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ON SEMI-RYAN COMPLEX SUBMANIFOLDS IN AN INDEFINITE COMPLEX SPACE FORM

YOUNG SUK CHOI, JUNG-HWAN KWON AND YOUNG JIN SUH

ABSTRACT. The purpose of this paper is to study several classes of an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex space form $M_{0+t}^{n+p}(c)$ of index $2t$, and of an n-dimensional space-like complex hypersurface of a complex Minkowski space C_1^{n+1} in terms of $RS=0$.

1. Introduction. The theory of indefinite complex submanifolds of an indefinite complex space form is one of the most interesting topics in differential geometry, and it has been investigated by many geometers from various points of view (see [**1**], [**3**], [**6**], [**7**], [**10**] and [**14**]–[**17**], etc.).

Let $M_t^m(c)$ be an m-dimensional semi-definite complex space form of constant holomorphic sectional curvature c and of index $2t$, $0 \leqq t \leqq m$. As is well known, it globally consists of three kinds of complex space forms: the semi-definite complex projective space $CP_t^m(c)$, the semi-definite complex Euclidean space C_t^m, or the semi-definite complex hyperbolic space $CH_t^m(c)$, according to whether $c>0$, $c=0$ or $c<0$.

Let M be a semi-definite Kaehler manifold, and let us denote by R and S the Riemannian curvature tensor and the Ricci tensor on M, respectively. Recently, the present authors [**5**] have given a complete classification of semi-symmetric complex hypersurfaces in a semi-definite complex space form $M_1^m(c)$. Here the notion of semi-symmetric means $R(X,Y)R=0$ for any vector fields X and Y on M. In this paper we want to introduce another notion of $R(X,Y)S=0$, which is said to be

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semi-Ryan. Such a *semi-Ryan* condition is a much more generalized notion than the semi-symmetric condition $R(X,Y)R = 0$ for any vector fields X, Y on M (see [**17**], [**18**] and [**19**]).

Now the above notion of semi-symmetric and semi-Ryan is just denoted by $RR = 0$ or $RS = 0$, respectively. With such a geometric condition $RS = 0$, we give a complete classification of space-like *semi-Ryan* Kaehler submanifolds in $M^{n+p}_{0+t}(c)$ as follows:

Theorem 1. *Let M be an n-dimensional complete space-like semi-Ryan complex submanifold of an $(n+p)$-dimensional indefinite complex space form $M^{n+p}_{0+t}(c)$ of index $2t$. If its totally real holomorphic bisectional curvature is nonvanishing at all of its points, then M is Einstein.*

Moreover, for a space-like complex hypersurface satisfying $RS = 0$ in a complex Minkowski space C^{n+1}_1, we proved the following:

Theorem 2. *Let M be an n-dimensional space-like semi-Ryan complex hypersurface of C^{n+1}_1. Then M is cylindrical.*

2. Semi-definite Kaehler manifolds. This section is concerned with recalling basic formulas on semi-definite Kaehler manifolds. Let M be a complex $m(\geqq 2)$-dimensional semi-definite Kaehler manifold equipped with semi-definite Kaehler metric tensor g and almost complex structure J. For the semi-definite Kaehler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2t$, $0 \leqq t \leqq m$. In the case where t is contained in the range $0 < t < m$, M is called an *indefinite Kaehler manifold* and the structure $\{g, J\}$ is called an *indefinite Kaehler structure* and, in particular, in the case where $t = 0$ or m, M is only called a *Kaehler manifold*, and then the structure $\{g, J\}$ is called a *Kaehler structure*.

Now we can choose a local field $\{E_\alpha\} = \{E_A, E_{A^*}\} = \{E_1, \ldots, E_m, E_{1*}, \ldots, E_{m*}\}$ of orthonormal frames on a neighborhood of M, where $E_{A^*} = JE_A$ and $A^* = m + A$. Here the indices $A, B, \ldots$ run from 1 to m and the indices $\alpha, \beta, \ldots$ run from 1 to $2m = m^*$. We set $U_A = (E_A - iE_{A^*})/\sqrt{2}$ and $\overline{U}_A = (E_A + iE_{A^*})/\sqrt{2}$, where i denotes the imaginary unit. Then $\{U_A\}$ constitutes a local field of unitary frames on the neighborhood of M. This is a complex linear frame which is orthonormal with respect to the semi-definite Kaehler metric, that is, $g(U_A, \overline{U}_B) = \varepsilon_A \delta_{AB}$, where

$$\varepsilon_A = 1 \quad \text{or} \quad -1,$$

according to whether

$$1 \leqq A \leqq m-t \quad \text{or} \quad m-t+1 \leqq A \leqq m.$$

Let $\{\theta_\alpha\} = \{\theta_A, \theta_{A^*}\}$, $\{\theta_{\alpha\beta}\}$ and $\{\Theta_{\alpha\beta}\}$ be the canonical form, the connection form and the curvature form on M, respectively, with respect to the local field $\{E_\alpha\} = \{E_A, E_{A^*}\}$ of orthonormal frames. Then we have the structure equations

(2.1)

$$d\theta_\alpha + \sum_\beta \varepsilon_\beta \theta_{\alpha\beta} \wedge \theta_\beta = 0, \qquad \theta_{\alpha\beta} - \theta_{\alpha^*\beta^*} = 0,$$

$$\theta_{\alpha^*\beta} + \theta_{\alpha\beta^*} = 0, \qquad \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \qquad \theta_{\alpha\beta^*} - \theta_{\beta\alpha^*} = 0,$$

$$d\theta_{\alpha\beta} + \sum_\gamma \varepsilon_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} = \Theta_{\alpha\beta}, \qquad \Theta_{\alpha\beta} = -\frac{1}{2} \sum_{\gamma,\delta} \varepsilon_\gamma \varepsilon_\delta K_{\alpha\beta\gamma\delta} \theta_\gamma \wedge \theta_\delta,$$

where $K_{\alpha\beta\gamma\delta}$ denotes the components of the Riemannian curvature tensor R of M.

Now let $\{\omega_A\}$ be the dual coframe field with respect to the local field $\{U_A\}$ of unitary frames on the neighborhood of M given by

$$\omega_A = (\theta_A + i\theta_{A^*})/\sqrt{2}.$$

Then $\{\omega_A\} = \{\omega_1, \dots, \omega_m\}$ consists of complex-valued 1-forms of type $(1,0)$ on M such that $\omega_A(U_B) = \varepsilon_A \delta_{AB}$ and $\{\omega_A, \bar{\omega}_A\} = \{\omega_1, \dots, \omega_m, \bar{\omega}_1, \dots, \bar{\omega}_m\}$ are linearly independent. The semi-definite Kaehler metric g of M can be expressed as $g = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_A$. Associated with the frame field $\{U_A\}$, there exist complex-valued forms ω_{AB} given by

$$\omega_{AB} = \theta_{AB} + i\theta_{A^*B},$$

which are usually called *connection forms* on M such that they satisfy the structure equations of M;

$$\begin{aligned} d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\ d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\ \Omega_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R_{\overline{A}BC\overline{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned} \tag{2.2}$$

where $\Omega = (\Omega_{AB})$, $\Omega_{AB} = \Theta_{AB} + i\Theta_{A^*B}$ (respectively $R_{\overline{A}BC\overline{D}}$) denotes the curvature form (respectively the components of the semi-definite Riemannian curvature tensor R) of M. So, by (2.1) and (2.2), we obtain

$$R_{\overline{A}BC\overline{D}} = -\{(K_{ABCD} + K_{A^*BC^*D}) + i\,(K_{A^*BCD} - K_{ABC^*D})\}. \tag{2.3}$$

The second relation of equation (2.2) means that the skew-Hermitian symmetry of Ω_{AB}, which is equivalent to the symmetric condition $R_{\overline{A}BC\overline{D}} = \overline{R}_{\overline{B}AD\overline{C}}$. Moreover, the first Bianchi identity $\sum_B \varepsilon_B \Omega_{AB} \wedge \omega_B = 0$ is given by the exterior differential of the first and third equations of (2.2), which implies the further symmetric relations

$$R_{\overline{A}BC\overline{D}} = R_{\overline{A}CB\overline{D}} = R_{\overline{D}CB\overline{A}} = R_{\overline{D}BC\overline{A}}.$$

Now, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$S = \sum_{A,B} \varepsilon_A \varepsilon_B (S_{A\overline{B}} \omega_A \otimes \bar{\omega}_B + S_{\overline{A}B} \bar{\omega}_A \otimes \omega_B),$$

where $S_{A\overline{B}} = \sum_C \varepsilon_C R_{\overline{C}CA\overline{B}} = S_{\overline{B}A} = \overline{S}_{\overline{A}B}$. The scalar curvature r is also given by $r = 2\sum_A \varepsilon_A S_{A\overline{A}}$. An n-dimensional semi-definite Kaehler manifold M is said to be *Einstein* if the Ricci tensor S is given by

$$S_{A\overline{B}} = \frac{r}{2n}\,\varepsilon_A \delta_{AB}.$$

The components $R_{\overline{A}BC\overline{D};E}$ and $R_{\overline{A}BC\overline{D};\overline{E}}$ of the covariant derivative of the Riemannian curvature tensor R are defined by

$$\begin{aligned}\sum_E \varepsilon_E(R_{\overline{A}BC\overline{D};E}\omega_E + R_{\overline{A}BC\overline{D};\overline{E}}\bar{\omega}_E) \\ = dR_{\overline{A}BC\overline{D}} - \sum_E \varepsilon_E(R_{\overline{E}BC\overline{D}}\bar{\omega}_{EA} + R_{\overline{A}EC\overline{D}}\omega_{EB} \\ + R_{\overline{A}BE\overline{D}}\omega_{EC} + R_{\overline{A}BC\overline{E}}\bar{\omega}_{ED}).\end{aligned}$$

The second Bianchi identity is given by

$$R_{\overline{A}BC\overline{D};E} = R_{\overline{A}BE\overline{D};C}.$$

Let M be an m-dimensional semi-definite Kaehler manifold of index $2t$, $0 \leqq t \leqq m$. A plane section P of the tangent space T_xM of M at any point x is said to be *nondegenerate* provided that $g_x|_P$ is nondedgenerate. It is easily seen that P is nondegenerate if and only if it has a basis $\{X, Y\}$ such that

$$g(X,X)\,g(Y,Y) - g(X,Y)^2 \neq 0.$$

If the nondegenerate plane P is invariant by the complex structure J, it is said to be *holomorphic*. It is also trivial that the plane P is holomorphic if and only if it contains a vector X in P such that $g(X,X) \neq 0$. For the nondegenerate plane P spanned by X and Y in P, the sectional curvature $K(P)$ is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}.$$

The holomorphic plane spanned by a space-like or time-like vector X and JX is said to be *space-like* or *time-like*, respectively. The sectional curvature $K(P)$ of the holomorphic plane P is called the *holomorphic sectional curvature*, which is denoted by $H(P)$. The semi-definite Kaehler manifold M is said to be *of constant holomorphic sectional curvatuare* if its holomorphic sectional curvatures $H(P)$ are constant for all holomorphic planes at all points of M. Then M is called a *semi-definite complex space form*, which is denoted by $M_t^m(c)$ provided that it is of constant holomorphic sectional curvature c, of complex dimension m and of index $2t(\geqq 0)$. It is seen in Wolf [**20**] that the standard models of semi-definite complex space forms are the following three kinds: the semi-definite complex projective space $CP_t^m(c)$, the semi-definite complex Euclidean space C_t^m or the semi-definite complex hyperbolic space $CH_t^m(c)$, according to whether $c > 0$, $c = 0$ or $c < 0$. For any integer $q(0 \leqq t \leqq m)$, it is also seen by [**20**] that they are complete simply connected semi-definite complex space forms of dimension m and of index $2t$. The Riemannian curvature tensor $R_{\overline{A}BC\overline{D}}$ of $M_t^m(c)$ is given by

$$R_{\overline{A}BC\overline{D}} = \frac{c}{2}\varepsilon_B\varepsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}). \tag{2.4}$$

Now let M be an m-dimensional semi-definite Kaehler manifold of an index $2t$ equipped with semi-definite Kaehler structure $\{g, J\}$. We can choose a local field of $\{E_\alpha\} = \{E_A, E_{A^*}\}$ of orthonormal frames on the neighborhood of M such that $g(E_A, E_B) = \varepsilon_A\delta_{AB}$. Let $\{U_A\}$ be a local field of unitary frames associated with the orthonormal frames $\{E_A, E_{A^*}\}$ on the neighborhood of M stated above in the first of this section. This is a complex linear frame, which is orthonormal with respect to the semi-definite Kaehler metric, that is, $g(U_A, \overline{U}_B) = \varepsilon_A\delta_{AB}$.

Given two holomorphic planes P and Q in T_xM at any point x in M, the holomorphic bisectional curvature $H(P,Q)$ determined by the two planes P and Q of M is defined by

$$H(P,Q) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{2.5}$$

where X, respectively Y, is a nonzero vector in P, respectively Q. In particular, the holomorphic bisectional curvature $H(P,Q)$ is said to be *space-like* or *time-like* if P and Q are both space-like or either P or Q is time-like. It is a simple matter to verify that the righthand side in (2.5) depends only on P and Q and so it is well defined. It may be denoted by $H(P,Q) = H(X,Y)$. It is easily seen that $H(P,P) = H(P) = H(X,X) =: H(X)$ is the holomorphic sectional curvature determined by the holomorphic plane P, where X is a nonzero vector in P. We denote by P_A the holomorphic plane $[E_A, JE_A]$ spanned by E_A and $JE_A = E_{A^*}$. We set

$$H(P_A, P_B) = H_{AB}(A \neq B), \qquad H(P_A, P_A) = H(P_A) = H_{AA} = H_A.$$

The *holomorphic bisectional curvature* $H_{AB}(A \neq B)$ and the *holomorphic sectional curvature* H_A are given by

$$H_{AB} = \frac{g(R(E_A, JE_A)JE_B, E_B)}{g(E_A, E_A)g(E_B, E_B)} = -\varepsilon_A\varepsilon_B K_{AA^*BB^*}(A \neq B),$$

$$H_A = \frac{g(R(E_A, JE_A)JE_A, E_A)}{g(E_A, E_A)g(E_A, E_A)} = -K_{AA^*AA^*}.$$

By (2.3) we have

$$H_{AB} = \varepsilon_A\varepsilon_B R_{\overline{A}AB\overline{B}}(A \neq B), \qquad H_A = R_{\overline{A}AA\overline{A}}. \tag{2.6}$$

3. Space-like complex submanifolds. This section is concerned with space-like complex submanifolds of an indefinite Kaehler manifold. First of all, the basic formulas for the theory of space-like complex submanifolds are prepared.

Let M' be an $(n+p)$-dimensional connected indefinite Kaehler manifold of index $2p$ with indefinite Kaehler structure (g', J'). Let M be an n-dimensional connected space-like complex submanifold of M', and let g be the induced Kaehler metric tensor of index $2p$ on M from g'. We can choose a local field $\{U_A\} = \{U_j, U_x\} = \{U_1, \ldots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \ldots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated:

$$A, B, C, \ldots = 1, \ldots, n, n+1, \ldots, n+p;$$

$$i, j, k, \ldots = 1, \ldots, n; \quad x, y, z, \ldots = n+1, \ldots, n+p.$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_j, \omega_y\}$ be its dual frame fields. Then the indefinite Kaehler metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_A$, where $\{\varepsilon_A\} = \{\varepsilon_j, \varepsilon_y\}$. The connection forms on M' are denoted by $\{\omega_{AB}\}$. The canonical forms ω_A and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$\begin{gathered} d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\ d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\ \Omega'_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{gathered} \tag{3.1}$$

where Ω'_{AB} (respectively $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form (respectively the components of the indefinite Riemannian curvature tensor R') of M'.

Restricting these forms to the submanifold M, we have

$$\omega_x = 0, \tag{3.2}$$

and the induced Kaehler metric tensor g of M is given by $g = 2\sum_j \varepsilon_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame field due to $\{U_j\}$, which consists of complex-valued 1-forms of type $(1,0)$ on M. Moreover, $\omega_1, \dots, \omega_n$, $\bar{\omega}_1, \dots, \bar{\omega}_n$ are linearly independent, and $\{\omega_j\}$ is the canonical form on M. It follows from (3.2) and Cartan's lemma that the exterior derivative of (3.2) gives rise to

$$\omega_{xi} = \sum_j \varepsilon_j h_{ij}{}^x \omega_j, \quad h_{ij}{}^x = h_{ji}{}^x. \tag{3.3}$$

The quadratic form $\alpha = \sum_{i,j,x} \varepsilon_i \varepsilon_j \varepsilon_x h_{ij}{}^x \omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle NM on M in M' is called the *second fundamental form* of the submanifold M. Therefore, the structure equations for M are similarly given by

$$\begin{aligned} d\omega_i + \sum_j \varepsilon_j \omega_{ij} \wedge \omega_j = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0,\\ d\omega_{ij} + \sum_k \varepsilon_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij},\\ \Omega_{ij} = \sum_{k,m} \varepsilon_k \varepsilon_m R_{\bar{i}jk\bar{m}} \omega_k \wedge \bar{\omega}_m. \end{aligned} \tag{3.4}$$

Moreover, the following relationships are obtained:

$$\begin{aligned} d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy},\\ \Omega_{xy} = \sum_{k,m} \varepsilon_k \varepsilon_m R_{\bar{x}yk\bar{m}} \omega_k \wedge \bar{\omega}_m, \end{aligned} \tag{3.5}$$

where Ω_{xy} is called the *normal curvature form* of M. For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that we have the Gauss equation

$$R_{\bar{i}jk\bar{m}} = R'_{\bar{i}jk\bar{m}} - \sum_x \varepsilon_x h_{jk}{}^x \bar{h}_{im}{}^x. \tag{3.6}$$

And, by means of (3.3) and (3.5), we have

$$R_{\bar{x}yk\bar{m}} = R'_{\bar{x}yk\bar{m}} + \sum_r \varepsilon_r h_{kr}{}^x \bar{h}_{rm}{}^y. \tag{3.7}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum_K \varepsilon_k R'_{\bar{j}ik\bar{k}} - h_{i\bar{j}}{}^2, \qquad r = 2\Big(\sum_{k,j} \varepsilon_k \varepsilon_j R'_{\bar{k}kj\bar{j}} - h_2\Big), \tag{3.8}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{x,r} \varepsilon_x \varepsilon_r h_{ir}{}^x \bar{h}_{rj}{}^x$ and $h_2 = \sum_j \varepsilon_j h_{j\bar{j}}{}^2$.

Now the components $h_{ijk}{}^x$ and $h_{ij\bar{k}}{}^x$ of the covariant derivative of the second fundamental form on M are given by

$$\begin{aligned} &\sum_k \varepsilon_k (h_{ijk}{}^x \omega_k + h_{ij\bar{k}}{}^x \bar{\omega}_k) \\ &\quad = dh_{ij}{}^x - \sum_k \varepsilon_k (h_{kj}{}^x \omega_{ki} + h_{ik}{}^x \omega_{kj}) + \sum_y \varepsilon_y h_{ij}{}^y \omega_{xy}. \end{aligned} \tag{3.9}$$

Then, substituting $dh_{ij}{}^x$ in this definition into the exterior derivative of (3.3) and using (3.1)–(3.4) and (3.8), we have

$$h_{ijk}{}^x = h_{ikj}{}^x, \qquad h_{ij\bar{k}}{}^x = -R'_{\bar{x}ij\bar{k}}. \tag{3.10}$$

Similarly, the components $h_{ijkm}{}^x$ and $h_{ijk\bar{m}}{}^x$, respectively $h_{ij\bar{k}m}{}^x$ and $h_{ij\bar{k}\bar{m}}{}^x$, of the covariant derivative of $h_{ijk}{}^x$, respectively $h_{ij\bar{k}}{}^x$, can be defined by

$$\begin{aligned} &\sum_m \varepsilon_m (h_{ijkm}{}^x \omega_m + h_{ijk\bar{m}}{}^x \bar{\omega}_m) \\ &\quad = dh_{ijk}{}^x - \sum_m \varepsilon_m (h_{mjk}{}^x \omega_{mi} + h_{imk}{}^x \omega_{mj} + h_{ijm}{}^x \omega_{mk}) \\ &\qquad + \sum_y \varepsilon_y h_{ijk}{}^y \omega_{xy}, \end{aligned}$$

$$\begin{aligned} &\sum_m \varepsilon_m (h_{ij\bar{k}m}{}^x \omega_m + h_{ij\bar{k}\bar{m}}{}^x \bar{\omega}_m) \\ &\quad = dh_{ij\bar{k}}{}^x - \sum_m \varepsilon_m (h_{mj\bar{k}}{}^x \omega_{mi} + h_{im\bar{k}}{}^x \omega_{mj} + h_{ij\bar{m}}{}^x \bar{\omega}_{mk}) \\ &\qquad + \sum_y \varepsilon_y h_{ij\bar{k}}{}^y \omega_{xy}. \end{aligned}$$

Differentiating (3.9) exteriorly and using the properties $d^2 = 0$, (3.4), (3.5), (3.8), (3.9) and (3.10), we have the following Ricci formula for the second fundamental form

$$h_{ijkm}{}^x = h_{ijmk}{}^x, \qquad h_{ij\bar{k}\bar{m}}{}^x = h_{ij\bar{m}\bar{k}}{}^x,$$

$$\begin{aligned} h_{ijk\bar{m}}{}^x - h_{ij\bar{m}k}{}^x = \sum_n \varepsilon_n (R_{\bar{m}ki\bar{n}} h_{nj}{}^x + R_{\bar{m}kj\bar{n}} h_{in}{}^x) \\ - \sum_y \varepsilon_y R_{\bar{x}yk\bar{m}} h_{ij}{}^y. \end{aligned}$$

In particular, let the ambient space M' be an $(n+p)$-dimensional semi-definite complex space form $M_{s+t}^{n+p}(c)$ of constant holomorphic sectional curvature c and of index $2(s+t)$, $0 \leqq s \leqq n$, $0 \leqq t \leqq p$. Then we get

$$R_{\bar{i}jk\bar{m}} = \frac{c}{2}\,\varepsilon_j \varepsilon_k (\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm}) - \sum_x \varepsilon_x h_{jk}{}^x \bar{h}_{im}{}^x, \tag{3.11}$$

$$S_{i\bar{j}} = \frac{(n+1)c}{2}\,\varepsilon_i \delta_{ij} - h_{i\bar{j}}{}^2, \tag{3.12}$$

$$r = n(n+1)c - 2h_2, \tag{3.13}$$

$$h_{ij\bar{k}}{}^x = 0, \tag{3.14}$$

$$\begin{aligned} h_{ijk\bar{m}}{}^x = \frac{c}{2}\,(\varepsilon_k h_{ij}{}^x \delta_{km} - \varepsilon_i h_{jk}{}^x \delta_{im} + \varepsilon_j h_{ki}{}^x \delta_{jm}) \\ - \sum_{n,y} \varepsilon_n \varepsilon_y (h_{ni}{}^x h_{jk}{}^y + h_{nj}{}^x h_{ki}{}^y + h_{nk}{}^x h_{ij}{}^y) \bar{h}_{nm}{}^y. \end{aligned} \tag{3.15}$$

For the sake of brevity, a tensor $h_{i\bar{j}}{}^{2m}$ and a function h_{2m} on M for any integer $m (\geqq 2)$ are introduced as follows:

$$h_{i\bar{j}}{}^{2m} = \sum_{i_1,\ldots,i_{m-1}} \varepsilon_{i_1} \cdots \varepsilon_{i_{m-1}} h_{i\bar{i}_1}{}^2 h_{i_1\bar{i}_2}{}^2 \cdots h_{i_{m-1}\bar{j}}{}^2,$$

$$h_{2m} = \sum_i \varepsilon_i h_{i\bar{i}}{}^{2m}.$$

In particular, if M is a hypersurface, then a tensor ${h_{ij}}^{2m+1}$ on M is introduced as follows:

$$ {h_{ij}}^{2m+1} = \sum_k \varepsilon_k {h_{i\bar{k}}}^{2m} h_{kj}. $$

4. Examples of indefinite Einstein complex submanifolds. We give here some examples of indefinite Einstein submanifolds of an indefinite complex space form.

Example 4.1. The indefinite Euclidean space C_s^n of index $2s$ is a totally geodesic complex hypersurface of C_s^{n+1} or C_{s+1}^{n+1} in a natural way.

Example 4.2. For an indefinite complex projective space $CP_s^{n+1}(c)$ of index $2s$ and of constant holomorphic sectional curvature c, if $\{z_1, \dots, z_s, z_{s+1}, \dots, z_{n+2}\}$ is the usual homogeneous coordinate system of $CP_s^{n+1}(c)$, then for each j fixed, the equation $z_j = 0$ defines a totally geodesic complex hypersurface identifiable with $CP_s^n(c)$ or $CP_{s-1}^n(c)$ according to whether $s+1 \leqq j \leqq n+2$ or $1 \leqq j \leqq s$. By taking into account that $CH_s^n(-c)$ is obtained from $CP_{n-s}^n(c)$ by reversing the sign of its indefinite Kaehler metric, the previous discussion shows that $CH_s^n(-c)$ is a totally geodesic complex hypersurface of both $CH_s^{n+1}(-c)$ and $CH_{s+1}^{n+1}(-c)$.

Example 4.3. Let Q_s^n be an indefinite complex hypersurface of $CP_s^{n+1}(c)$ defined by the equation

$$ -\sum_{j=1}^{s} {z_j}^2 + \sum_{k=s+1}^{n+2} {z_k}^2 = 0 $$

in the homogeneous coordinate system of $CP_s^{n+1}(c)$. Then Q_s^n is a complete indefinite complex hypersurface of index $2s$ and, moreover, in a similar way to Kobayashi and Nomizu [**8**, Chapter 11, Example 10.6] it is Einstein, and then the Ricci tensor S satisfies $S = ncg/2$. This is called an *indefinite complex quadric*.

Note that Q^n_s can also be constructed as an indefinite Einstein complex hypersurface of $CH^{n+1}_{s+1}(-c)$.

Remark 4.1. Szabó [**18**] showed that a complete Einstein complex hypersurface of a complex space form $M^{n+1}(c)$ is totally geodesic or $c > 0$. In the latter case, M is locally congruent to the complex quadric Q^n. Example 4.3 means that the situation of indefinite Einstein complex hypersurfaces is quietly different from those of the definite cases.

Remark 4.2. An indefinite Einstein complex hypersurface of an indefinite complex space form is investigated in detail by Montiel and Romero [**10**] (cf. Romero's survey [**14**]).

Example 4.4 ([10]). Let us consider an indefinite complex hypersurface of $CP^{2n+1}_{n+1}(c)$ defined by the equation

$$\sum_{j=1}^{n+1} z_j z_{n+1+j} = 0$$

in the homogeneous coordinate system of $CP^{2n+1}_{n+1}(c)$. It is a complete complex hypersurface of index $2n$, which is denoted by $Q^{2n^*}_n$. It is easily seen that the Ricci tensor S satisfies $S = (n+1)cg$, and hence it is Einstein.

A similar discussion as in Example 4.3 shows that $Q^{2n^*}_n$ is also an indefinite complete Einstein complex hypersurface of $CH^{2n+1}_{n+1}(-c)$.

Example 4.5. For the homogeneous coordinate systems $\{z_1, \dots, z_s, z_{s+1}, \dots, z_{n+1}\}$ of $CP^n_s(c)$ and $\{w_1, \dots, w_t, w_{t+1}, \dots, w_{m+1}\}$ of $CP^m_t(c)$, a mapping f of $CP^n_s(c) \times CP^m_t(c)$ into $CP^{N(n,m)}_{R(n,m,s,t)}(c)$ with

$$N(n,m) = n + m + nm, \qquad R(n,m,s,t) = s(m-t) + t(n-s) + s + t$$

is defined by

$$f(z,w) = (z_a w_u, z_r w_x, z_b w_y, z_s w_v),$$

where

$$a, b, \dots = 1, \dots, s; \quad r, s, \dots = s+1, \dots, n+1,$$
$$x, y, \dots = 1, \dots, t; \quad u, v, \dots = t+1, \dots, m+1.$$

Then f is a well-defined holomorphic mapping, and it is seen that f is also an isometric imbedding which is called an *indefinite Segre imbedding*. In the case of $n = m$, it is Einstein and the Ricci tensor S satisfies $S = (n+1)cg/2$. In particular, if $s = t = 0$, then f is a classical Segre imbedding (cf. Nakagawa and Takagi [**11**]). This example is due to Ikawa, Nakagawa and Romero [**6**].

As the simplest case in the definite product ones, $CP^1(c) \times CP^1(c)$ is the complex quadric Q^2 in $CP^3(c)$. In the indefinite case, however, we can consider two product manifolds, $CP^1_1(c) \times CP^1_1(c)$ and $CP^1_1(c) \times CP^1(c)$, which are mutually different complex quadric in $CP^3_2(c)$. In fact, it is seen in Montiel and Romero [**10**] that they are denoted by Q^2_2 and $Q^{2^*}_2$, respectively.

By using that an indefinite complex hyperbolic space $CH^n_s(-c)$ is obtained from $CP^n_{n-s}(c)$ by changing the metric by its negative, another indefinite Segre imbedding

$$f : CH^n_s(-c) \times CH^m_t(-c) \longrightarrow CH^{N(n,m)}_{S(n,m,s,t)}(-c)$$

is given, where $S(n,m,s,t) = (n-s)(m-t)+st+s+t$. In the case where $n = m$, it is Einstein, and the Ricci tensor S satisfies $S = -(n+1)cg/2$. In particular, for $s = t = 0$, we have a holomorphic isometric imbedding f of $CH^n(-c) \times CH^m(-c)$ into $CH^{N(n,m)}_{nm}(-c)$.

Example 4.6. Let h_j be holomorphic functions of C. The range of indices are given as follows:

$$i, j, \ldots = 1, \ldots, n; \quad a, b, \ldots = 1, \ldots, s;$$
$$x, y, \ldots = s+1, \ldots, n; \quad A, B, \ldots = 1, \ldots, 2n.$$

In a $(2n+1)$-dimensional complex manifold C^{2n+1} with the standard basis, a Hermitian form F is defined by

$$F(z, w) = -\sum_a z_a w_a + \sum_x z_x w_x + \sum_{j^*} z_{j^*} w_{j^*} + z_{2n+1} w_{2n+1},$$

where $j^* = n + j$ and $z = (z_a, z_x, z_{j^*}, z_{2n+1}) = (z_A, z_{2n+1})$, $w = (w_A, w_{2n+1})$ are in C^{2n+1}. The scalar product defined by the real part $\mathrm{Re}\, F$ is an indefinite Riemannian metric of index $2s$ on C^{2n+1} and

$(C^{2n+1}, \mathrm{Re}\, F)$ is a flat indefinite complex space form, which is denoted by C_s^{2n+1}. Let h_j be holomorphic functions of C into C. For the complex coordinate system (z_A, z_{2n+1}) of C_s^{2n+1}, let $M = M_s^{2n}(h_j, c_j)$ be the complex hypersurface in C_s^{2n+1}, given by the equation

$$z_{2n+1} = \sum_j h_j(z_j + c_j z_{j^*})$$

for any complex number c_j. Then $M_2^{2n}(h_j, c_j)$ is a family of complete connected hypersurfaces of index $2s$ in C_s^{2n+1} if $|c_a| \geqq 1$ for any a. Furthermore, it is diffeomorphic to C^{2n}. If all functions h_x are linear and if $|c_a| = 1$, then $M_s^{2n}(h_j, c_j)$ is Ricci flat. In particular, it is not flat provided that there is an index a such that h_a is not linear. $M_n^{2n}(h_j, 1)$ is also a complete connected complex hypersurface of index $2n$ in C_n^{2n+1}, which is Ricci flat. These examples are due to Aiyama, Ikawa, Kwon and Nakagawa [**1**].

In particular, $M_n^{2n}(z^p, 1)$ for any integer $p (\geqq 2)$ is a complete connected complex hypersurface of index $2n$ in C_n^{2n+1}, which are Ricci flat but not flat. This is due to Romero [**15**].

5. Normal curvature tensor. In this section we introduce the concept of the normal curvature tensor on the space-like complex submanifold in an indefinite Kaehler manifold.

Let M' be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$ equipped with indefinite Kaehler structure $\{g', J'\}$, and let M be an n-dimensional space-like complex submanifold of M' endowed with induced Kaehler structure $\{g, J\}$ from the indefinite Kaehler structure $\{g', J'\}$. Let us denote by $\nabla^\perp$ the normal connection on M, namely, it is the mapping of $TM \times NM$ into NM defined by

$$\nabla^\perp(X, V) = \nabla^\perp{}_X V = \text{the normal part of } \nabla'_X V$$

for any tangent vector field X in TM and any normal vector field V in NM, where ∇' is the Kaehler connection on M' and TM and NM are the tangent bundle and the normal bundle of M, respectively [**14**]. The normal curvature tensor $R^\perp$ on M is defined by

$$R^\perp(X, Y)V = (\nabla^\perp X \nabla^\perp Y - \nabla^\perp{}_Y \nabla^\perp{}_X - \nabla^\perp{}_{[X,Y]})V,$$

where $X, Y \in TM$ and $V \in NM$. If it satisfies

$$R^{\perp}(X, Y)V = f\ g(X, JY)J'V,$$

where f is any function on M, then the normal connection $\nabla^{\perp}$ is said to be *proper*. In particular, if f is a nonzero constant or 0 on M, then it is said to be *semi-flat* or *flat*, respectively.

Remark 5.1. The concept of the semi-flatness of the normal connection of the complex submanifold in a complex projective space is introduced, and the justification of the concept is given by Yano and Kon [**21**].

On the other hand, the proper case is treated by Ki and Nakagawa [**7**].

Remark 5.2. In the semi-Riemannian geometry, the shape operator A on the indefinite Einstein hypersurface M of index $2s$ in $M_{s+1}^{n+1}(c)$ cannot be necessarily diagonalized. By the classification of the self-adjoint endomorphisms of a scalar product, we have the following properties:

(1) A is diagonalizable,

(2) A is not diagonalizable, but either $\varepsilon_{n+1}h_2 < 0$ or $h_2 = 0$ and not totally geodesic.

An indefinite Einstein hypersurface is said to be *proper* if the shape operator A is diagonalizable (see [**3**], [**7**]). The terminology "proper" of the normal connection is named after the concept.

Now, in order to consider the normal curvature transformation, we see the local version of the normal curvature tensor. Let M be an n-dimensional space-like complex submanifold of an $(n+p)$-dimensional indefinite Kaehler manifold M' of index $2p$ with almost complex structure J'. We can choose a local field $\{E_A, E_{A^*}\} = \{E_1, \ldots, E_m, E_{1^*}, \ldots, E_{m^*}\}$, $m = n + p$, of orthonormal frames on a neighborhood of M' in such a way that, restricted to M, $E_1, \ldots, E_n$, $E_{1^*}, \ldots, E_{n^*}$ are tangent to M and the others are normal to M, where $E_{A^*} = J'E_A$. Let $\{U_A\} = \{U_j, U_y\}$ be the local field of unitary frames associated with the orthonormal frame. With respect to the frame field, let $\{\omega_A\} = \{\omega_j, \omega_y\}$ be its canonical form on the ambient space.

Restricting these forms to the submanifold M, we have $\omega_x = 0$. Then U_j is a local unitary frame field with respect to this induced metric and $\{\omega_j\}$ is the canonical form on M. From the structure equations of the ambient space, it follows that the structure equations for M and the equation on the normal bundle NM on M are expressed as

$$d\omega_{xy} + \sum_z \varepsilon_z \omega_{xz} \wedge \omega_{zy} = \Omega_{xy},$$
$$\Omega_{xy} = \sum_{k,m} \varepsilon_k \varepsilon_m R_{\bar{x}yk\bar{m}} \omega_k \wedge \bar{\omega}_m,$$

where Ω_{xy} is called the *normal curvature form* of M and $R_{\bar{x}yk\bar{m}}$ denotes the components of the normal curvature form Ω_{xy} on M. It is seen that its components are given as the ones of the extension of complex linearity of the normal curvature tensor $R^{\perp}$ on M. By means of (3.3) and (3.5), we have

$$R_{\bar{x}yk\bar{m}} = R'_{\bar{x}yk\bar{m}} + \sum_j \varepsilon_j h^x_{kj} \bar{h}_{jm}{}^y. \tag{5.1}$$

By the property of (5.1) of the normal curvature tensor, we can define a linear transformation T_N on the np-dimensional complex vector space Ξ^{np} consisting of tensors (ξ_{xk}) at each point on M by

$$T_N(\xi_{xk}) = (\eta_{xk}), \quad \eta_{xk} = \sum_{y,m} \varepsilon_y \varepsilon_m R_{\bar{x}yk\bar{m}} \xi_{ym}.$$

We denote by $(R_{ym}{}^{xk})$ the matrix of the linear transformation T_N. The linear operator defined by the $np \times np$ Hermitian matrix $(R_{ym}{}^{xk})$ is called the *normal curvature operator* on M. Then T_N is the self-adjoint operator with respect to the definite metric canonically defined on Ξ^{np}. We assume that the matrix $(R_{ym}{}^{xk})$ is diagonalizable. In this case, we can suitably choose an indefinite unitary frame field $\{U_A\} = \{U_j, U_y\}$ in such a way that it satisfies

$$R_{\bar{x}yk\bar{m}} = \varepsilon_x \varepsilon_k f_{xk} \delta_{ym}{}^{xk} = \varepsilon_x \varepsilon_k f_{xk} \delta_{xy} \delta_{km}, \tag{5.2}$$

where every eigenvalue f_{xk} of T_N is a real valued function on M. By (5.1) and (5.2), we have

$$R'_{\bar{x}yk\bar{m}} = \varepsilon_x \varepsilon_k f_{xk} \delta_{xy} \delta_{km} - \sum_j \varepsilon_j h_{kj}{}^x \bar{h}_{jm}{}^y.$$

Remark 5.3. In the space-like complex hypersurface M, the normal connection is always proper.

6. Semi-definite complex submanifolds with $RS = 0$. This section is concerned with indefinite complex submanifolds in an indefinite complex space form satisfying the *semi-Ryan* condition $R(X, Y)S = 0$ for any tangent vectors X and Y on M. Let M be an n-dimensional semi-definite complex submanifold of index $2s$ of a semi-definite complex space form $M' = M^{n+p}_{s+t}(c)$, $0 \leqq s \leqq n$, $0 \leqq t \leqq p$. We denote by R or S the Riemannian curvature tensor or the Ricci tensor on M, respectively. Assume that the submanifold M satisfies the condition

$$R(X, Y)S = 0, \quad X, Y \in TM.$$

It is equivalent to

$$\sum_r \varepsilon_r (R_{\bar{i}jr\bar{k}} S_{m\bar{r}} - R_{\bar{i}jm\bar{r}} S_{r\bar{k}}) = 0 \quad \text{or} \quad S_{\bar{i}jk\bar{m}} - S_{\bar{i}j\bar{m}k} = 0. \tag{6.1}$$

By (3.11) and (3.12), we have

$$\begin{aligned} &c(\varepsilon_j \delta_{jm} h_{k\bar{i}}{}^2 - \varepsilon_i \delta_{ik} h_{j\bar{m}}{}^2) \\ &\qquad + 2 \sum_{x,n} \varepsilon_x \varepsilon_n (\bar{h}_{in}{}^x h_{n\bar{m}}{}^2 h_{jk}{}^x - \bar{h}_{im}{}^x h_{k\bar{n}}{}^2 h_{jn}{}^x) = 0. \end{aligned} \tag{6.2}$$

Putting $j = m$ in (6.2), multiplying ε_j and summing up with respect to the index j, we have

$$\begin{aligned} &c(n h_{i\bar{j}}{}^2 - h_2 \varepsilon_j \delta_{ij}) \\ &\qquad + 2\left(\sum_{x,k,m} \varepsilon_x \varepsilon_k \varepsilon_m h_{ik}{}^x h_{m\bar{k}}{}^2 \bar{h}_{mj}{}^x - h_{i\bar{j}}{}^4 \right) = 0. \end{aligned} \tag{6.3}$$

Theorem 6.1. *Let M be an n-dimensional semi-definite complex submanifold of index $2s$ in $M^{n+p}_{s+t}(c)$, $c \neq 0$. If M satisfies $RS = 0$ and if the Ricci tensor commutes with any shape operator in the direction of any unit normal field, then M is Einstein.*

Proof. Since M satisfies the condition $RS = 0$, the equation (6.3) holds. Now, let A^x be the shape operator associated with any unit normal field E_x. Under the assumption about any shape operator, we see $SA^x - A^x S = 0$. According to (3.12), we have

$$\sum_j \varepsilon_j h_{i\bar{j}}{}^2 h_{jk}{}^x = \sum_j \varepsilon_j h_{ij}{}^x h_{k\bar{j}}{}^2. \tag{6.4}$$

From (6.3) and (6.4), it follows that we have $c(nh_{i\bar{j}}{}^2 - h_2 \varepsilon_i \delta_{ij}) = 0$, which means that M is Einstein. This completes the proof. □

In the case where M is a hypersurface, it is easily seen that the Ricci tensor commutes with the shape operator. Thus, as a direct consequence of Theorem 6.1, we can prove

Corollary 6.2. *Let M be a semi-definite complex hypersurface of index $2s$ in $M^{n+1}_{s+t}(c)$, $0 \leqq s \leqq n$, $t = 0$ or 1, $c \neq 0$. If M satisfies the condition $RS = 0$, then M is Einstein.*

Remark 6.1. Corollary 6.2 is already proved by Aiyama, Ikawa, Kwon and Nakagawa [**1**].

Corollary 6.3. *Let M be an n-dimensional space-like complex submanifold of $M^{n+p}_p(c)$, $c \neq 0$. If M satisfies the condition $RS = 0$, and if the normal connection is proper, then M is Einstein.*

Proof. Let (M', g') be an $(n+p)$-dimensional indefinite Kaehler manifold of index $2p$, and let M be an n-dimensional space-like complex submanifold of M'. For the normal curvature tensor $R^{\perp} = \{R_{\bar{x}yk\bar{m}}\}$ on M, the normal curvature operator T_N on the np-dimensional complex vector space $\Xi^{np}_o = N_o M^C \times T_o M^C$ at any point x on M is defined by

$$T_N(\xi_{xk}) = (\eta_{xk}), \qquad \eta_{xk} = \sum_{y,m} \varepsilon_y \varepsilon_m R_{\bar{x}yk\bar{m}} \xi_{ym},$$

where $N_o M$ is the normal space at o to M in the tangent space $T_o M'$ and V^C denotes the complexification of a real vector space V. We denote by $(R_{yn}{}^{xk})$ the matrix of the linear transformation T_N.

By the assumption that the normal connection is proper, the linear transformation T_N is diagonalizable, and it satisfies

$$R_{\bar{x}yk\bar{m}} = \varepsilon_x \varepsilon_k f_{xk} \delta_{xy} \delta_{km}.$$

Since T_N is self-adjoint, every eigenvalue f_{xk} of T_N is a real-valued function on M. So we have by (5.1)

$$R'_{\bar{x}yk\bar{m}} = \varepsilon_x \varepsilon_k f_{xk} \delta_{xy} \delta_{km} - \sum_j \varepsilon_j h_{kj}{}^x \bar{h}_{jm}{}^y.$$

From this, together with (2.4), it follows that

$$\sum_j \varepsilon_j h_{ij}{}^x \bar{h}_{jk}{}^y = \varepsilon_x \varepsilon_i \left(f_{xi} - \frac{c}{2} \right) \delta_{xy} \delta_{ik}. \tag{6.5}$$

Transvecting (6.5) with $\varepsilon_k \varepsilon_y h_{km}{}^y$ or $\varepsilon_i \varepsilon_x \bar{h}_{mi}{}^x$, and summing up with respect to m and y or i and x, respectively, we have

$$\sum_j \varepsilon_j h_{ij}{}^x h_{k\bar{j}}{}^2 = \left(f_{xi} - \frac{c}{2} \right) h_{ik}{}^x,$$
$$\sum_j \varepsilon_j h_{j\bar{m}}{}^2 \bar{h}_{jk}{}^y = \left(f_{ym} - \frac{c}{2} \right) \bar{h}_{mk}{}^y,$$

which implies that the Ricci tensor commutes with any shape operator. So, by Theorem 6.1, M is Einstein. This completes the proof. □

Now let M be an n-dimensional space-like complex submanifold of an $(n+p)$-dimensional indefinite complex space form $M_t^{n+p}(c)$, $0 < t \leqq p$. Then the matrix $(h_{i\bar{j}}{}^2)$ is a Hermitian one, whose eigenvalues μ_j's are real value functions on M, say $h_{i\bar{j}}{}^2 = \mu_i \delta_{ij}$. Therefore, (6.3) is given by

$$c(n h_{i\bar{j}}{}^2 - h_2 \delta_{ij}) + 2 \left(\sum_{x,k,m} \varepsilon_x h_{ik}{}^x h_{m\bar{k}}{}^2 \bar{h}_{mj}{}^x - h_{i\bar{j}}{}^4 \right) = 0.$$

Accordingly, we have

$$c(n h_4 - h_2{}^2) + 2 \left(\sum_{x,i,j,k,m} \varepsilon_x h_{ik}{}^x h_{m\bar{k}}{}^2 \bar{h}_{mj}{}^x h_{j\bar{i}}{}^2 - h_6 \right) = 0.$$

The second term on the righthand side of the above equation is given by

$$\begin{aligned}
2\Big(\sum_{x,i,j,k,m} \varepsilon_x {h_{ik}}^x {h_{m\bar{k}}}^2 {\bar{h}_{mj}}^x {h_{j\bar{i}}}^2 - h_6\Big) \\
&= 2\Big(\sum_{x,i,k} \varepsilon_x \mu_i \mu_k {h_{ik}}^x {\bar{h}_{ik}}^x - \sum_{x,i,k} \varepsilon_x {\mu_i}^2 {h_{ik}}^x {\bar{h}_{ik}}^x) \\
&= 2\sum_{x,i,k} \varepsilon_x (\mu_i\mu_k - \mu_k^2) {h_{ik}}^x {\bar{h}_{ik}}^x \\
&= -\sum_{x,i,k} \varepsilon_x (\mu_i - \mu_k)^2 {h_{ik}}^x {\bar{h}_{ik}}^x.
\end{aligned}$$

On the other hand, it is easily seen that we have

$$nh_4 - {h_2}^2 \,=\, \frac{1}{2}\sum_{i\neq k}(\mu_i - \mu_k)^2.$$

Thus (6.3) is reformed as

$$\Big(\sum_{i\neq k} c - 2\sum_{x,i\neq k} \varepsilon_x {h_{ik}}^x {\bar{h}_{ik}}^x\Big)(\mu_i - \mu_k)^2 \,=\, 0,$$

from which together with (3.6) it follows that we have

$$\sum_{i\neq k} R_{\bar{i}ik\bar{k}}(\mu_i - \mu_k)^2 \,=\, 0. \tag{6.6}$$

Now a nondegenerate totally real plane $[X, Y]$ is defined by a nondegenerate plane $\{X, Y\}$ of the orthonormal pair X and Y, and its image $\{JX, JY\}$ by the almost complex structure J on M. For two holomorphic planes $P = [X, JX]$ and $Q = [Y, JY]$ where X and Y are orthonormal vectors, the totally real bisectional curvature $H(P, Q)$ is defined by

$$H(P, Q) = H(X, Y) = g\,(R(X, JX)JY, Y).$$

Accordingly, we have by (2.6)

$$H(E_i, E_k) \,=\, R_{\bar{i}ik\bar{k}}, \quad i \neq k. \tag{6.7}$$

By (6.6) and (6.7) we can prove the main theorem in Section 1.

Theorem 6.4. *Let M be an n-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex space form $M^{n+p}_{0+t}(c)$, $0 < t \leqq p$. If M satisfies the condition $RS = 0$, and if its totally real bisectional curvature is nonvanishing at all of its points, then M is Einstein.*

As a direct consequence of Theorem 6.4, the following corollaries are derived. In fact, by the Gauss equation (3.6), we have

$$R_{\bar{i}ik\bar{k}} = R'_{\bar{i}ik\bar{k}} - \sum_x \varepsilon_x h_{ik}{}^x \bar{h}_{ik}{}^x, \quad i \neq k.$$

Since the ambient space is the semi-definite complex space form, it is reformed as

$$2R_{\bar{i}ik\bar{k}} = c - 2\sum_x \varepsilon_x h_{ik}{}^x \bar{h}_{ik}{}^x, \quad i \neq k.$$

So we can prove

Corollary 6.5. *Let M be an n-dimensional complex submanifold of $M^{n+p}(c)$, $c < 0$. If M satisfies the condition $RS = 0$, then M is Einstein.*

Corollary 6.6. *Let M be an n-dimensional space-like complex submanifold of $M^{n+p}_p(c)$, $c > 0$. If M satisfies the condition $RS = 0$, then M is Einstein.*

Remark 6.3. Corollary 6.6 was already proved by Aiyama, Nakagawa and the second author [**2**]. For Corollary 6.5 in the case where $c > 0$, there exists a counter example (see Example 4.5). On the other hand, for Corollary 6.6 in the case where $c < 0$ we also have a counter example. See also Example 4.5.

Under such a situation it seems to be interesting if we consider the case where the ambient space is indefinite complex Euclidean. This will be discussed in much more detail in the last section.

7. Complex Minkowski spaces. In this section, let M be an n-dimensional space-like complex hypersurface of an $(n+1)$-dimensional indefinite complex Euclidean space C_1^{n+1} of index 2. It is said to be *p-cylindrical* if it contains a p-dimensional totally geodesic submanifold through any point on M in C_1^{m+1}, which is not contained in a $(p+1)$-dimensional totally geodesic submanifold.

An $(n-1)$-cylindrical complex hypersurface M in C_1^{n+1} is simply said to be a *cylinder*. It is evident that a cylinder M satisfies the condition $RS = 0$, but not Einstein (see [**1**]).

Theorem 7.1. *Let M be an n-dimensional space-like complex hypersurface of C_1^{n+1}. If M satisfies the condition $RS = 0$, then M is cylindrical.*

Proof. From (6.2) we have $\bar{h}_{im}{}^3 h_{jk} - \bar{h}_{im} h_{jk}{}^3 = 0$ because of $c = 0$. Transvecting the above equation with $\bar{h}_{kl} h_{hm}{}^3$ and putting $i = h$ and $j = l$, and summing up with respect to i and j, we have

$$h_2 h_6 - {h_4}^2 = 0.$$

Let μ_j be an eigenvalue of the Hermitian matrix $H_2 = (h_{i\bar{j}}{}^2)$ which are real valued functions on M. By the straightforward calculation, we get

$$h_2 h_6 - {h_4}^2 = \frac{1}{2} \sum_{i \neq j} \mu_i \mu_j (\mu_i - \mu_j)^2.$$

Since all eigenvalues are nonpositive, the righthand side of the above equation is nonnegative and therefore we have

$$\mu_i \mu_j (\mu_i - \mu_j) = 0, \quad i \neq j.$$

For any indices i and j, the above equation means that there exist at most two distinct eigenvalues 0 and $\mu (\neq 0)$ of the Hermitian matrix H_2. At any point x in M, we denote by $p(x)$ the multiplicity of the eigenvalue 0. Then we have

$$h_2(x) \, = \, (n-p)(x)\mu(x).$$

Since the function h_2 is smooth and the eigenvalue μ is continuous on M, $p(x)$ must be continuous, which implies that the multiplicity of 0 is constant on M. We denote by $\{\omega_j\}$ the canonical form associated with the frame $\{U_j\}$ and let $\{\omega_{ij}\}$ be the connection form. Without loss of generality, we may suppose that $\mu_j = {h_{j\bar{j}}}^2$, where the range of indices $a, b, \ldots$ run from 1 to p and the indices $r, s, \ldots$ run from $p+1$ to n. Let T be a p-dimensional distribution on M defined by $\omega_r = \omega_{n+1} = 0$. Then we have

$$\mu = {h_{a\bar{a}}}^2 = 0,$$

which implies that we get $h_{aj} = 0$ for any indices a and j. Thus the connection form ω_{n+1j} is given by

$$\omega_{n+1a} = \sum_j h_{aj}\omega_j = 0, \qquad \omega_{n+1r} = \sum_j h_{rj}\omega_j = \sum_s h_{rs}\omega_s \tag{7.1}$$

with the help of (3.3). By the structure equation we have

$$\sum_A \omega_{n+1A} \wedge \omega_{Aa} \;=\; \Omega'_{n+1a} \;=\; 0,$$

because the ambient space is a complex Minkowski one, and due to equation (7.1). Hence, we have by (7.1) $\sum_r \omega_{n+1r} \wedge \omega_{ra} = 0$, from which together with (7.1), it follows that we have $\sum_{r,s} h_{rs}\omega_s \wedge \omega_{ra} = 0$, and hence we have

$$\sum_s \omega_s \wedge \left(\sum_r h_{sr}\omega_{ra} \right) \;=\; 0,$$

which along with Cartan's lemma yields

$$\sum_r h_{sr}\omega_{ra} = \sum_t f_{sta}\omega_t, \qquad f_{sta} \;=\; f_{tsa}, \tag{7.2}$$

where f_{st} are smooth functions. Because of ${h_{r\bar{s}}}^2 = \sum_t h_{rt}\bar{h}_{ts} = \mu\delta_{rs}$, transvecting $\bar{h}_{us}$ with (7.2) and summing up with respect to s, we have

$$\mu\,\omega_{ta} \;=\; \sum_{r,s} \bar{h}_{ts} f_{sra}\omega_r.$$

Thus we have

$$\omega_{ra} = \sum_s k_{ras}\omega_s, \qquad k_{ras} = \frac{1}{\mu}\,\sum_t \bar{h}_{rt} f_{tsa}. \tag{7.3}$$

We denote by ∇ the Riemannian connection on M. Then we have

$$\begin{aligned}\nabla E_A &= \sum_j \omega_{ja} E_j \\ &= \sum_b \omega_{ba} E_b + \sum_r \omega_{ra} E_r \\ &= \sum_c \omega_{ca} E_c + \sum_{r,s} k_{ras} \omega_s E_r,\end{aligned}$$

which implies that $[E_a, E_b]$ for any indices a and b are also contained in the distribution T. Namely, it is integrable. Along its integral submanifold N we see by (7.1) and (7.3)

$$\omega_{ra} = 0 = \omega_{n+1a},$$

which implies that N is the p-dimensional totally geodesic submanifold in M and in C_1^{n+1}. It is easily seen that there does not exist a $(p+1)$-dimensional totally geodesic submanifold of M which contains N. This completes the proof. $\square$

Remark 7.1. In his paper [**19**], Takahashi proved a complex hypersurface M satisfying the condition $RS = 0$ in a complex Euclidean space C^{n+1} is a cylinder if it is not totally geodesic. Theorem 7.1 is the complex version of this property.

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Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea
E-mail address: yjsuh@bh.kyungpook.ac.kr

Department of Mathematics Education, Taegu University, Taegu 705-714, Korea
E-mail address: jhkwon@biho.taegu.ac.kr

Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea
E-mail address: yjsuh@bh.kyungpook.ac.kr