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On semigroups with an infinitesimal operator

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Abstract. Let $\{F^t : t \geq 0\}$ be an iteration semigroup of linear continuous set-valued functions. If the semigroup has an infinitesimal operator then it is a uniformly continuous semigroup majorized by an exponential semigroup. Moreover, for sufficiently small t every linear selection of F^t is invertible and there exists an exponential semigroup $\{f^t : t \geq 0\}$ of linear continuous selections f^t of F^t.

If X is a nonempty set, then $n(X)$ denotes the set of all nonempty subsets of X. All linear spaces are over $\mathbb{R}$.

We say that a nonempty subset C of a linear space is a *cone* if $tC \subset C$ for every $t > 0$.

Let X, Y be linear spaces and C be a convex cone in X. The set-valued function (abbreviated to s.v. function) $F : C \to n(Y)$ is called *superadditive* if

$$F(x) + F(y) \subset F(x + y) \quad \text{for all } x, y \in C. \tag{1}$$

F is said to be *additive* if equality holds in (1), and $\mathbb{Q}^+$-*homogeneous* if

$$F(\lambda x) = \lambda F(x) \quad \text{for all } x \in C,\ \lambda \in \mathbb{Q}^+, \tag{2}$$

where $\mathbb{Q}^+$ is the set of all positive rational numbers. F is *linear* if it is additive and (2) is satisfied for all $\lambda > 0$.

If X is a linear topological space, then $b(X)$ denotes the set of all bounded elements of $n(X)$, and $c(X)$ stands for the family of all compact elements of $n(X)$.

Now let X, Y be topological spaces. An s.v. function $F : X \to n(Y)$ is called *lower semicontinuous* at $x_0 \in X$ if for every open set G in Y such that $F(x_0) \cap G \neq \emptyset$ there exists a neighbourhood U of x_0 in X such that $F(x) \cap G \neq \emptyset$ for $x \in U$. We say that F is lower semicontinuous in a set $A \subset X$ if F is lower semicontinuous at every point $x \in A$.

We say that $F : X \to n(Y)$ is *upper semicontinuous* at $x_0 \in X$ if for every open set $G \subset Y$ such that $F(x_0) \subset G$ there exists a neighbourhood U of x_0

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in X such that $F(x) \subset G$ for $x \in U$; F is called upper semicontinuous in a set $A \subset X$ if it is upper semicontinuous at every point of A, and *continuous* if it is both lower and upper semicontinuous.

We recall a set-valued version of the Banach–Steinhaus theorem.

LEMMA 1 (Lemma 4 in [8]). *Let X, Y be normed spaces, $C \subset X$ a convex cone of the second category in C, and $\{F_i : i \in I\}$ a family of superadditive, $\mathbb{Q}^+$-homogeneous and lower semicontinuous s.v. functions $F_i : C \to n(Y)$. If $\bigcup_{i\in I} F_i(x) \in b(Y)$ for every $x \in C$ then there exists a constant $M \in (0,\infty)$ such that*

$$\sup_{i\in I} \|F_i(x)\| \le M\|x\|, \qquad x \in C,$$

where $\|F_i(x)\| = \sup\{\|y\| : y \in F_i(x)\}$.

Applying Lemma 1 to one s.v. function $F : C \to b(Y)$, we can define the norm of F by

$$\|F\| := \inf\{M > 0 : \forall_{x\in C}\ \|F(x)\| \le M\|x\|\}. \tag{3}$$

REMARK 1. There exists an open convex cone which is of the first category in itself.

Proof. This example is adapted from [8]. Let $C(\mathbb{R},\mathbb{R})$ denote the space of all bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and let $K := \{x \in C(\mathbb{R},\mathbb{R}) : \operatorname{supp} x \in b(\mathbb{R})\}$ and $\|x\| := \sup\{|x(t)| : t \in \mathbb{R}\}$ for $x \in K$. Observe that $(K, \|\cdot\|)$ is a normed space. Therefore K is an open convex cone in the space K. On the other hand, from the proof of Remark 1 in [8] it follows that K is of the first category in K. ■

Since not every convex cone with nonempty interior is of the second category in itself, in order to define the norm of a linear continuous s.v. function defined on such a cone, the following lemma will be useful.

LEMMA 2. *Let X be a normed space, $C \subset X$ a convex cone with nonempty interior, and $F : C \to b(X)$ a superadditive, $\mathbb{Q}^+$-homogeneous and upper semicontinuous s.v. function. Then there exists a constant $M > 0$ such that*

$$\|F(x)\| \le M\|x\| \quad \text{for } x \in C.$$

Proof. Fix $x_0 \in \operatorname{Int} C$ and $\varepsilon > 0$. Then there exists $r > 0$ such that $B(x_0, r) \subset C$, where $B(x_0, r)$ is the open ball with center at x_0 and radius r. Since F is upper semicontinuous at x_0, there exists $0 < \delta < r$ such that

$$F(y) \subset F(x_0) + B(0,\varepsilon) \quad \text{for } y \in B(x_0,\delta).$$

Let $x \in B(0,\delta) \cap C$. Then $x + x_0 \in B(x_0,\delta) \cap C = B(x_0,\delta)$ and

$$F(x + x_0) \subset F(x_0) + B(0,\varepsilon).$$

By the above relation and superadditivity of F,

$$F(x) + F(x_0) \subset F(x + x_0) \subset F(x_0) + B(0, \varepsilon) \subset F(x_0) + \operatorname{cl} B(0, \varepsilon)$$

for all $x \in B(0, \delta) \cap C$. On account of the Rådström theorem (see [5]) we obtain

$$F(x) \subset \operatorname{cl} B(0, \varepsilon) \quad \text{for all } x \in B(0, \delta) \cap C.$$

This means that $\lim_{x \to 0,\, x \in C} F(x) = \{0\}$ and according to Lemma 2 in [8], the proof is complete.

For a function F satisfying the assumptions of the above lemma we can define the norm $\|F\|$ in the same way as in (3).

If Y is a normed space, then h denotes the Hausdorff distance derived from the norm in Y.

LEMMA 3. *Let X, Y be normed spaces and $C \subset X$ be a convex cone of the second category in C. Let $\{F_t : t > 0\}$ be a family of superadditive, $\mathbb{Q}^+$-homogeneous and lower semicontinuous s.v. functions $F_t : C \to b(Y)$. If there exists an s.v. function $G : C \to b(Y)$ such that*

$$\lim_{t \to 0} h(F_t(x), G(x)) = 0 \quad \textit{for all } x \in C, \tag{4}$$

then there exist $M, T \in (0, \infty)$ such that

$$\|F_t\| \le M \quad \textit{for every } t \in (0, T].$$

Proof. Assume that $G : C \to b(Y)$ is an s.v. function satisfying (4) and suppose that the assertion of the lemma is false. Then for every $n \in \mathbb{N}$ there exists $t_n \in (0, 1/n)$ such that

$$\|F_{t_n}\| > n.$$

Therefore, according to Lemma 1, there exists an element $x_0 \in C$ for which the set $\bigcup_{n \in \mathbb{N}} F_{t_n}(x_0)$ is not bounded. Thus

$$\sup\{\|F_{t_n}(x_0)\| : n \in \mathbb{N}\} = \infty. \tag{5}$$

On the other hand, condition (4) implies that there exists $T > 0$ with

$$F_t(x_0) \subset G(x_0) + S \quad \text{for } t \in (0, T), \tag{6}$$

where S is the closed unit ball in Y. Therefore

$$\|F_{t_n}(x_0)\| \le \|G(x_0)\| + 1 \quad \text{for every } n > 1/T,$$

which contradicts (5).

From now on, Id stands for the map $x \mapsto \{x\}$, called the set-valued identity.

COROLLARY 1. *Let X be a normed space, $C \subset X$ a convex cone of the second category in C, and $\{F_t : t > 0\}$ a family of superadditive, $\mathbb{Q}^+$-homogeneous and lower semicontinuous set-valued functions $F_t : C \to b(X)$,*

$t > 0$. *If there exists an s.v. function* $G : C \to b(X)$ *such that*

$$\lim_{t\to 0} h\left(\frac{1}{t}\,(F_t(x)-x), G(x)\right) = 0 \qquad \textit{for } x \in C,$$

then

$$\lim_{t\to 0} \|F_t - \mathrm{Id}\| = 0.$$

Proof. According to Lemma 3, there exist positive constants T, M such that

$$\left\|\frac{1}{t}\,(F_t - \mathrm{Id})\right\| \le M \qquad \text{for all } t \in (0, T].$$

Therefore

$$\|F_t - \mathrm{Id}\| \le tM \qquad \text{for } t \in (0, T],$$

which proves the corollary.

LEMMA 4. *Let X be a normed space, $C \subset X$ a convex cone with nonempty interior, and $\{F_t : t > 0\}$ a family of linear continuous s.v. functions $F_t : C \to b(X)$, $t > 0$, such that $\lim_{t\to 0}\|F_t - \mathrm{Id}\| = 0$. Then there exists a constant $T > 0$ such that each linear selection of F_t $(0 < t < T)$ is invertible.*

Proof. According to Lemma 5 in [8] there exists a positive constant M such that for every linear continuous s.v. function F,

$$h(F(x), F(y)) \le M\|F\|\,\|x-y\|, \qquad x, y \in C. \tag{7}$$

By our assumptions, there exists $T > 0$ such that

$$\|F_t - \mathrm{Id}\| < \frac{1}{2M}, \qquad 0 < t < T. \tag{8}$$

Fix $t \in (0, T)$ and let f_t be a linear selection of F_t. Since $\operatorname{Int} C \ne \emptyset$, we have $X = C - C$. Thus there exists a unique linear extension $\widehat{f_t}$ of f_t to the space X, which is defined as follows:

$$\widehat{f_t}(x-y) = f_t(x) - f_t(y), \qquad x, y \in C.$$

By (7) and (8), for all $x, y \in C$,

$$\begin{aligned}\|\widehat{f_t}(x-y) - (x-y)\| &= \|(f_t(x)-x) - (f_t(y)-y)\| \le M\|f_t - \mathrm{id}\|\,\|x-y\| \\ &\le M\|F_t - \mathrm{Id}\|\,\|x-y\| < \frac{1}{2}\,\|x-y\|,\end{aligned}$$

and therefore $\|\widehat{f_t} - \mathrm{id}\| < 1$, which completes the proof. ∎

Combining Lemma 4 with Corollary 1, we get

COROLLARY 2. *Let X be a normed space, $C \subset X$ a convex cone of the second category in C with nonempty interior, and $\{F_t : t > 0\}$ a family of linear continuous s.v. functions $F_t : C \to b(X)$, $t > 0$. Assume that there* *exists an s.v. function* $G : C \to b(X)$ *such that*

$$\lim_{t\to 0} h\left(\frac{1}{t}\,(F_t(x) - x), G(x)\right) = 0 \quad \text{for } x \in C.$$

Then there exists a constant $T > 0$ *such that each linear selection of* F_t, $0 < t < T$, *is invertible.*

The composition $G \circ F$ of s.v. functions $F : X \to n(Y)$ and $G : Y \to n(Z)$ is the s.v. function given as follows:

$$(G \circ F)(x) := G(F(x)) \quad \text{for } x \in X.$$

A family $\{F^t : t \geq 0\}$ of s.v. functions $F^t : X \to n(X)$ is called an *iteration semigroup* if

$$F^t \circ F^s = F^{t+s} \quad \text{for all } s, t \geq 0.$$

Let $\{F^t : t \geq 0\}$ be an iteration semigroup of s.v. functions, defined on a cone C in a normed space X with values in $b(C)$. It is *continuous* if the s.v. function $t \mapsto F^t(x)$ is continuous for every $x \in C$. The semigroup $\{F^t : t \geq 0\}$ *has an infinitesimal operator* if there exists an s.v. function $G : C \to b(X)$ such that $\lim_{t\to 0} h(t^{-1}(F^t(x)-x), G(x)) = 0$, for every $x \in C$. Then G is called an infinitesimal operator of the semigroup.

Let C be a convex cone of the second category in itself with nonempty interior. By Corollary 1, a semigroup of linear continuous s.v. functions $F^t : C \to b(C)$ which has an infinitesimal operator is uniformly continuous, that is, $\lim_{t\to 0} \|F^t - \mathrm{Id}\| = 0$. Moreover, for sufficiently small t every linear selection of F^t is invertible (see Corollary 2).

According to Lemmas 4 and 5 of [8] we obtain a more general version of Theorem 1 of [2] (the proof runs in much the same way).

THEOREM 1. *Let* X *be a normed space, and* $C \subset X$ *a convex cone of the second category in* C *with nonempty interior. If* $\{F^t : t \geq 0\}$ *is an iteration semigroup of linear continuous s.v. functions* $F^t : C \to c(C)$, $t \geq 0$, *satisfying the conditions*

(i) $F^0 = \mathrm{Id}$,
(ii) $\lim_{t\to 0} \|F^t - \mathrm{Id}\| = 0$,

then there exist constants $M > 0$ *and* $\omega \geq 0$ *with the property that*

$$\|F^t\| \leq M e^{\omega t}, \quad t \geq 0. \tag{9}$$

Moreover, if B *is a bounded subset of* C, *then*

$$\forall_{s_0\geq 0}\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in B}\forall_{s\geq 0}\ (|s - s_0| < \delta \Rightarrow h(F^s(x), F^{s_0}(x)) < \varepsilon). \tag{10}$$

REMARK 2. An iteration semigroup satisfying the assumptions of Theorem 1 is continuous.

Corollary 3. *Let X be a normed space, and $C \subset X$ a convex cone of the second category in C with nonempty interior. If $\{F^t : t \geq 0\}$ is an iteration semigroup of linear continuous s.v. functions $F^t : C \to c(C)$, $t \geq 0$, satisfying the conditions*

(i) $F^0 = \mathrm{Id}$,

(ii) *there exists an s.v. function $G : C \to b(X)$ such that*
$$\lim_{t\to 0} t^{-1}(F^t(x) - x) = G(x) \quad \text{for every } x \in C,$$

then there exists constants $M > 0$ and $\omega \geq 0$ with the property that
$$\|F^t\| \leq Me^{\omega t}, \quad t \geq 0.$$
Moreover, if B is a bounded subset of C, then
$$\forall_{s_0\geq 0}\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x\in B}\forall_{s\geq 0}\ (|s - s_0| < \delta \Rightarrow h(F^s(x), F^{s_0}(x)) < \varepsilon).$$

If X is a linear space, then we say that an iteration semigroup $\{F^t : t\geq 0\}$ is *concave* if
$$F^{\lambda s+(1-\lambda)t}(x) \subset \lambda F^s(x) + (1-\lambda)F^t(x)$$
for all $s, t \geq 0$, $\lambda \in [0, 1]$ and $x \in X$.

Observe that a concave iteration semigroup satisfying the assumptions of Theorem 1 in [3] also fulfils the assumptions of the above corollary.

Let C be a convex cone in a normed space X. If $g : C \to C$ is an additive and positively homogeneous continuous operator, then $e^{tg} : C \to C$ is defined as follows:
$$e^{tg}(x) = \sum_{i=0}^{\infty} \frac{t^i g^i(x)}{i!}, \quad x \in C,\ t \geq 0.$$

Lemma 5. *Let X be a Banach space, C a convex cone in X with nonempty interior, and T a positive number. Let $\{f^t : t \in [0, T]\}$ be a family of linear continuous operators from C into C satisfying the conditions*

(i) $f^0 = \mathrm{id}$,

(ii) $f^t \circ f^s = f^{t+s}$ *for* $t, s, t + s \in [0, T]$,

(iii) $\lim_{t\to 0} \|f^t - \mathrm{id}\| = 0$.

Then there exists a unique linear continuous operator $g : X \to X$ such that $f^t(x) = e^{tg}(x)$ for all $x \in C$ and $t \in [0, T]$.

Proof. Let $\widetilde{f}^t$ be an extension of f^t to $X = C - C$ defined as follows:
$$\widetilde{f}^t(x_1 - x_2) = f^t(x_1) - f^t(x_2), \quad t \in [0, T],\ x_1, x_2 \in C.$$

Note that $\{\widetilde{f}^t : t \in [0, T]\}$ is a family of linear continuous operators satisfying (i)–(iii). Indeed, for all $x_1, x_2 \in C$, $t, s, t + s \in [0, T]$,
$$\begin{aligned}(\widetilde{f}^t \circ \widetilde{f}^s)(x_1 - x_2) &= \widetilde{f}^t(\widetilde{f}^s(x_1 - x_2)) = \widetilde{f}^t(f^s(x_1) - f^s(x_2))\\ &= \widetilde{f}^t(f^s(x_1)) - \widetilde{f}^t(f^s(x_2)) = f^t(f^s(x_1)) - f^t(f^s(x_2))\\ &= f^{t+s}(x_1) - f^{t+s}(x_2) = \widetilde{f}^{t+s}(x_1 - x_2).\end{aligned}$$

By Lemma 5 in [8], there exists $M > 0$ such that for every $t \geq 0$ and $x, y \in C$,

$$\|(f^t - \mathrm{id})(x) - (f^t - \mathrm{id})(y)\| \leq M\|f^t - \mathrm{id}\|\|x - y\|$$

and consequently $\|\widetilde{f}^t - \mathrm{id}\| \leq M\|f^t - \mathrm{id}\|$, which together with condition (iii) gives

$$\lim_{t\to 0} \|\widetilde{f}^t - \mathrm{id}\| = 0.$$

Take any $t > T$ and $n \in \mathbb{N}$ large enough that $t/n \in [0, T]$ and define $\widetilde{f}^t := (\widetilde{f}^{t/n})^n$. This function is well defined. If $n, m \in \mathbb{N}$ are so chosen that $t/n, t/m \in [0, T]$ then $t/nm \in [0, T]$. Hence

$$(\widetilde{f}^{t/m})^m = [(\widetilde{f}^{t/nm})^n]^m = [(\widetilde{f}^{t/nm})^m]^n = (\widetilde{f}^{t/n})^n.$$

Since $\{\widetilde{f}^t : t \geq 0\}$ is a uniformly continuous iteration semigroup of linear continuous operators, there exists a unique linear continuous operator $g : X \to X$ such that $\widetilde{f}^t = e^{tg}$ for $t \geq 0$ (cf. Corollary 1.4 in [4]), which completes the proof.

An element x of a nonempty set A in a linear space X is called an *extreme point* of A if there is no $\lambda \in (0, 1)$ and two different $x_1, x_2 \in A$ such that $x = \lambda x_1 + (1 - \lambda)x_2$. We denote by $\mathrm{Ext}\, A$ the set of all extreme points of A.

Let X be a nonempty set, Y a linear space, and $F : X \to n(Y)$ an s.v. function. We say that a selection f of F is *extreme* if $f(x) \in \mathrm{Ext}\, F(x)$ for all $x \in X$.

Remark 3. Let X, Y be normed spaces, $C \subset X$ an open convex cone, and $x_0 \in C$. If f is an additive selection of an additive lower semicontinuous s.v. function $F : C \to n(Y)$ such that $f(x_0) \in \mathrm{Ext}\, F(x_0)$ then f is extreme, linear and continuous.

Proof. According to Nikodem's theorem (Th. 5.4 in [1]) a selection f such that $f(x_0) \in \mathrm{Ext}\, F(x_0)$ is unique and $f(x) \in \mathrm{Ext}\, F(x)$ for $x \in C$. Since $f(x) \in F(x)$ for each $x \in C$ and F is continuous, by Theorems 5.2 and 5.3 in [1], f is linear continuous.

The following lemma is a generalization of Lemma 4 in [2]. The separability of X is not necessary.

Lemma 6. *Let X be a normed space, C an open convex cone in X, and $x_0 \in C$. Let $F, G : C \to n(C)$ be additive lower semicontinuous s.v. functions such that every extreme additive selection of F is invertible. Then for each additive selection h of $F \circ G$ such that*

$$h(x_0) \in \mathrm{Ext}\, (F \circ G)(x_0)$$

there exist unique additive selections f and g of F and G respectively for which

$$h = f \circ g.$$

Moreover, f and g are extreme, linear and continuous.

Proof. There exists a point $y_0 \in G(x_0)$ such that $h(x_0) \in \operatorname{Ext} F(y_0)$. According to Nikodem's theorem (Th. 5.4 in [1]) there exists exactly one additive selection f of F such that $h(x_0) = f(y_0)$ and $f(x) \in \operatorname{Ext} F(x)$ for $x \in C$.

We will show that $y_0 \in \operatorname{Ext} G(x_0)$ and it is unique.

Suppose that $\lambda \in (0,1)$, $y_1, y_2 \in G(x_0)$ and $y_0 = \lambda y_1 + (1-\lambda)y_2$. Then $f(y_1), f(y_2) \in F(G(x_0))$ and

$$h(x_0) = f(y_0) = \lambda f(y_1) + (1-\lambda) f(y_2) \in \operatorname{Ext}\,(F \circ G)(x_0),$$

hence $f(y_1) = f(y_2) = f(y_0)$. Since f is invertible we have $y_1 = y_2 = y_0$.

Now suppose that there exists $z \neq y_0$ such that $z \in \operatorname{Ext} G(x_0)$ and $h(x_0) \in \operatorname{Ext} F(z)$. Then for every $\lambda \in (0,1)$,

$$h(x_0) = \lambda h(x_0) + (1-\lambda)h(x_0) \in \lambda F(z) + (1-\lambda)F(y_0) = F(\lambda z + (1-\lambda)y_0).$$

Since $h(x_0)$ is an extreme point of $F(G(x_0))$ it cannot be expressed as a convex combination of elements of the set $F(\lambda z + (1-\lambda)y_0) \subset F(G(x_0))$. Hence $h(x_0) \in \operatorname{Ext} F(\lambda z + (1-\lambda)y_0)$. On account of Nikodem's theorem there exists a unique extreme additive selection $\widetilde{f}$ of F such that $h(x_0) = \widetilde{f}(\lambda z + (1-\lambda)y_0)$. Remark 3 shows that $\widetilde{f}$ is linear and therefore

$$h(x_0) = \widetilde{f}(\lambda z + (1-\lambda)y_0) = \lambda \widetilde{f}(z) + (1-\lambda)\widetilde{f}(y_0). \tag{11}$$

Since $h(x_0) \in \operatorname{Ext}\,(F \circ G)(x_0)$ and $\widetilde{f}(z), \widetilde{f}(y_0) \in (F \circ G)(x_0)$, (11) shows that $h(x_0) = \widetilde{f}(z) = \widetilde{f}(y_0)$ and so $z = y_0$.

Again by Nikodem's theorem there exists exactly one additive selection g of the additive s.v. function G such that $y_0 = g(x_0)$ and $g(x) \in \operatorname{Ext} G(x)$ for $x \in C$.

Therefore $h(x_0) = f(y_0) = f(g(x_0)) = (f \circ g)(x_0)$ and h, $f \circ g$ are additive selections of $F \circ G$, which yields $h = f \circ g$. On account of Remark 3 both f and g are linear continuous, which completes the proof.

By induction, we get the following corollary.

COROLLARY 4. *Let X be a normed space, C an open convex cone in X, $x_0 \in C$, and $n \geq 2$ a positive integer. Let $F_1, \ldots, F_n : C \to c(C)$ be additive lower semicontinuous s.v. functions such that every extreme additive selection of F_i is invertible for $i \in \{2, \ldots, n\}$. Then for every additive selection h of $F_n \circ \cdots \circ F_1$ satisfying*

$$h(x_0) \in \operatorname{Ext}(F_n \circ \cdots \circ F_1)(x_0)$$

there exist unique additive selections f_i *of* F_i, $i \in \{1, \dots, n\}$, *such that*

$$h = f_n \circ \dots \circ f_1.$$

Moreover each f_i, $i \in \{1, \dots, n\}$, *is extreme, linear and continuous.*

Combining Lemma 4 and the above corollary gives us a lemma which will be useful later.

LEMMA 7. *Let* X *be a Banach space and* $C \subset X$ *an open convex cone. Let* $\{F^t : t \geq 0\}$ *be an iteration semigroup of linear continuous s.v. functions* $F^t : C \to c(C)$, $t \geq 0$, *satisfying the condition*

$$\lim_{t \to 0} \|F^t - \mathrm{Id}\| = 0.$$

Then every extreme linear selection of F^t $(t > 0)$ *is invertible.*

Proof. According to Lemma 4, there exists a constant $T > 0$ such that each linear continuous selection of F^t $(0 < t < T)$ is invertible.

It remains to show that the assertion is true for $t > T$. Fix $t_0 > T$, $x_0 \in C$ and a linear selection f of F^{t_0} such that $f(x_0) \in \mathrm{Ext}\, F^{t_0}(x_0)$. Let $n \in \mathbb{N}$ be large enough that $t_0/n \in (0, T)$. Then $f(x_0) \in \mathrm{Ext}\, F^{t_0}(x_0) = \mathrm{Ext}\, (F^{t_0/n})^n(x_0)$. Hence, on account of Corollary 4, there exist unique linear selections $f_1, \dots, f_n$ such that

$$f = f_n \circ \dots \circ f_1.$$

Since each function f_i $(i \in \{1, \dots, n\})$ is invertible, so is f.

Note that if $\{F^t : t \geq 0\}$ is an iteration semigroup with an infinitesimal operator, then every extreme linear selection of F^t is invertible.

The next theorem is a refinement of Theorem 2 in [2]; the assumption that there exists a finite cone-basis is omitted. Moreover the assertion is stronger.

THEOREM 2. *Let* X *be a Banach space and* $C \subset X$ *an open convex cone. Let* $\{F^t : t \geq 0\}$ *be an iteration semigroup of linear continuous s.v. functions* $F^t : C \to c(C)$, $t \geq 0$, *satisfying the conditions*

(i) $F^0 = \mathrm{Id}$,
(ii) $\lim_{t \to 0} \|F^t - \mathrm{Id}\| = 0$.

Then for every $t_0 > 0$, $x_0 \in C$ *and* $y_0 \in \mathrm{Ext}\, F^{t_0}(x_0)$ *there exists exactly one iteration semigroup* $\{f^t : t \geq 0\}$ *of linear selections* f^t *of* F^t *with the property* $f^{t_0}(x_0) = y_0$.

Moreover, f^t *is extreme for every* $t \in [0, t_0]$ *and there exists a unique linear continuous operator* $g : X \to X$ *such that* $f^t(x) = e^{tg}(x)$ *for all* $t \geq 0$ *and* $x \in C$.

Proof. Let $\{F^t : t \geq 0\}$ be an iteration semigroup satisfying our assumptions. By Lemma 7 every extreme linear selection of F^t is invertible for $t > 0$. Fix $t_0 > 0$, $x_0 \in C$ and $y_0 \in \operatorname{Ext} F^{t_0}(x_0)$. Then there exists exactly one extreme linear selection a^{t_0} of F^{t_0} such that

$$a^{t_0}(x_0) = y_0. \tag{12}$$

Let $t, s, t+s \in [0, t_0]$. Then $\operatorname{Ext} F^{t_0}(x) = \operatorname{Ext}(F^{t_0-(t+s)} \circ F^{t+s})(x)$ for all $x \in C$. On account of Lemma 6, there exist unique extreme linear selections $a^{t_0-(t+s)}$, f^{t+s} of $F^{t_0-(t+s)}$, F^{t+s} respectively, such that

$$a^{t_0} = a^{t_0-(t+s)} \circ f^{t+s}. \tag{13}$$

Similarly, there exist unique extreme linear selections g^t, h^s of F^t, F^s respectively, such that

$$a^{t_0} = a^{t_0-(t+s)} \circ g^t \circ h^s. \tag{14}$$

Since $a^{t_0-(t+s)}$ is invertible, from (13) and (14) we conclude that for every $t, s, t+s \in [0, t_0]$,

$$f^{t+s} = g^t \circ h^s. \tag{15}$$

In this way we have defined the families of linear functions $\{f^t : t \in [0, t_0]\}$, $\{g^t : t \in [0, t_0]\}$ and $\{h^t : t \in [0, t_0]\}$ satisfying the Pexider equation (15). Taking in (15) $s = 0$ and next $t = 0$ we obtain

$$f^t = g^t \circ h^0 = g^t \circ \mathrm{id} = g^t \qquad \text{for } t \in [0, t_0],$$
$$f^s = g^0 \circ h^s = \mathrm{id} \circ h^s = h^s \qquad \text{for } s \in [0, t_0].$$

Thus for all $t, s \in [0, t_0]$ such that $t + s \in [0, t_0]$ we have $f^t = g^t = h^t$.

Therefore $\{f^t : t \in [0, t_0]\}$ is a family of extreme linear continuous selections of functions from $\{F^t : t \in [0, t_0]\}$, respectively, such that

$$f^{t+s} = f^t \circ f^s \qquad \text{for } t, s, t+s \in [0, t_0]. \tag{16}$$

Moreover, since $a^{t_0}(x_0) = y_0$, substituting $t + s = t_0$ in (13) we obtain $f^{t_0}(x_0) = y_0$.

Observe that $\lim_{t\to 0} \|f^t - \mathrm{id}\| = 0$, since $\|f^t - \mathrm{id}\| \le \|F^t - \mathrm{Id}\|$. Therefore, by Lemma 5, there exists a unique linear continuous operator g such that $f^t = e^{tg}$ for $t \in [0, t_0]$ and $e^{t_0 g}(x) = f^{t_0}(x) = y_0$.

Take $t > t_0$ and $x \in C$. There exists $n \in \mathbb{N}$ large enough to have $t/n \in [0, t_0]$. Since $f^{t/n}(x) \in \operatorname{Ext} F^{t/n}(x)$ and $f^t = (f^{t/n})^n$, we have $f^t(x) \in F^t(x)$, $x \in C$, which finishes our proof.

Corollary 5. *Let X be a Banach space and $C \subset X$ an open convex cone. Let $\{F^t : t \ge 0\}$ be an iteration semigroup of linear continuous s.v. functions $F^t : C \to c(C)$, $t \ge 0$, satisfying the conditions*

(i) $F^0 = \mathrm{Id}$,

(ii) *there exists a set-valued function $G : C \to b(X)$ such that*

$$\lim_{t\to 0} \frac{1}{t}(F^t(x) - x) = G(x) \qquad \text{for } x \in C.$$

Then for every $t_0 > 0$, $x_0 \in C$ and $y_0 \in \operatorname{Ext} F^{t_0}(x_0)$ there exists exactly one iteration semigroup $\{f^t : t \geq 0\}$ of linear selections f^t of F^t with the property $f^{t_0}(x_0) = y_0$.

Moreover, f^t is extreme for $t \in [0, t_0]$ and there exists a unique linear continuous operator $g : X \to X$ such that $g(x) \in G(x)$ and $f^t(x) = e^{tg}(x)$ for every $x \in C$ and $t \geq 0$.

Proof. Let $\{F^t : t \geq 0\}$ be an iteration semigroup satisfying our assumptions. According to Corollary 1, the semigroup also fulfills all assumptions of the above theorem. Therefore there exists a unique linear continuous operator $g : X \to X$ such that $f^t(x) = e^{tg}(x)$ for every $x \in C$ and $t \geq 0$. Hence

$$\frac{e^{tg}(x) - x}{t} = \frac{f^t(x) - x}{t} \in \frac{F^t(x) - x}{t}, \qquad x \in C,\ t \geq 0. \tag{17}$$

Fix any $\varepsilon > 0$ and $x \in C$. According to assumption (ii), there exists $T_1 > 0$ such that

$$\frac{F^t(x) - x}{t} \subset G(x) + \varepsilon S \quad \text{for all } 0 < t < T_1. \tag{18}$$

Combining (17) with (18), we can write

$$\frac{e^{tg}(x) - x}{t} \in G(x) + \varepsilon S \quad \text{for all } t \in (0, T_1).$$

Consequently,

$$g(x) \in G(x) + \varepsilon S.$$

Since $G(x)$ is compact, as a limit of a sequence in the complete space $c(C)$,

$$g(x) \in G(x), \qquad x \in C,$$

and the proof is complete.

COROLLARY 6. *Let X be a Banach space and $C \subset X$ a convex cone with nonempty interior. Let $\{F^t : t \geq 0\}$ be an iteration semigroup of linear continuous s.v. functions $F^t : C \to c(C)$, $t \geq 0$, satisfying the conditions*

(i) $F^0 = \mathrm{Id}$,

(ii) $F^t(\operatorname{Int} C) \subset \operatorname{Int} C$, $t > 0$,

(iii) *there exists a set-valued function $G : C \to c(C)$ such that*

$$\lim_{t \to 0} \frac{1}{t}\,(F^t(x) - x) = G(x) \quad \text{for } x \in C.$$

Then for every $t_0 > 0$, $x_0 \in \operatorname{Int} C$ and $y_0 \in \operatorname{Ext} F^{t_0}(x_0)$ there exists exactly one iteration semigroup $\{f^t : t \geq 0\}$ of linear selections f^t of F^t with the property $f^{t_0}(x_0) = y_0$.

Moreover, $f^t(x) \in \operatorname{Ext} F^t(x)$ for $x \in \operatorname{Int} C$ and $t \in [0, t_0]$, and there exists a unique linear continuous operator $g : X \to X$ such that $g(x) \in G(x)$ $(x \in C)$ and $f^t = e^{tg}$ for $t \geq 0$.

Proof. Define $\widehat{F}^t = F^t|_{\operatorname{Int} C}$ for all $t \geq 0$. It is easy to observe that $\{\widehat{F}^t : t \geq 0\}$ is an iteration semigroup of linear s.v. functions satisfying all assumptions of Theorem 2.

Then for every $t_0 > 0$, $x_0 \in \operatorname{Int} C$ and $y_0 \in \operatorname{Ext} \widehat{F}^{t_0}(x_0)$ there exists exactly one iteration semigroup $\{\widehat{f}^t : t \geq 0\}$ of linear continuous selections $\widehat{f}^t$ of $\widehat{F}^t$ with the property $\widehat{f}^{t_0}(x_0) = y_0$.

Moreover, there exists a unique linear continuous operator $g : X \to X$ such that $g(x) \in G(x)$ and $\widehat{f}^t(x) = e^{tg}(x)$ for $x \in \operatorname{Int} C$ and $t \geq 0$.

Since C is a convex cone with nonempty interior, $\overline{C} = \overline{\operatorname{Int} C}$. Therefore we can uniquely extend every function $\widehat{f}^t$ to a linear continuous function $f^t : C \to X$, $t \geq 0$. It is also easily seen that $f^t(x) = e^{tg}(x)$ for $x \in C$ and $t \geq 0$.

Fix $t \geq 0$. We will show that f^t is a selection of F^t. Take $x \in C \setminus \operatorname{Int} C$ and a sequence of elements $\{x_n : n \in \mathbb{N}\}$ of the cone $\operatorname{Int} C$. Then, by the closedness of $F^t(x)$ and the continuity of F^t,

$$f^t(x) = \lim_{n\to\infty} \widehat{f}^t(x_n) \in \lim_{n\to\infty} \widehat{F}^t(x_n) = F^t(x). \ \blacksquare$$

From the above theorem and Theorem 1 of [3], one can obtain a similar result for concave iteration semigroups.

COROLLARY 7. *Let X be a Banach space and $C \subset X$ a closed convex cone with nonempty interior. Let $\{F^t : t \geq 0\}$ be a concave iteration semigroup of linear continuous s.v. functions $F^t : C \to c(C)$, $t \geq 0$, such that $F^0 = \operatorname{Id}$ and $F^t(\operatorname{Int} C) \subset \operatorname{Int} C$ for $t > 0$. Then for every $t_0 > 0$, $x_0 \in \operatorname{Int} C$ and $y_0 \in \operatorname{Ext} F^{t_0}(x_0)$ there exists exactly one iteration semigroup $\{f^t : t \geq 0\}$ of linear selections f^t of F^t with the property $f^{t_0}(x_0) = y_0$.*

Moreover, $f^t(x) \in \operatorname{Ext} F^t(x)$ for $x \in \operatorname{Int} C$ and $t \in [0, t_0]$ and there exists a unique linear continuous operator $g : X \to X$ such that

$$g(x) \in \bigcap_{t \geq 0} \frac{1}{t} \left(F^t(x) - x \right)$$

and $f^t(x) = e^{tg}(x)$ for all $t \geq 0$ and $x \in C$.

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