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ON SEMIGROUPS WITH INVOLUTION

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A semigroup S with an involution * is called a *special involution semigroup* if and only if, for every finite nonempty subset T of S,

$$(\exists t \in T)(\forall u, v \in T) \quad tt^* = uv^* \Rightarrow u = v.$$

It is shown that a semigroup is inverse if and only if it is a special involution semigroup in which every element invariant under the involution is periodic. Other examples of special involution semigroups are discussed; these include free semigroups, totally ordered cancellative commutative semigroups and certain semigroups of matrices. Some properties of the semigroup algebras of special involution semigroups are also derived. In particular, it is shown that their real and complex semigroup algebras are semiprimitive.

1. DEFINITIONS

The notation and terminology is that of [**1**] throughout.

Let S be a semigroup, written multiplicatively. An *involution* on S is an anti-isomorphism of period two; that is, a mapping $^*\colon S \to S$, $x \mapsto x^*$ such that, for all $x, y \in S$,

$$(xy)^* = y^*x^* \quad \text{and} \quad x^{**} = x.$$

Now suppose that S admits an involution *. We call S an *involution semigroup*. An element $x \in S$ is termed *hermitian* if and only if $x^* = x$. Note that, for all $x \in S$, the elements xx^* and x^*x are hermitian; moreover, every power of a hermitian element is again hermitian. We call S *hermitian-periodic* if and only if every hermitian element of S is periodic; and we call S a *special involution semigroup* if and only if, for every finite nonempty subset T of S,

$$(\exists t \in T)(\forall u, v \in T) \quad tt^* = uv^* \Rightarrow u = v.$$

A condition similar to but apparently weaker than this second condition was recently introduced by Shehadah [**7**].

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We are grateful to Mr T. Lavers for providing the basic version of Example 4.

2. Inverse semigroups

In this section we give a new characterisation of inverse semigroups. Our starting-point is the following result [**1**, Section 2.3, Exercise 7(a)]. A proof is provided for convenience.

Lemma 1. *Let S be a semigroup. Then S is inverse if and only if S is a regular involution semigroup in which every idempotent is hermitian.*

Proof: If S is inverse then it has the stated properties with inversion $(x \mapsto x^{-1})$ as involution. Assume, conversely, that S is a regular semigroup with an involution * such that, for all $e = e^2 \in S$, $e^* = e$. Let $a \in S$ and let $axa = a$, $xax = x$, $aya = a$, $yay = y$ for some $x, y \in S$. To show that S is inverse it is enough to show that $x = y$. By hypothesis, $(ax)^* = ax$ and $(ay)^* = ay$. Hence

$$x = xax = x(ax)^* = xx^*a^* = xx^*a^*y^*a^* = xy^*a^* = x(ay)^* = xay.$$

Similarly, since $(xa)^* = xa$ and $(ya)^* = ya$, we have that $y = xay$. The result follows. $\square$

Remark. The argument above is essentially due to Penrose [**6**]. It should be noted that on an inverse semigroup there may exist involutions other than inversion with respect to which every idempotent is hermitian: for example, if S is the group of invertible $n \times n$ matrices over a field, where $n \geqslant 2$, then the transpose mapping $(A \mapsto A^t)$ is an involution on S and preserves the identity matrix; but there exists $A \in S$ with $A^t \neq A^{-1}$.

We now come to the main result. For the sake of completeness, we include a proof of the first part, which effectively comprises [**3**, Lemma 2.1].

Theorem 1.

(i) *If S is an inverse semigroup then S is a hermitian-periodic special involution semigroup with respect to inversion.*

(ii) *If S is a hermitian-periodic special involution semigroup then S is inverse and the given involution coincides with inversion.*

Proof: (i) Let S be an inverse semigroup and consider the involution defined by $x \mapsto x^{-1}$ $(x \in S)$. If $x \in S$ is such that $x = x^{-1}$ then $x^3 = x$. Thus S is hermitian-periodic. Now let T be a finite nonempty subset of S and let $t \in T$ be chosen so that tt^{-1} is maximal in $\{xx^{-1} : x \in T\}$, under the natural partial ordering of the idempotents of S. Suppose that $x, y \in T$ are such that $tt^{-1} = xy^{-1}$. Then $xx^{-1}tt^{-1} = xy^{-1} = tt^{-1}$; that is, $tt^{-1} \leqslant xx^{-1}$. Hence $tt^{-1} = xx^{-1}$, by the choice of t. Further, since $tt^{-1} = \left(tt^{-1}\right)^{-1} = \left(xy^{-1}\right)^{-1} = yx^{-1}$, a similar argument yields $tt^{-1} = yy^{-1}$. Thus $xx^{-1} = xy^{-1} = yy^{-1}$. Consequently, $xy^{-1}x = xx^{-1}x = x$ and

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$y^{-1}xy^{-1} = y^{-1}yy^{-1} = y^{-1}$, from which we see that $x = \left(y^{-1}\right)^{-1} = y$. Thus S is a special involution semigroup.

(ii) Let S be a hermitian-periodic special involution semigroup, with respect to the involution $*$. First, we show that

$$(\forall h = h^* \in S) \quad h = h^3. \tag{2.1}$$

Let $h = h^* \in S$. By hypothesis, $\langle h \rangle$ is finite. Hence there exist positive integers m and n such that $h^m = h^{m+n}$. Choose m to be minimal with this property. Write $T\colon = \{h^i : i \geqslant m-1\}$ if $m \geqslant 2$ and $T\colon = \langle h \rangle$ if $m = 1$. Then, since T is finite and every power of h is hermitian,

$$(\exists t \in T)(\forall p, q \in T) \quad t^2 = pq \Rightarrow p = q. \tag{2.2}$$

Suppose that $m \geqslant 2$. Then $t = h^i$ for some $i \geqslant m-1$. Since $2i \geqslant 2(m-1) \geqslant m$, we have that $h^{2i} = h^{2i+n} = h^{2(m-1)+k}$, where $k = 2i - 2(m-1) + n$ $(\geqslant 1)$. Thus $t^2 = h^{2i} = h^{m-1}h^{m-1+k}$ and so, by (2.2), $h^{m-1} = h^{m-1+k}$. But this contradicts the choice of m. Thus $m = 1$. Hence $h = h^{n+1}$, $T = \langle h \rangle$ and $t = h^i$ for some $i \in \{1, 2, \dots, n\}$. Then $t^2 = h^{n+2i} = h^{n+i-1}h^{i+1}$ and so, by (2.2), $h^{n+i-1} = h^{i+1}$. Consequently,

$$h = h^{2n+1} = h^{n+i-1}h^{n-i+2} = h^{i+1}h^{n-i+2} = h^{n+3} = h^3.$$

This establishes (2.1).

We now prove that S is regular. Let $x \in S$. Then, by (2.1),

$$xx^* = (xx^*)^3.$$

Write $U\colon = \{x, xx^*, xx^*x, (xx^*)^2\}$. Since U is finite,

$$(\exists u \in U)(\forall p, q \in U) \quad uu^* = pq^* \Rightarrow p = q. \tag{2.3}$$

If $u = xx^*$ or $u = (xx^*)^2$ then $uu^* = (xx^*)^2 = x(xx^*x)^*$ and so, by (2.3), $x = xx^*x$. On the other hand, if $u = x$ or $u = xx^*x$ then $uu^* = (xx^*)^3 = (xx^*)(xx^*)^2$ and so, by (2.3), $xx^* = (xx^*)^2$, from which it follows that $uu^* = (xx^*)^2$ and therefore (as before) that $x = xx^*x$. Thus

$$(\forall x \in S) \quad x = xx^*x. \tag{2.4}$$

Next, we show that every idempotent in S is hermitian. Let $e = e^2 \in S$ and write $V\colon = \{e, ee^*\}$. Then

$$(\exists v \in V)(\forall p, q \in V) \quad vv^* = pq^* \Rightarrow p = q.$$

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If $v = e$ then $vv^* = ee^* = e(ee^*)^*$ and so $e = ee^* = e^*$. Alternatively, if $v = ee^*$ then $vv^* = (ee^*)^2 = ee^*$, by (2.4), and so $e = e^*$, as before.

It now follows from Lemma 1 that S is inverse.

To complete the proof, we note that if $x \in S$, then, by (2.4), $x = xx^*x$ and so also $x^* = x^*xx^*$, from which we see that $x^* = x^{-1}$. ∎

COROLLARY. *Every periodic (in particular, finite) special involution semigroup is inverse.*

3. Further examples

In this section we give examples of special involution semigroups that are not inverse.

EXAMPLE 1: Let S denote the Rees matrix semigroup $\mathcal{M}(G; 2, 2; P)$, where G is an infinite cyclic group, with generator a and identity e, and P is the matrix $\begin{bmatrix} e & a \\ a & e \end{bmatrix}$. Then S is completely simple (and so regular) [1, Chapter 3]. Define $*$ on S by the rule that

$$(a^r; i, j)^* = (a^r; j, i) \qquad (i, j \in (1, 2); r \in \mathbb{Z}).$$

It is routine to verify that $*$ is an involution on S.

Let T be a finite nonempty subset of S and let $r \in \mathbb{Z}$ be defined by

$$r: = \min\{s : (a^s; i, j) \in T \text{ for some } i, j\}.$$

Choose $i, j \in \{1, 2\}$ such that $(a^r; i, j) \in T$ and suppose that $(a^s; k, l), (a^t; m, n) \in T$ are such that

$$(a^r; i, j)(a^r; i, j)^* = (a^s; k, l)(a^t; m, n)^*.$$

Then $(a^{2r}; i, i) = (a^{s+t+u}; k, m)$, where $u = 1$ if $l \neq n$ and $u = 0$ if $l = n$. Hence

$$i = k = m, \tag{3.1}$$

and $s + t - 2r = -u$. Now $s + t - 2r \geqslant 0$, since $r \leqslant s$ and $r \leqslant t$; also $-u \leqslant 0$. Thus $s + t - 2r = 0 = u$; that is,

$$r = s = t \quad \text{and} \quad l = n. \tag{3.2}$$

Hence, from (3.1) and (3.2), $(a^s; k, l) = (a^t; m, n)$. This shows that S is special with respect to $*$.

EXAMPLE 2: Let X be a nonempty set and let $\mathcal{F}_X$ denote the free semigroup on X. An arbitrary element $w \in \mathcal{F}_X$ can be written uniquely in the form $w = x_1 x_2 \ldots x_n$

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for some positive integer n and some $x_1, x_2, \ldots, x_n \in X$. We denote n by $|w|$. Further, we define $w^* \in \mathcal{F}_X$ by $w^* : = x_n x_{n-1} \ldots x_1$. It is easy to see that $^* : \mathcal{F}_X \to \mathcal{F}_X$, given by $w \mapsto w^*$, is an involution on $\mathcal{F}_X$.

Let T be a nonempty subset of $\mathcal{F}_X$ (not necessarily finite) and let $t \in T$ be such that

$$|t| = \min\{|w| : w \in T\}.$$

Suppose that $tt^* = uv^*$ for some $u, v \in T$. Then $2|t| = |u| + |v|$ and so, since $|t| \leqslant |u|$ and $|t| \leqslant |v|$, we have that $|t| = |u| = |v|$. It follows that $t = u = v$. Thus $\mathcal{F}_X$ is a special involution semigroup.

EXAMPLE 3: A cancellative commutative semigroup S is said to be *totally ordered* if and only if, as a set, it admits a total ordering $\leqslant$ and the multiplication is such that

$$(\forall a, b, c \in S) \quad a < b \Rightarrow ac < bc. \tag{3.3}$$

Let S be a totally ordered cancellative commutative semigroup and let T be a finite nonempty subset of S. Choose t to be the least element of T under the ordering. Suppose that $t^2 = uv$ for some $u, v \in T$. Then $t \leqslant u$ and $t \leqslant v$. But if either $t < u$ or $t < v$ then, from (3.3) and transitivity, $t^2 < uv$, which is false. Hence $t = u = v$. This shows that S is special with respect to the identity automorphism.

The free commutative semigroup $\mathcal{FC}_X$ on a nonempty set X is a particular case: for we may assume that X is totally ordered and the order on X can be extended (lexicographically) to a total ordering on $\mathcal{FC}_X$ with respect to which (3.3) holds.

EXAMPLE 4: (T. Lavers). Let $\mathbb{H}$ denote the division ring of all real quaternions. Thus $\mathbb{H}$ is a four-dimensional algebra over the real field, with basis $\{1, i, j, k\}$, where $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. For an arbitrary element $x = a + bi + cj + dk \in \mathbb{H}$ (a, b, c, d real) we define $\overline{x}$ to be $a - bi - cj - dk$. Then $x\overline{x} = a^2 + b^2 + c^2 + d^2$, a non-negative real number, and so

$$x\overline{x} = 0 \Longleftrightarrow x = 0. \tag{3.4}$$

Moreover, the following hold:

$$(\forall x, y \in \mathbb{H}) \quad x = \overline{\overline{x}}, \quad \overline{x + y} = \overline{x} + \overline{y}, \quad \overline{xy} = \overline{y}\,\overline{x}. \tag{3.5}$$

Clearly, we may regard the complex field $\mathbb{C}$ as the subalgebra of $\mathbb{H}$ spanned by $\{1, i\}$; and the operation $x \mapsto \overline{x}$ restricted to $\mathbb{C}$ is then just complex conjugation.

Let M_n denote the ring of all $n \times n$ matrices over $\mathbb{H}$. For $X = [x_{rs}] \in M_n$ we define $X^* \in M_n$ by taking the (r, s)th entry of X^* to be $\overline{x_{sr}}$. From (3.5) it follows that, for all $X, Y \in M_n$,

$$X^{**} = X, \quad (X + Y)^* = X^* + Y^*, \quad (XY)^* = Y^* X^*.$$

Now let S be a subsemigroup of the multiplicative semigroup of M_n that is closed under the unary operation $^*: X \mapsto X^*$. Then * is an involution on S. We prove that it is special.

First, we note that, for all $X = [x_{rs}] \in M_n$, $\operatorname{trace}(XX^*) = \sum_{r,s=1}^{n} x_{rs}\overline{x_{rs}}$, a non-negative real number; and, by (3.4),

$$\operatorname{trace}(XX^*) = 0 \iff X = 0. \tag{3.6}$$

Consider a finite nonempty subset T of S. Choose $A \in T$ such that

$$\operatorname{trace}(AA^*) = \max\{\operatorname{trace}(BB^*) : B \in T\}.$$

Suppose that $AA^* = BC^*$ for some $B, C \in T$. Then $AA^* = (AA^*)^* = CB^*$ and so, in M_n, $(B-C)(B-C)^* = BB^* + CC^* - 2AA^*$. Hence

$$0 \leqslant \operatorname{trace}\left((B-C)(B-C)^*\right) = \operatorname{trace}(BB^*) + \operatorname{trace}(CC^*) - 2\operatorname{trace}(AA^*) \leqslant 0.$$

Thus $\operatorname{trace}\left((B-C)(B-C)^*\right) = 0$ and so, from (3.6), $B = C$.

The following are therefore special involution semigroups: the multiplicative semigroups of all $n \times n$ matrices over

(a) the semiring of all positive integers,
(b) the ring of all integers,
(c) the rational, real and complex fields,
(d) the division ring of all real quaternions.

Yet another case is the multiplicative semigroup of all $n \times n$ doubly stochastic matrices. (A real square matrix A is *doubly stochastic* if and only if (i) each entry is non-negative, (ii) the sum of the entries in each row is 1, (iii) the sum of the entries in each column is 1.)

4. Semigroup algebras

Let F be a field and let S be a semigroup. We denote the semigroup algebra of S over F by $F[S]$ [**1**, Section 5.2].

The next result provides some motivation for the study of special involution semigroups. It is a direct generalisation of [**3**, Lemma 2.3], which, in turn, extends [**5**, Theorem 3.2].

Lemma 2. *Let S be a special involution semigroup and let F be a subfield of the complex field that is closed under complex conjugation. Then $F[S]$ has no nonzero nil right ideals.*

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Proof: Let * denote the involution on S and let $\overline{\alpha}$ denote the complex conjugate of $\alpha \in F$. Then $\dagger\colon F[S] \to F[S]$ defined by

$$\left(\sum_i \alpha_i x_i\right)^\dagger = \sum_i \overline{\alpha_i} x_i^* \qquad (\alpha_i \in F,\, x_i \in S)$$

is an (algebra) involution. We show that

$$(\forall a \in F[S]) \quad aa^\dagger = 0 \Rightarrow a = 0. \tag{4.1}$$

Let $a \in F[S] \setminus \{0\}$. Then $a = \sum_{i=1}^{n} \alpha_i x_i$ for some positive integer n, some distinct $x_1, x_2, \dots, x_n \in S$ and some nonzero $\alpha_1, \alpha_2, \dots, \alpha_n \in F$. Hence

$$aa^\dagger = \sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} (x_i x_j^*). \tag{4.2}$$

Since S is a special involution semigroup, we can assume without loss of generality that

$$(\forall i, j \in \{1, 2, \dots, n\}) \quad x_1 x_1^* = x_i x_j^* \Rightarrow i = j. \tag{4.3}$$

Assume further that the x_i are numbered so that, for some r such that $1 \leqslant r \leqslant n$,

$$x_1 x_1^* = x_2 x_2^* = \dots = x_r x_r^* \neq x_i x_i^* \quad \text{if} \quad i > r.$$

Then, from (4.2) and (4.3), the coefficient of $x_1 x_1^*$ in $aa^\dagger$ is $\sum_{i=1}^{r} |\alpha_i|^2$. But this is nonzero and so $aa^\dagger \neq 0$. Thus (4.1) holds.

Now suppose that A is a nonzero nil right ideal of $F[S]$. Let $a \in A \setminus \{0\}$. Then $aa^\dagger \in A$ and so there exists a positive integer k such that $(aa^\dagger)^k = 0$. Write $m\colon = \min\{k\colon (aa^\dagger)^k = 0\}$. By (4.1), $m \geqslant 2$. Now put $b\colon = (aa^\dagger)^{m-1}$. Then $b = b^\dagger$ and so $bb^\dagger = b^2 = (aa^\dagger)^{2m-2} = 0$, since $2m - 2 \geqslant m$. Thus, by (4.1), $b = 0$, contrary to the choice of m. Hence no such right ideal A exists. ∎

In particular, for S as above, $\mathbb{C}[S]$ is semiprime and so from [**3**, Lemma 1.1] we can deduce the following theorem.

Theorem 2. *Let S be a special involution semigroup and let F be a field of characteristic zero. Then $F[S]$ is semiprime.*

Again, for S as above, we note that S^1 is also a special involution semigroup. (If $S \neq S^1$ we extend the involution * on S by taking $1^* = 1$.) Hence, by Lemma 2, $\mathbb{Q}[S^1]$ has no nonzero nil ideals. But $F[S^1]$ is semiprime for all fields F of characteristic zero, by Theorem 2. Thus, from [**3**, Lemma 1.2] we obtain Theorem 3 below.

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THEOREM 3. *Let S be a special involution semigroup and let F be a field of characteristic zero that is not algebraic over its prime subfield. Then $F[S]$ is semiprimitive (that is, has zero Jacobson radical).*

In particular, $\mathbb{R}[S]$ and $\mathbb{C}[S]$ are semiprimitive.

For S an inverse semigroup, this result is due to Domanov [**2**]. On the other hand, for S the multiplicative semigroup of $n \times n$ matrices over a subfield of $\mathbb{C}$ closed under complex conjugation (see Example 4), the result is a special case of [**4**, Theorem 3.6].

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