Journal of Algebraic Combinatorics, 22, 139–149, 2005

On Semi-Pseudo-Ovoids

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Received May 12, 2004; Revised December 7, 2004; Accepted December 7, 2004

Abstract. In this paper we introduce semi-pseudo-ovoids, as generalizations of the semi-ovals and semi-ovoids. Examples of these objects are particular classes of SPG-reguli and some classes of m-systems of polar spaces. As an application it is proved that the axioms of pseudo-ovoid $O(n, 2n, q)$ in PG$(4n-1, q)$ can be considerably weakened and further a useful and elegant characterization of SPG-reguli with the polar property is given.

Keywords: semi-pseudo-ovoid, egg, SPG-regulus, polar property

1. Introduction

1.1. Pseudo-ovals and pseudo-ovoids

In PG$(2n+m-1, q)$ consider a set $O(n, m, q)$ of $q^m + 1$ $(n-1)$-dimensional subspaces PG$^{(0)}(n-1, q)$, PG$^{(1)}(n-1, q)$, $\ldots$, PG$^{(q^m)}(n-1, q)$, every three of which generate a PG$(3n-1, q)$ and such that each element PG$^{(i)}(n-1, q)$ of $O(n, m, q)$ is contained in a PG$^{(i)}(n+m-1, q)$ having no points in common with any PG$^{(j)}(n-1, q)$ for $j \neq i$. It is easy to check that PG$^{(i)}(n+m-1, q)$ is uniquely determined, $i = 0, \ldots, q^m$. The space PG$^{(i)}(n+m-1, q)$ is called the *tangent space* of $O(n, m, q)$ at PG$^{(i)}(n-1, q)$, $i = 0, \ldots, q^m$. For $n = m$ such a set $O(n, n, q)$ is called a *pseudo-oval* or a *generalized oval* or an $[n-1]$-*oval* of PG$(3n-1, q)$; a generalized oval of PG$(2, q)$ is just an oval of PG$(2, q)$. For $n \neq m$ such a set $O(n, m, q)$ is called a *pseudo-ovoid* or a *generalized ovoid* or an $[n-1]$-*ovoid* or an *egg* of PG$(2n+m-1, q)$; a [0]-ovoid of PG$(3, q)$ is just an ovoid of PG$(3, q)$.

In Payne and Thas [9] (Theorem 8.7.2) it is proved that either $ma = n(a+1)$ or $n = m$, with a an odd natural number, and that for q even we have either $n = m$ or $2n = m$. Also, in Payne and Thas [9] (Chapter 8) many other properties of $O(n, m, q)$ appear. It is still an open question whether or not for q odd we have $m \in \{n, 2n\}$.

Pseudo-ovals and pseudo-ovoids play an important role in the theory of finite generalized quadrangles, as in Payne and Thas [9] (Theorem 8.7.1) it is shown that their study is equivalent to the study of finite translation generalized quadrangles.

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1.2. Semi-pseudo-ovoids

A *semi-pseudo-ovoid* or a *semi-egg* of $\mathrm{PG}(h, q)$ is a non-empty set O of η mutually skew $(n-1)$-dimensional subspaces, denoted $\mathrm{PG}^{(i)}(n-1, q)$, $i = 1, \ldots, \eta$, with $h > 2n-1$, so that for every i the union of all n-dimensional subspaces containing $\mathrm{PG}^{(i)}(n-1, q)$ and disjoint from $\mathrm{PG}^{(j)}(n-1, q)$, for every $j \neq i$, is an $(h-n)$-dimensional subspace $\mathrm{PG}^{(i)}(h-n, q)$ of $\mathrm{PG}(h, q)$. The space $\mathrm{PG}^{(i)}(h-n, q)$ is called the *tangent space*, or just the *tangent*, of O at $\mathrm{PG}^{(i)}(n-1, q)$.

For $n = 1$ semi-pseudo-ovoids are just semi-ovals and semi-ovoids; see Thas [11] and Buekenhout [2] for motivation, examples and existence.

It is also clear that pseudo-ovals and pseudo-ovoids provide examples of semi-pseudo-ovoids.

We now describe a method to construct a new semi-pseudo-ovoid from a given one. Let O be a semi-pseudo-ovoid consisting of $(n-1)$-dimensional subspaces of $\mathrm{PG}(h, q)$. Let $\pi \in O$ and assume that any n-dimensional subspace containing any element γ of $O - \{\pi\}$ and any point of π, has a point in common with at least one element of $O - \{\pi, \gamma\}$. Then $O - \{\pi\}$ is still a semi-pseudo-ovoid of $\mathrm{PG}(h, q)$.

2. The main inequalities

2.1. Main theorem

Theorem 2.1 *If O is a semi-pseudo-ovoid consisting of η $(n-1)$-dimensional subspaces of* $\mathrm{PG}(h, q)$, *then*

$$1 + q^{h-2n+1} \leq \eta \leq 1 + q^{\frac{h+1}{2}}.$$

It follows that $h \leq 4n - 1$.

Proof: Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a semi-pseudo-ovoid in $\mathrm{PG}(h, q)$ consisting of η $(n-1)$-dimensional subspaces. The tangent space of O at π_i will be denoted by τ_i, with $i = 1, 2, \ldots, \eta$. Further, let $\tilde{O} = \pi_1 \cup \pi_2 \cup \cdots \cup \pi_\eta$.

Consider an n-dimensional subspace β with $\pi_1 \subset \beta \subset \tau_1$ and let γ be an $(n+1)$-dimensional subspace with $\beta \subset \gamma$, $\gamma \not\subset \tau_1$. Each n-dimensional subspace δ of γ containing π_1, with $\delta \neq \beta$, contains a point of $\tilde{O} - \pi_1$. Hence

$$|(\tilde{O} \cap \gamma) - \pi_1| \geq q.$$

There are exactly $\frac{q^{h-n}-1}{q-1} - \frac{q^{h-2n}-1}{q-1}$ spaces γ. It follows that

$$|\tilde{O}| \geq \frac{q^n - 1}{q-1} + q\frac{q^{h-n} - q^{h-2n}}{q-1},$$

that is,

$$|\tilde{O}| \geq \frac{q^n - 1}{q - 1}(1 + q^{h-2n+1}).$$

Consequently,

$$|O| \geq 1 + q^{h-2n+1}. \tag{1}$$

Next, let x_i be any point of $\mathrm{PG}(h, q) - \tilde{O}$, and let t_i be the number of n-dimensional subspaces ξ on x_i with $\pi_j \subset \xi \subset \tau_j$ for some j. First we count the number of pairs (x_i, ξ), with $x_i \in \xi$, ξ n-dimensional, and $\pi_j \subset \xi \subset \tau_j$ for some j. We obtain

$$\sum_i t_i = \eta q^n \frac{q^{h-2n+1} - 1}{q - 1}. \tag{2}$$

Next, we count the number of ordered triples (x_i, ξ, ξ'), with $x_i \in \xi$, $x_i \in \xi'$, $\xi \neq \xi'$, ξ and ξ' n-dimensional, $\pi_j \subset \xi \subset \tau_j$ for some j, and $\pi_{j'} \subset \xi' \subset \tau_{j'}$ for some j'. We obtain

$$\sum_i t_i(t_i - 1) = \eta(\eta - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{3}$$

Hence

$$\sum_i t_i^2 = \eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{4}$$

The number of points x_i is equal to

$$d = |\mathrm{PG}(h, q) - \tilde{O}| = \frac{q^{h+1} - 1}{q - 1} - \eta\frac{q^n - 1}{q - 1}. \tag{5}$$

Now we have $d \sum_i t_i^2 - (\sum_i t_i)^2 \geq 0$, and so, by (2), (4) and (5)

$$\left(\frac{q^{h+1} - 1}{q - 1} - \eta\frac{q^n - 1}{q - 1}\right)\eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1} - \left(\eta q^n \frac{q^{h-2n+1} - 1}{q - 1}\right)^2 \geq 0,$$

that is, $\eta^2 - 2\eta - (q^{h+1} - 1) \leq 0$, and so,

$$\eta \leq 1 + q^{\frac{h+1}{2}}. \tag{6}$$

Finally, from (1) and (6) follows that $1 + q^{h-2n+1} \leq 1 + q^{\frac{h+1}{2}}$, and so

$$h \leq 4n - 1.$$

□

2.2. Pseudo-ovoids

In this section we will show that by Theorem 2.1 the original definition of pseudo-ovoid $O(n, 2n, q)$ can be considerably weakened.

Theorem 2.2 *If for a semi-pseudo-ovoid O consisting of η $(n-1)$-dimensional subspaces of* PG(h, q) *we have $h = 4n - 1$, then $\eta = 1 + q^{2n}$ and so O is a pseudo-ovoid.*

Proof: Assume that $h = 4n - 1$ for the semi-pseudo-ovoid O. Then $h - 2n + 1 = (h+1)/2$, and so by Theorem 2.1 we have $\eta = 1+q^{h-2n+1} = 1+q^{2n}$. As we have equality in (6), we also have $d \sum_i t_i^2 - (\sum_i t_i)^2 = 0$, with the notation of the proof of Theorem 2.1. So t_i is a constant. Hence

$$t_i = \frac{\sum_i t_i}{d} = q^n + 1$$

for all i. As $\eta = 1 + q^{h-2n+1}$, each n-dimensional subspace containing $\pi_i \in O$, but not contained in the tangent space of O at π_i, contains exactly one point of $\tilde{O} - \pi_i$, where $\tilde{O}$ is the set of all points in all elements of O, with $i = 1, 2, \ldots, q^{2n} + 1$. It follows that any three distinct elements of O generate a $(3n - 1)$-dimensional subspace of PG(h, q). Consequently O is a pseudo-ovoid of PG$(4n - 1, q)$. □

Remark

(a) It follows that an egg $O(n, 2n, q)$ is a set of $(n-1)$-dimensional subspaces of PG$(4n - 1, q)$, such that for each $\pi_i \in O(n, 2n, q)$ the union of all n-dimensional subspaces containing π_i but skew to all elements of $O(n, 2n, q) - \{\pi_i\}$ is a $(3n - 1)$-dimensional subspace of PG$(4n - 1, q)$.
(b) A *weak egg* of PG$(4n - 1, q)$ is a set of $1 + q^{2n}$ $(n - 1)$-dimensional subspaces of PG$(4n - 1, q)$, every three of which generate a $(3n - 1)$-dimensional subspace. It is an open question whether or not each weak egg is an egg; see Lavrauw [5].

3. Interpretation of the equalities

We will use the notation introduced in Section 2.

Theorem 3.1 *For a semi-pseudo-ovoid O we have $\eta = 1 + q^{\frac{h+1}{2}}$ if and only if each point not in an element of O is on a constant number of tangent spaces. This constant equals $1 + q^{\frac{h-2n+1}{2}}$.*

Proof: Each point not in an element of O is on a constant number of tangent spaces if and only if t_i is a constant in the proof of Theorem 2.1, if and only if $d\sum_i t_i^2 - (\sum_i t_i)^2 = 0$, if and only if $\eta = 1 + q^{\frac{h+1}{2}}$. In such a case

$$t_i = \frac{\sum_i t_i}{d} = 1 + q^{\frac{h-2n+1}{2}}.$$

□

Theorem 3.2 *For a semi-pseudo-ovoid O we have $\eta = 1 + q^{\frac{h+1}{2}}$ if and only if each hyperplane not containing a tangent space of O, contains a constant number of elements of O. This constant equals $1 + q^{\frac{h-2n+1}{2}}$.*

Proof: Let γ_i be any hyperplane not containing a tangent space of the semi-pseudo-ovoid O. The number of elements of O in γ_i will be denoted by u_i. Now we count the number of pairs (γ_i, π), with $\pi \in O$ in γ_i. We obtain

$$\sum_i u_i = \eta q^n \frac{q^{h-2n+1} - 1}{q - 1}. \tag{7}$$

Next we count the number of ordered triples (γ_i, π, π'), with $\pi \neq \pi'$, $\pi \in O$, $\pi' \in O$ and π, π' in γ_i. We obtain

$$\sum_i u_i(u_i - 1) = \eta(\eta - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{8}$$

From (7) and (8) it follows that

$$\sum_i u_i^2 = \eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1}. \tag{9}$$

The number of hyperplanes of PG(h, q) not containing a tangent space τ_j equals $\frac{q^{h+1}-1}{q-1} - \eta\frac{q^n-1}{q-1} = g$. As $g\sum_i u_i^2 - (\sum_i u_i)^2 \geq 0$, we obtain

$$\left(\frac{q^{h+1} - 1}{q - 1} - \eta\frac{q^n - 1}{q - 1}\right)\eta(\eta + q^n - 1)\frac{q^{h-2n+1} - 1}{q - 1} - \left(\eta q^n \frac{q^{h-2n+1} - 1}{q - 1}\right)^2 \geq 0,$$

that is, $\eta^2 - 2\eta - (q^{h+1} - 1) \leq 0$, or equivalently,

$$\eta \leq 1 + q^{\frac{h+1}{2}}. \tag{10}$$

We have equality in (10) if and only if u_i is a constant. In such a case this constant equals

$$u_i = \frac{\sum_i u_i}{g} = 1 + q^{\frac{h-2n+1}{2}}.$$

The theorem is proved. □

Corollary 3.3 *If for a semi-pseudo-ovoid O we have $\eta = 1 + q^{\frac{h+1}{2}}$, then $\tilde{O}$ ($\tilde{O}$ is the union of the elements of O) has two intersection numbers with respect to hyperplanes. Hence $\tilde{O}$ defines a projective linear two-weight code and a strongly regular graph.*

Proof: If the hyperplane γ contains a tangent space of O, then $\gamma \cap \tilde{O}$ is the disjoint union of one element of O and $q^{\frac{h+1}{2}}$ $(n-2)$-dimensional subspaces; if γ does not contain a tangent space of O, then $\gamma \cap \tilde{O}$ is the disjoint union of $1 + q^{\frac{h-2n+1}{2}}$ elements of O and $q^{\frac{h+1}{2}} - q^{\frac{h-2n+1}{2}}$ $(n-2)$-dimensional subspaces. The fact that $\tilde{O}$ defines a projective linear two-weight code and a strongly regular graph now follows from Calderbank and Kantor [3]. □

Theorem 3.4 *A semi-pseudo-ovoid O is either a pseudo-oval or a pseudo-ovoid if and only if $\eta = 1 + q^{h-2n+1}$.*

Proof: Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a semi-pseudo-ovoid. Then, by the proof of Theorem 2.1, $\eta = 1 + q^{h-2n+1}$ if and only if any n-dimensional subspace containing π_i, but not contained in the tangent space of O at π_i, has exactly one point in common with $\tilde{O} - \pi_i$, for all $i = 1, 2, \ldots, \eta$, that is, if and only if any three distinct elements of O generate a $(3n-1)$-dimensional subspace, that is, if and only if O is either a pseudo-oval or a pseudo-ovoid. □

4. Translation duals

If O is a pseudo-ovoid consisting of $q^{2n}+1$ $(n-1)$-dimensional subspaces of $\mathrm{PG}(4n-1, q)$, then the tangent spaces of O form a pseudo-ovoid O^* in the dual space of $\mathrm{PG}(4n-1, q)$; see Payne and Thas [9] (Theorem 8.7.2). The pseudo-ovoid O^* is called the *translation dual* of O. If q is even, then for every known pseudo-ovoid O we have $O \cong O^*$; for q odd, there are examples with $O \ncong O^*$, see e.g. Payne [8]. Now we extend the notion of translation dual to semi-pseudo-ovoids.

Lemma 4.1 *Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$, with $\eta = 1 + q^{\frac{h+1}{2}}$, be a semi-pseudo-ovoid in* $\mathrm{PG}(h, q)$ *and let τ_i be the tangent space of O at π_i, $i = 1, 2, \ldots, \eta$. If γ is a hyperplane of τ_i not containing π_i, with $i \in \{1, 2, \ldots, \eta\}$, then there is at least one τ_j, with $j \neq i$, for which $\tau_j \cap \gamma$ is $(h-2n)$-dimensional, that is, for which $\tau_j \cap \gamma = \tau_i \cap \tau_j$.*

Proof: Assume, by way of contradiction, that for any $j \neq i$ we have that $\tau_j \cap \gamma$ is $(h-2n-1)$-dimensional. Now we count in two ways the number of pairs (z, τ_j), with

$z \in \gamma - \pi_i$, $z \in \tau_j$, and $j \neq i$. We obtain

$$\frac{q^{h-n} - q^{n-1}}{q-1} \cdot q^{\frac{h-2n+1}{2}} = q^{\frac{h+1}{2}} \cdot \frac{q^{h-2n} - 1}{q-1},$$

clearly a contradiction. □

Theorem 4.2 *Let O be a semi-pseudo-ovoid in* PG(h, q), *with* $|O| = 1 + q^{\frac{h+1}{2}}$. *Then the tangent spaces of O form a semi-pseudo-ovoid O^* in the dual space of* PG(h, q).

Proof: Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$, with $\eta = 1 + q^{\frac{h+1}{2}}$, and let τ_i be the tangent space of O at π_i, $i = 1, 2, \ldots, \eta$. By Lemma 4.1 the space π_i is the intersection of all hyperplanes γ of τ_i, for which the space $\langle \gamma, \tau_j \rangle$ generated by γ and τ_j is PG(h, q), for all $j \neq i$. It follows that the tangent spaces of O form a semi-pseudo-ovoid O^* in the dual space of PG(h, q). □

The semi-pseudo-ovoid O^* will be called the *translation dual* of the semi-pseudo-ovoid O.

5. Particular semi-pseudo-ovoids

5.1. α-Regular semi-pseudo-ovoids

A semi-pseudo-ovoid O in PG(h, q) is called *α-regular* if any n-dimensional subspace containing any element $\pi \in O$ but not contained in the tangent space of O at π, has a point in common with exactly α elements of $O - \{\pi\}$. Any pseudo-oval and pseudo-ovoid is 1-regular. α-Regular semi-ovals were studied in Blokhuis and Szönyi [1].

It is easily deduced, by considering all n-dimensional subspaces containing a given $\pi \in O$, that if the semi-pseudo-ovoid O is α-regular, then $|O| - 1 = \alpha q^{h-2n+1}$.

Further Theorem 2.1 has an immediate corollary bounding α.

Corollary 5.1 *If O is an α-regular semi-pseudo-ovoid consisting of* PG$(n-1, q)$ *in* PG(h, q), *then* $\alpha \leq q^{2n-\frac{h+1}{2}}$.

Proof: Since $|O| - 1 = \alpha q^{h-2n+1}$, and $|O| - 1 \leq q^{\frac{h+1}{2}}$ by Theorem 2.1, we obtain $\alpha \leq q^{2n-\frac{h+1}{2}}$. □

5.2. SPG-reguli satisfying the polar property

An SPG-regulus is a set $\mathcal{R}$ of $(n-1)$-dimensional subspaces $\pi_1, \ldots, \pi_r$, $r > 1$, of PG(h, q), satisfying:

(a) $\pi_i \cap \pi_j = \emptyset$ for all $i \neq j$.

(b) If $PG(n, q)$ contains π_i, then it has a point in common with 0 or α ($\alpha > 0$) spaces in $\mathcal{R}\backslash\{\pi_i\}$. If $PG(n, q)$ contains π_i and has no point in common with π_j for all $j \neq i$, then it is called a *tangent* of $\mathcal{R}$ at π_i.
(c) If the point x of $PG(h, q)$ is not contained in an element of $\mathcal{R}$ it is contained in a constant number θ ($\theta \geq 0$) of tangents of $\mathcal{R}$.

SPG-reguli were introduced by Thas in [12] and give rise to semipartial geometries.

An SPG-regulus satisfies the *polar property* if $h > 2n - 1$ and the union of tangents at each element π_i of $\mathcal{R}$ is a $PG^{(i)}(h - n, q) =: \tau_i$ ($i \in \{1, \ldots, r\}$) which will be called the *tangent space* of $\mathcal{R}$ at π_i; see De Winter and Thas [4]. Clearly SPG-reguli satisfying the *polar property* are exactly α-regular semi-pseudo-ovoids O, such that the number of tangent spaces on any point not in an element of O, is a constant.

Theorem 5.2 *A semi-pseudo-ovoid O is an SPG-regulus satisfying the polar property if and only if $|O| = 1 + q^{\frac{h+1}{2}}$.*

Proof: First, suppose that O is an SPG-regulus satisfying the polar property. Then $|O| = 1 + q^{\frac{h+1}{2}}$ by Thas [12].

Next, suppose that O is a semi-pseudo-ovoid satisfying $|O| = 1 + q^{\frac{h+1}{2}}$. Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ and let τ_i be the tangent space of O at π_i, $i = 1, 2, \ldots, \eta$. Further, let $PG(n, q)$ contain π_i, with $PG(n, q) \not\subset \tau_i$. Let α be the number of elements of $O - \{\pi_i\}$ intersecting $PG(n, q)$. Now we count in two ways the number of pairs (π_j, ϕ), with $\pi_j \subset \phi$, $j \neq i$, ϕ a hyperplane containing $PG(n, q)$. We obtain

$$\alpha \frac{q^{h-2n+1} - 1}{q - 1} + (q^{\frac{h+1}{2}} - \alpha)\frac{q^{h-2n} - 1}{q - 1} = \frac{q^{h-n} - q^{n-1}}{q - 1} \cdot q^{\frac{h-2n+1}{2}}.$$

Hence $\alpha = q^{2n - \frac{h+1}{2}}$. As α is independent from i and the choice of $PG(n, q)$, it follows that O is an SPG-regulus. □

The problem on weak eggs in $PG(4n - 1, q)$ mentioned in Section 2.2 now generalizes in a natural way to the following problem. Suppose $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ is a set of $\eta = 1 + q^{\frac{h+1}{2}}$ mutually disjoint $(n - 1)$-dimensional spaces in $PG(h, q)$. Further suppose that every n-dimensional space containing π_i, with $i = 1, 2, \ldots, \eta$, intersects either 0 or $\alpha = q^{2n - \frac{h+1}{2}}$ elements of $O - \{\pi_i\}$. It is an open problem whether or not O is a semi-pseudo-ovoid (and hence an SPG-regulus). Notice that for $h = 4n - 1$ this problem is exactly the problem for weak eggs mentioned before.

Theorem 5.3 *If O is a semi-pseudo-ovoid with $|O| = 1 + q^{\frac{h+1}{2}}$, then the translation dual O^* of O is also an SPG-regulus satisfying the polar property.*

Proof: Immediate from Theorems 4.2 and 5.2. □

A *generalized semi-pseudo-ovoid* $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ in $PG(h, q)$ is a set of η mutually disjoint $(n - 1)$-dimensional spaces in $PG(h, q)$, with $h > 2n - 1$, such that the union of the

n-dimensional subspaces containing π_i, $i = 1, 2, \ldots, \eta$, and disjoint from all π_j, $j \neq i$, contains an $(h-n)$-dimensional space.

There is a variant of Theorem 5.2 for generalized semi-pseudo-ovoids that can be useful, as it is sometimes easy to check that on any π_i there is an $(h-n)$-dimensional space τ_i disjoint from π_j, for every $j \neq i$, but difficult to check that every n-dimensional space containing π_i, but not contained in τ_i, has non-empty intersection with $\tilde{O} - \pi_i$.

Theorem 5.4 *Let O be a generalized semi-pseudo-ovoid in* $\mathrm{PG}(h, q)$, *with* $|O| = 1+q^{\frac{h+1}{2}}$. *Then O is an SPG-regulus satisfying the polar property.*

Proof: Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ with $\eta = 1 + q^{\frac{h+1}{2}}$, and let τ_i be a fixed $(h-n)$-dimensional space containing π_i and disjoint from π_j, $j \neq i$, for $i = 1, 2, \ldots, \eta$. Following the proof of Theorem 3.2 we see that every hyperplane containing no τ_i, $i = 1, 2, \ldots, \eta$, contains exactly $1 + q^{\frac{h-2n+1}{2}}$ elements of O. Now let $\mathrm{PG}(n, q)$ be any n-dimensional space containing π_i and having non-empty intersection with $\tilde{O} - \pi$. As in the proof of Theorem 5.2 we find that $\mathrm{PG}(n, q)$ intersects exactly $\alpha = q^{2n-\frac{h+1}{2}}$ elements of $O - \{\pi_i\}$. We now easily obtain that there are exactly $\frac{q^{h-2n+1}-1}{q-1}$ n-dimensional spaces containing π_i and having empty intersection with $\tilde{O} - \pi_i$. We conclude that τ_i is the union of all n-dimensional spaces containing π_i and having empty intersection with $\tilde{O} - \pi_i$, $i = 1, 2, \ldots, \eta$, that is, O is a semi-pseudo-ovoid. Applying Theorem 5.2 finishes the proof. $\square$

As an application we give a very short proof of a theorem of Luyckx [6] and provide a variant on Theorem 2.2, but first we give the definition of an m-system.

An *m-system* $\mathcal{M}$ of a finite (non-singular) classical polar space P is a set, of maximal possible size, of mutually disjoint totally singular m-dimensional subspaces of P with the property that no generator (that is, a maximal totally singular subspace) of P that contains an element of $\mathcal{M}$ intersects any other element of $\mathcal{M}$. We have $|\mathcal{M}| = |P|/|\text{generator}|$, as is shown in Shult and Thas [10] where m-systems were introduced.

Each m-system $\mathcal{M}$ of the polar space P, for which any $(m+1)$-dimensional subspace containing any $\pi \in \mathcal{M}$ and not contained in $\pi^\perp$ if P is defined by a polarity, or not contained in the tangent space of P at π if P is a quadric in even dimension over a field with characteristic two, has a point in common with at least one element of $\mathcal{M} - \{\pi\}$, provides an example of a semi-pseudo-ovoid.

Corollary 5.5 (Luyckx [6]) *Let O be an m-system of the polar space $P \in \{Q^-(2n+1, q), W_{2n+1}(q), H(2n, q)\}$, but not a spread of $W_{2n+1}(q)$. Then O is an SPG-regulus of the ambient space of P satisfying the polar property.*

Proof: If we denote the ambient space of P as $\mathrm{PG}(h, q)$ then in each case there holds $|O| = 1 + q^{\frac{h+1}{2}}$. Furthermore, the definition of an m-system implies immediately that O is a generalized semi-pseudo-ovoid. The result now follows from Theorem 5.4. $\square$

The following variant of Theorem 2.2 shows that in the original definition of pseudo-ovoid $O(n, 2n, q)$ the restriction that every three distinct elements of $O(n, 2n, q)$ should generate a PG$(3n - 1, q)$ is superfluous.

Corollary 5.6 *Let O be a generalized semi-pseudo-ovoid consisting of $1 + q^{2n}$ mutually disjoint $(n - 1)$-dimensional spaces in* PG$(4n - 1, q)$. *Then O is a pseudo-ovoid.*

Proof: Theorem 5.4 implies that O is an SPG-regulus with $\alpha = 1$. Hence every three distinct elements of O generate a $(3n - 1)$-dimensional space, that is, O is a pseudo-ovoid. $\square$

6. Derivation of semi-pseudo-ovoids

In this final section we show how new semi-pseudo-ovoids can be constructed from old ones without changing the size of the semi-pseudo-ovoid.

Theorem 6.1 *Let $O = \{\pi_1, \pi_2, \ldots, \pi_\eta\}$ be a semi-pseudo-ovoid consisting of $(n - 1)$-dimensional spaces in* PG(h, q), *with $\eta = 1 + q^{\frac{h+1}{2}}$. Let τ_i be the tangent space of O at π_i. Suppose that the tangent spaces $\tau_1, \tau_2, \ldots, \tau_s$ have a* PG$(h - 2n, q) =: \Pi$ *in common. If $\{\overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_s}\}$ is a set of mutually disjoint $(n - 1)$-dimensional spaces covering exactly the same point set as $\pi_1 \cup \pi_2 \cup \cdots \cup \pi_s$, then $\overline{O} = (O \cup \{\overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_s}\}) - \{\pi_1, \pi_2, \ldots, \pi_s\}$ is also a semi-pseudo-ovoid and hence an SPG-regulus satisfying the polar property.*

Proof: Clearly τ_i has empty intersection with the elements of $\overline{O} - \{\pi_i\}$, if $i \notin \{1, 2, \ldots, s\}$. Furthermore it is obvious that the $(h - n)$-dimensional space $\langle \bar{\pi}_j, \Pi \rangle$ has empty intersection with the elements of $\overline{O} - \{\overline{\pi_j}\}$, $j = 1, 2, \ldots, s$. We conclude that $\overline{O}$ is a generalized semi-pseudo-ovoid with $|\overline{O}| = 1 + q^{\frac{h+1}{2}}$. Theorem 5.4 finishes the proof. $\square$

This theorem generalizes a result from De Winter and Thas [4], where this is shown to be true if O is a set of $1 + q^3$ lines in PG$(5, q)$ arising from a Buekenhout-Metz unital in PG$(2, q^2)$. It is also not so difficult to see that it is a generalization of a result of Luyckx and Thas [7] on derivation of m-systems as well.

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