin with the following
PROPOSITION 9. If a ring R is local, then any finite flat R-module M is always free.
PROOF. Suppose that M is not free but flat. Denote by n the minimum
number of elements generating M and by s the minimum number of non zero

n elements ai of R such that aiui = 0 for not all a1 = 0 and a minimal base
i=1

(u1, u2i • • • , un) of M. By our assumption there exists such a positive integer s.
s Now we may assume ± aiui = 0 for all ai ~ 0 and M= (u1, , us, uS+1, , un).

i=1
t s

Again, by applying (I), we obtain ui = ri ju' j, ri jai = 0, for ri j E R, u'j M.
j=1 i=1
n t n

If we set u' j = r,kuk, rjk E R, then we have ui = ri jr~kuk. Since (u1, • • • ,
k=1 j=1 k=1

t

u,~) is minimal, ~rsjr~s is a unit of R, and so at least one rsjo of rsj's is a unit j=1
of R. If s =1, then a1= 0. This is a contradiction. If s > 1, then we have
s-1 s

as = biai, bi R, as rai = 0. If we set u'i = ui+bius, for 1 <_ i <_ s-1, then
i=1 i=1
s-1

we have au'i = 0 and M= (u'1, ... , us_1i us, , un). This is also a contradic-tion. Thus M must be free.

THEOREM 4.3' A local ring R is a valuation ring if and only if w. gl. dim
R <_ 1. Especially it is a field if and only if w. gl. dim R = 0.
PROOF. The only if part is well known (cf. [2, VI, 2.9]). Hence we have only to show the if part. If w. gl. dim R _< 1, then any ideal of R is R-flat. By Proposition 9 any finitely generated ideal of R is free, hence it is generated by a single non zero divisor. Thus R is a valuation ring. Suppose that w. gl. dim R' = 0. If R is not a field, there exists a non unit a 0 of R. Then we have w. dimRR/(a) = 0. Again, by Proposition 9, R/(a) is free. This is obviously a contradiction. Consequently R must be a field.
The following proposition is a special case of [4, Theorem 5].
PROPOSITION 10. A ring R is regular if and only if w. gl. dim R =0.
PROOF. Obvious by Theorem 1, 4 and (II).
PRoPosITIoN 11. For any ring R we have w. gl. dim R <_ 1 if and only if the quotient ring Rm of R with respect to any maximal ideal m of R is a valuation ring.
PROOF. This follows from Theorem 4 and (II).
We shall now give a characterization of semi-hereditary rings, which is a generalization of Hattori's result (cf. [5] and [6]).
THEOREM 5. For any ring R with the total quotient ring K, the following

3) This theorem and Proposition 9 may be known. However, as these could not be found in any papers, the proofs of these are given here.

Semi-hereditary rings 117

conditions are equivalent :
1) R is a semi-hereditary ring.
2) w. gl. dim R 1 and w. gl. dim K= 0.
3) For any torsion free R-module M, we have w. dimRM= 0.
PROOF.4' The equivalence of 1) and 2) follows from Theorem 2 and Propo-sitions 11 and 12. Also the implication 3) ---> 2) is obvious by (III). Hence it suffices to prove the implication 2) --* 3). If M is a torsion-free R-module, then for any maximal ideal m of R Mm is a torsion free R11-module by Proposition 7. Since Rm is a valuation ring, any finite torsion-free R-module, is projective (cf. [2, VII, 4.1]). Since Torn commutes with direct limites, we obtain w. dimnmM,n = 0. Then by applying (II) to M we obtain w. dimRM= 0.
It is shown in [2, VII, 4.1] that an integral domain R is a Prufer ring if and only if any finite torsion-free R-module is projective. However a finite torsion-free module over a semi-hereditary ring which is not an integral domain is not always projective.
COROLLARY. For any ring ,R with the total quotient ring K, the following statements are equivalent :
1) R is a direct sum of a finite number of Prufer rings.
2) w. gl. dim R < 1 and gl. dim K= 0.
3) Any finite torsion free R-module is projective.
PROOF. The implications 1) 2) --* 3) are obvious by Theorem 4. Hence
we have only to prove the implication 3) --~ 2). Assume that R satisfies the con-dition 3). Then, by Theorem 5, we have w. gl. dim R <_ 1 and w. gl. dim K= 0. Hence it suffices to show gl. dim K= 0, that is, that K is semi-simple. If we set pn, = {a; as = 0 for some s R-m, a R } for any maximal ideal m of R, then P,,, is a prime ideal of R. Since any R/P,,, is a torsion-free R-module generated by a single element, it is projective by our assumption, and so p,n is a direct summand of R. Accordingly we have p,,, = (e) for a suitable ldempotent em of R. If we set e n, =1-e,n for any m and denote by a the ideal generated by all e,n's, then a is not contained in any maximal ideal m of R, hence we have a = R. Then we have 1= ale,n1+a2e,n,+ +ane,,,n, a~ R, by choosing suitably a finite number of e,n's. Since e,n is contained in psuch that pm, ~ p, this shows that there is only a finite number of pm in R. Consequently K must be semi-simple, as any PmK is a maximal ideal of K. The following proposition is a slight generalization of [8, Theorem 1]. PROPOSITION 12. Let R be a semi-hereditary ring whose total quotient ring

4) This theorem can be proved by using the similar method as in [6]. However, in this case, we need to use the proof of the only if part of Theorem 2 to show the implication 1) -* 2). The condition d) in [6], Theorem 2 can not be generalized with-out assuming any condition for a ring.

118 S. ENDO

K is semi-simple. Then any finite R-module M is expressible as a direct sum of a torsion free module and a torsion module.

PROOF. By (III) we have an exact sequence

0 --* TorR(M, K/R) -p M--; M® K--~

R If we set M' = Image M in M® K, then we have also an exact sequence :

R

0 --> TorR(M, K/R) -; M--* M' -- 0 .

Since M' is a finite torsion-free R-module, it is projective by Corollary to Theorem
5. Then the above exact sequence splits and we have M TorR(M, K/R) M'o This proves our assertion.
Finally we shall give a characterization of quasi-regular rings.5' PROPOSITION 13. For any ring R the following conditions are equivalent:
1) R is a quasi-regular ring.
2) Any torsion free divisible R-module M is R -fiat.
3) For any R-modules M, N and any n >>-1 TorR(M, N) is a torsion R-module.. PROOF. Let K be the total quotient ring of R. By (III) we have
w. dimRM= w. dimKM for any torsion-free divisible R-module M. If R is quasi regular, then we have w. gl. dim K= 0 by Proposition 10. Hence w. dimRM= w. dimKM= 0. This shows 1) --k 2). Let M, N be any R-modules. Applying (II), we have (Torn(M, N))T ~ Torn(MT, NT) ~ (TorR(MT, NT))T. If R satisfies the condition 2), then T orn(MT, NT) = 0 for n ? 1 since MT, NT are regarded as torsion-free divisible R-modules. Therefore we have also (Torn(M, N))T = 0. Since (Torn(M, N))T ~ K® TorR(M, N) by (II), Torn(M, N) is a torsion R-module by (III). Thus 2) --~ 3) is shown. Let MK, NK be any K -module. If we regard MK, NK as R-modules and Tor(MK, NK) is a torsion R-module, we obtain. TorK(MK, NK) ~ K® TorR(MK, NK) = 0 by (II), (III). This proves 3) -; 1).

Kanto Gakuin University

References

[1] Y. Akizuki, The theory of local rings, Lecture notes at Univ. of Chicago, 1958. [2] H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, 1956. [3] S. Endo, Note on P. P. rings, Nagoya Math. J., 17 (1960), 167-170.
[4] M. Harada, Note on the dimension of modules and algebras, J. Inst. Poly. Osaka City Univ., 7 (1956), 17-27.
[5] M. Harada, A note on Hattori's theorems, J. Inst. Poly. Osaka City Univ., 9 (1958), 43-45.
[6] A. Hattori, On Prefer rings, J. Math. Soc. Japan, 9 (1957), 381-385.

5) The implications 1) -- 2) and 1) > 3) in this proposition were first proved by M. Harada [5].

[7]

[8]

[9]
[10]

Semi-hereditary rings

A. Hattori, A foundation of the torsion theory over general rings,
J., 17 (1960), 147-158.
I. Kaplansky, Modules over Dedekind rings and valuation rings, Math. Soc., 72 (1952), 327-340.
W. Krull, Idealtheorie, Berlin, 1935.
J. v. Neumann, On regular rings, Proc. Nat. Acad. Sci. U. S. A., 22

119

Nagoya Math,

Trans. Amer.

(1936),707-713,