Nonlinear Differ. Equ. Appl. (2018) 25:35

1021-9722/18/040001-23
published online July 12, 2018
https://doi.org/10.1007/s00030-018-0526-6

Nonlinear Differential Equations and Applications NoDEA

On semilinear elliptic equations with diffuse measures

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Abstract. We consider semilinear equation of the form $-Lu = f(x,u)+\mu$, where L is the operator corresponding to a transient symmetric regular Dirichlet form $\mathcal{E}$, μ is a diffuse measure with respect to the capacity associated with $\mathcal{E}$, and the lower-order perturbing term $f(x,u)$ satisfies the sign condition in u and some weak integrability condition (no growth condition on $f(x,u)$ as a function of u is imposed). We prove the existence of a solution under mild additional assumptions on $\mathcal{E}$. We also show that the solution is unique if f is nonincreasing in u.

Mathematics Subject Classification. Primary 35J61, Secondary 35R06.

Keywords. Semilinear elliptic equation, Dirichlet operator, Measure data.

1. Introduction

Let E be a locally compact separable metric space, m be a positive Radon measure on E such that supp $[m] = E$, and let $(\mathcal{E}, D(\mathcal{E}))$ be a regular transient symmetric Dirichlet form on $L^2(E;m)$. In this paper, we consider semilinear equations of the form

$$-Lu = f(\cdot, u) + \mu. \tag{1.1}$$

In (1.1), $f : E \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the so-called "sign condition":

$$f(x,0) = 0, \qquad f(x,y)y \le 0, \quad x \in E,\, y \in \mathbb{R}, \tag{1.2}$$

and μ is a diffuse measure on E with respect to the capacity associated with $\mathcal{E}$, i.e. a bounded signed Borel measure on E which charges no set of capacity zero. As for L, we assume that it is the operator corresponding to $\mathcal{E}$, i.e. the unique nonpositive self-adjoint operator on $L^2(E;m)$ such that

$$D(L) \subset D(\mathcal{E}), \qquad \mathcal{E}(u,v) = (-Lu, v), \quad u \in D(L)\,, v \in D(\mathcal{E}),$$

This work was supported by Narodowe Centrum Nauki (Grant No. 2016/23/B/ST1/01543).

where $(\cdot,\cdot)$ stands for the usual scalar product in $L^2(E;m)$ (see [15, Section 1.3]). Problems of the form (1.1) with f satisfying the sign conditions are called absorption problems. The model examples of (1.1) are

$$-\Delta u = f(\cdot,u)+\mu \quad \text{in } D, \quad u=0 \quad \text{on } \partial D, \tag{1.3}$$

where $E:=D$ is a bounded open subset of $\mathbb{R}^d$ and Δ is the Laplace operator, and

$$-\Delta^{\alpha/2} u = f(\cdot,u)+\mu \quad \text{in } D, \quad u=0 \quad \text{on } \mathbb{R}^d\setminus D, \tag{1.4}$$

where $\Delta^{\alpha/2}$ is the fractional Laplace operator with $\alpha\in(0,2)$.

The study of problems of the form (1.3) with $\mu\in L^1(D;dx)$ was initiated by Brezis and Strauss [9] (in fact, in [9] more general second-order elliptic differential operator is considered). In [9] it is proved that if f satisfies the sign condition and

$$\forall a>0 \quad F_a\in L^1(E;m), \qquad \text{where } F_a(x)=\sup_{|y|\le a}|f(x,y)|,\, x\in E \tag{1.5}$$

with $E=D$, then there exists a solution to (1.3) for μ belonging to some class which is "arbitrarily smaller" than $L^1(D;dx)$. If f satisfies stronger monotonicity condition:

$$(f(x,y_1)-f(x,y_2))(y_1-y_2)\le 0, \quad x\in E,\, y_1,y_2\in\mathbb{R}, \tag{1.6}$$

then the solution exists for any $\mu\in L^1(D;dx)$ and is unique. Later, Gallouët and Morel [16] proved the existence of a solution to (1.3) for any $\mu\in L^1(D;dx)$ and f satisfying (1.2), (1.5). Orsina and Ponce [28] have subsequently generalized and strengthened this result by showing that a solution to (1.3) exists for any diffuse measure μ and any f satisfying (1.2) and an integrability condition weaker than (1.5).

Equations of the form (1.1) in the case where L is a general, possibly nonlocal, operator associated with a transient regular Dirichlet form were considered by Klimsiak and Rozkosz [22,24] in case f satisfies the monotonicity condition, and by Klimsiak [20] in case f satisfies the sign condition (in fact, in [20] systems of equations with right-hand side satisfying a generalized sign condition are considered).

In [20,22,24] the proofs of the existence results rely heavily on probabilistic methods. In particular, we make an extensive use of the theory of backward stochastic differential equations and we use some results from stochastic analysis and probabilistic potential theory. In the present paper we give new, rather short analytical proofs of some of the results of [20,22]. We are motivated by the desire to make them accessible to people working in PDEs that are not familiar with probabilistic methods.

Let $D_e(\mathcal{E})$ denote the extended Dirichlet space of $(\mathcal{E},D(\mathcal{E}))$. In the present paper we provide a proof of the existence of a solution u in the sense of duality (or, equivalently, renormalized solution; see Sect. 3) to (1.1) for f satisfying (1.2) and (1.5) under the following additional assumption on $\mathcal{E}$:

$$\begin{aligned}&\text{if } \{u_n\}\subset D_e(\mathcal{E}) \text{ and } \sup_{n\ge1}\mathcal{E}(u_n,u_n)<\infty,\\ &\quad \text{then, up to a subsequence, } \{u_n\} \text{ converges } m\text{-a.e.}\end{aligned} \tag{1.7}$$

We also show that if u is a solution to (1.1), then $T_k(u) = ((-k)\vee u)\wedge k \in D_e(\mathcal{E})$ for every $k > 0$ and

$$\mathcal{E}(T_k(u), T_k(u)) \leq 2k\|\mu\|_{TV},$$

where $\|\mu\|_{TV}$ is the total variation norm of μ. Furthermore, if (1.6) is satisfied, then the solution u is unique.

Condition (1.7) holds true in many interesting situations. For instance, it holds if

$$\text{the embedding } V_1 \hookrightarrow L^2(E; m) \text{ is compact,} \tag{1.8}$$

where V_1 denotes the space $D(\mathcal{E})$ equipped with the norm determined by the form $\mathcal{E}_1(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)$. Another condition, which is often satisfied in practice and implies (1.7), is the so called absolute continuity condition saying that

$$R_\alpha(x,\cdot) \ll m \quad \text{for any } \alpha > 0 \text{ and } x \in E, \tag{1.9}$$

where $R_\alpha(x,\cdot)$ is the resolvent kernel associated with $\mathcal{E}$. For symmetric forms considered in this paper, condition (1.9) is equivalent to the condition

$$P_t(x,\cdot) \ll m \quad \text{for any } t > 0 \text{ and } x \in E, \tag{1.10}$$

where $P_t(x,\cdot)$ is the transition kernel associated with $\mathcal{E}$.

The main idea of our proofs resembles the idea used in case of problem (1.3) (see the proof of Theorem B.4 in Brezis et al. [8] and also Ponce [31, Chapter 19]). Let V denote the extended space $D_e(\mathcal{E})$ equipped with the norm determined by $\mathcal{E}$. We first prove the existence of a solution to (1.1) with $\mu \in \mathcal{M}_{0,b} \cap V'$, where $\mathcal{M}_{0,b}$ is the set of all diffuse measures on E and V' is the dual of V. This step can be viewed as some modification of the result of Brezis and Browder [7] on absorption problems (1.3) with $\mu \in H^{-1}(D)$. To get the existence for general $\mu \in \mathcal{M}_{0,b}$, we approximate it by a suitably chosen sequence $\{\mu_n\} \subset \mathcal{M}_{0,b} \cap V'$ and show that solutions u_n of (1.1) corresponding to the measures μ_n converge to a solution of (1.1). In this second step we use some a priori estimates for u_n in V and condition (1.7). In [25] it is proved that any $\mu \in \mathcal{M}_{0,b}$ admits decomposition of the form $\mu = g + \nu$ with $g \in L^1(E; m)$ and $\nu \in \mathcal{M}_{0,b} \cap V'$ (this generalizes the corresponding result proved by Boccardo et al. [5] for the form associated with Δ). Therefore, similarly to [8], in the second step of the proof it is enough to approximate by $\{\mu_n\}$ the measure $\mu = g \cdot m$. This, however, does not simplify the reasoning, so in the present paper we give a direct approximation of $\mu \in \mathcal{M}_{0,b}$ (without recourse to [25]).

In the present paper we confine ourselves to single equation with operator corresponding to symmetric regular Dirichlet forms. For results (proved with the help of probabilistic methods) for quasi-regular, possibly nonsymmetric forms, we refer the reader to [24], and for results for systems of equations to [20]. Also note that equations with $f = 0$ but Δ replaced by the Schrödinger operator are treated in [29] and [31, Chapter 22].

In the paper we deal exclusively with equations with diffuse measures. The theory of semilinear equations with general bounded measures is much more subtle. In this case (1.3) with f satisfying (1.6) need not have a solution (see [2,3,8]). Results on (1.3) with general bounded measure μ and f satisfying the monotonicity condition are found in [3,8,13], and for equations with f satisfying the sign condition (1.2) in [31, Chapter 21]. The Dirichlet problem for linear equations with nonlocal operators and bounded measure μ is studied in [19,26,30]. In Klimsiak [21] general equations of the form (1.1) with general bounded measure μ and f satisfying (1.6) are considered. The question whether one can extend the existence results of [31] to some nonlocal operators or extend some existence results of [21] to f satisfying (1.2) remains open.

2. Preliminaries

In this paper, E is a locally compact separable metric space and m is a Radon measure such that $\operatorname{supp}[m] = E$, i.e. m is a nonnegative measure on the σ-field of Borel subsets of E which is finite on compact sets and strictly positive on nonempty open sets.

In what follows $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric regular Dirichlet form on $L^2(E;m)$. We denote by $(\cdot,\cdot)$ the usual inner product in $L^2(E;m)$. As usual, for $\lambda \ge 0$ we set $\mathcal{E}_\lambda(u,v) = \mathcal{E}(u,v) + \lambda(u,v)$, $u,v \in D(\mathcal{E})$.

In the whole paper we assume that $(\mathcal{E}, D(\mathcal{E}))$ is transient. Recall that this means that there exists a bounded strictly m-a.e. positive $g \in L^1(E;m)$ such that

$$\int_E |u(x)|g(x)\, m(dx) \le \mathcal{E}(u,u), \quad u \in D(\mathcal{E}). \tag{2.1}$$

For an equivalent formulation, see [15, Section 1.5]. The extended Dirichlet space associated with $(\mathcal{E}, D(\mathcal{E}))$ (see [15, Section 1.5] for the definition) will be denoted by $(\mathcal{E}, D_e(\mathcal{E}))$. Note that $D_e(\mathcal{E})$ with the inner product $\mathcal{E}$ is a Hilbert space (see [15, Theorem 1.5.3]). In the sequel this space will be denoted by V. We denote by V' the dual space of V. The duality pairing between V' and V will be denoted by $\langle\langle \cdot,\cdot \rangle\rangle$.

In the paper we define 0-order quasi notions with respect to $\mathcal{E}$ (capacity $\mathrm{Cap}_{(0)}$, exceptional sets, nests, quasi-continuity) as in [15, Chapter 2, page 74]. Recall that $\mathrm{Cap}_{(0)}$ is defined as follows. For an open subset U of E, we set

$$\mathcal{L}_U^{(0)} = \{u \in D_e(\mathcal{E}) : u \ge 1 \ m\text{-a.e. on } U\}$$

and

$$\mathrm{Cap}_{(0)}(U) = \begin{cases} \inf\{\mathcal{E}(u,u) : u \in \mathcal{L}_U^{(0)}\} & \text{if} \quad \mathcal{L}_U^{(0)} \neq \emptyset, \\ \infty & \text{if} \quad \mathcal{L}_U^{(0)} = \emptyset. \end{cases}$$

Then, as usual, for an arbitrary $A \subset E$, we set

$$\mathrm{Cap}_{(0)}(A) = \inf\{\mathrm{Cap}_{(0)}(U) : U \text{ open}, \ U \supset A\}.$$

We say that $A \subset E$ is exceptional if $\mathrm{Cap}_{(0)}(A) = 0$, and we say that a property of points in E holds quasi-everywhere (q.e. in abbreviation) if it holds outside some exceptional subset of E.

For a measure μ on E and a function $u : E \to \mathbb{R}$, we use the notation

$$\langle \mu, u \rangle = \int_E u(x)\, \mu(dx),$$

whenever the integral is well defined. For a signed Borel measure μ, we denote by μ^+ and μ^- its positive and negative parts, and by $|\mu|$ the total variation measure, i.e. $|\mu| = \mu^+ + \mu^-$. We denote by $\mathcal{M}_b$ the space of all finite signed Borel measures on E endowed with the total variation norm $\|\mu\|_{TV} = |\mu|(E)$, and by $\mathcal{M}_{0,b}$ the subspace of $\mathcal{M}_b$ consisting of all measures charging no set of capacity $\mathrm{Cap}_{(0)}$ zero. Elements of $\mathcal{M}_{0,b}$ are called diffuse measures.

We write $\mu \in \mathcal{M}_{0,b} \cap V'$ if for some $c > 0$,

$$\langle |\mu|, |\tilde{u}| \rangle \le c\mathcal{E}(u,u)^{1/2}, \quad u \in D_e(\mathcal{E}),$$

where $\tilde{u}$ denotes a quasi-continuous m-version of u (see [15, Theorem 2.1.7]). Elements of $\mathcal{M}_{0,b} \cap V'$ are called measures of finite 0-order energy integral. If $\mu \in \mathcal{M}_{0,b} \cap V'$, then

$$\langle \mu, \tilde{u} \rangle = \langle\langle \mu, u \rangle\rangle, \quad u \in D_e(\mathcal{E}). \tag{2.2}$$

If $f \in V'$, then by Riesz's theorem there is a unique element $Gf \in D_e(\mathcal{E})$ such that

$$\mathcal{E}(Gf, u) = \langle\langle f, u \rangle\rangle, \quad u \in D_e(\mathcal{E}). \tag{2.3}$$

In particular, if $\mu \in \mathcal{M}_{0,b} \cap V'$, then the function $G\mu$ is well defined and belongs to V.

Let $\mathcal{B}^+(E)$ (resp. $\mathcal{B}_b(E)$) denote the set of all positive (resp. bounded) real Borel functions on E, and let $(G_\alpha)_{\alpha>0}$ denote the strongly continuous resolvent on $L^2(E;m)$ associated with $(\mathcal{E}, D(\mathcal{E}))$. Recall that αG_α is Markovian for each $\alpha > 0$, i.e. $0 \le \alpha G_\alpha f \le 1$ m-a.e. whenever $f \in L^2(E;m)$ and $0 \le f \le 1$ m-a.e. Since G_α is positivity preserving, we can extend it to any positive $f \in \mathcal{B}^+(E)$ by

$$G_\alpha f(x) = \lim_{n\to\infty} G_\alpha f_n(x) = \sup_{n\ge 1} G_\alpha f(x) \quad \text{for } m\text{-a.e. } x \in E, \tag{2.4}$$

where $\{f_n\} \subset L^2(E;m)$ is a nondecreasing sequence of positive functions converging m-a.e. to f. It is clear that $G_\alpha f$ does not depend on the choice of the sequence $\{f_n\}$. By the resolvent equation, if $\beta > \alpha > 0$, then $G_\alpha f \le G_\beta f$ m-a.e. for any $\mathcal{B}^+(E)$. Therefore for $f \in \mathcal{B}^+(E)$ we can set

$$Gf(x) := G_0 f(x) = \lim_{\alpha\downarrow 0} G_\alpha f(x) = \sup_{\alpha>0} G_\alpha f(x) \quad \text{for } m\text{-a.e. } x \in E. \tag{2.5}$$

By [15, Lemma 2.2.11], for $f \in \mathcal{B}^+(E)$ such that $f \cdot m \in V'$, Gf defined by (2.5) coincides with Gf of (2.3).

An increasing sequence $\{F_n\}$ of closed subsets of E is called a generalized nest if $\mathrm{Cap}_{(0)}(K \setminus F_n) \to 0$ for any compact $K \subset E$. A Borel measure μ on E is called smooth if there exists a generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot \mu \in \mathcal{M}_{0,b}$, $n \ge 1$. In particular each diffuse measure is smooth. If $\{F_n\}$ is a generalized nest such that $\mathbf{1}_{F_n} \cdot \mu \in \mathcal{M}_{0,b}$ then

$$\mu\Big(E\setminus\bigcup_{n=1}^{\infty}F_n\Big)=0 \tag{2.6}$$

(see [15, (2.2.18)]).

By the 0-order version of [15, Theorem 2.2.4] (see the remark following [15, Corollary 2.2.2]), for each smooth measure μ there exists a generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n}\cdot\mu\in\mathcal{M}_{0,b}\cap V'$. Therefore, for a positive smooth measure μ, we may define a function $G\mu$ with values in $[0,\infty]$ by

$$G\mu(x)=\lim_{n\to\infty}G\mu_n(x)=\sup_{n\ge1}G\mu_n(x),\quad x\in E, \tag{2.7}$$

where $\mu_n=\mathbf{1}_{F_n}\cdot\mu$ and $\{F_n\}$ is a generalized nest such that $\mu_n\in\mathcal{M}_{0,b}\cap V'$, $n\ge1$. Note that $G\mu$ is defined uniquely up to m-equivalence.

In the paper, for a function u on E, we denote by $\tilde{u}$ its quasi-continuous m-version (whenever it exists). We will freely use, without explicit mention, the following fact: if $u_1\le u_2$ m-a.e. and u_1,u_2 have quasi-continuous m-versions, then $\tilde{u}_1\le\tilde{u}_2$ q.e. (see [15, Lemma 2.1.4]).

Let $\beta>0$. In the proof of the lemma below we will need the symmetric form $\mathcal{E}^{(\beta)}$ defined by

$$\mathcal{E}^{(\beta)}(u,v)=\beta(u,v-\beta G_\beta v),\quad u,v\in L^2(E;m).$$

Since βG_β is a symmetric linear operator on $L^2(E;m)$, by [15, Lemma 1.4.1] there exists a unique nonnegative symmetric Radon measure σ on the product space $E\times E$ such that for any Borel functions $u,v\in L^1(E;m)$,

$$(u,\beta G_\beta)=\int_{E\times E}u(x)v(x)\,\sigma_\beta(dx\,dy). \tag{2.8}$$

Since βG_β is Markovian, from (2.8) it follows that $\sigma_\beta(E\times B)\le m(B)$ for any Borel $B\subset E$. Let s_β denote the Radon-Nikodym derivative of the measure $B\mapsto\sigma_\beta(E\times B)$ with respect to m. Then $0\le s_\beta\le1$ m-a.e., and by a direct computation one can check that for a Borel $u\in L^2(E;m)$ one can rewrite $\mathcal{E}^{(\beta)}(u,u)$ in the form

$$\begin{aligned}\mathcal{E}^{(\beta)}(u,u)=&\frac{\beta}{2}\int_{E\times E}(u(x)-u(y))^2\sigma_\beta(dx\,dy)\\&+\beta\int_E(u(x))^2(1-s_\beta(x))\,m(dx)\end{aligned} \tag{2.9}$$

(see [15, (1.4.8)]). The expression (2.9) can be extended to any Borel function u on E. Furthermore, by [15, Theorem 1.5.2(ii)], for any Borel $u\in D_e(\mathcal{E})$, $\mathcal{E}^{(\beta)}(u,u)$ increases to $\mathcal{E}(u,u)$ as $\beta\to\infty$.

Lemma 2.1. *Let $u\in D_e(\mathcal{E})$, and let $\psi:\mathbb{R}\to\mathbb{R}$ be an increasing function such that $\psi(0)=0$ and ψ is Lipschitz continuous with Lipschitz constant 1. Then*

$$\mathcal{E}(\psi(u),\psi(u))\le\mathcal{E}(u,\psi(u)).$$

Proof. By [15, Theorem 1.5.3(δ)], $\psi(u)\in D_e(\mathcal{E})$. Furthermore, for any $\beta>0$ we have

$$\mathcal{E}^{(\beta)}(u,\psi(u)) = \frac{1}{4}\{\mathcal{E}^{(\beta)}(u+\psi(u),u+\psi(u)) - \mathcal{E}^{(\beta)}(u-\psi(u),u-\psi(u))\}$$
$$= \frac{\beta}{2}\int_{E\times E}(u(x)-u(y))(\psi(u(x))-\psi(u(y)))\sigma_\beta(dx\,dy)$$
$$+\beta\int_E u(x)\psi(u(x))(1-s_\beta(x))\,m(dx)$$

and

$$\mathcal{E}^{(\beta)}(\psi(u),\psi(u)) = \frac{\beta}{2}\int_{E\times E}(\psi(u(x))-\psi(u(y)))^2\sigma_\beta(dx\,dy)$$
$$+\beta\int_E(\psi(u(x)))^2(1-s_\beta(x))\,m(dx).$$

From this we conclude that for any $\beta>0$,

$$\mathcal{E}^{(\beta)}(\psi(u),\psi(u)) \le \mathcal{E}^{(\beta)}(u,\psi(u)).$$

Letting $\beta\to\infty$ and using [15, Theorem 1.5.2(ii)] we obtain the desired inequality. □

For $k\ge 0$ and $u:E\to\mathbb{R}$, we write

$$T_k(u)(x) = ((-k)\vee u(x))\wedge k,\quad x\in E.$$

Since $\psi(y) = ((-k)\vee y)\wedge y$, $y\in\mathbb{R}$, satisfies the assumptions of Lemma 2.1, if $u\in D_e(\mathcal{E})$, then $T_k u\in D_e(\mathcal{E})$ and for every $k\ge 0$,

$$\mathcal{E}(T_k(u),T_k(u)) \le \mathcal{E}(u,T_k(u)). \tag{2.10}$$

Lemma 2.2. *Let $\mu\in\mathcal{M}^+_{0,b}$. Then $G\mu$ has a quasi-continuous m-version.*

Proof. Let $\{H_n\}$ be a generalized nest such that $\mu_n = \mathbf{1}_{H_n}\mu\in\mathcal{M}_{0,b}\cap V'$, $n\ge 1$, and $\mu(E\setminus H_n)\le 2^{-3n}$. Set $u_n = G\mu_n$, $u = G\mu$. It is clear that $u_n\nearrow u$ m-a.e. By the 0-order version of [15, (2.1.10)], for all $\varepsilon,\delta>0$ we have

$$\mathrm{Cap}_{(0)}(\tilde u_{n+1}-\tilde u_n>\varepsilon) = \mathrm{Cap}_{(0)}(T_{\varepsilon+\delta}(\tilde u_{n+1}-\tilde u_n)>\varepsilon)$$
$$\le \varepsilon^{-2}\mathcal{E}(T_{\varepsilon+\delta}(u_{n+1}-u_n),T_{\varepsilon+\delta}(u_{n+1}-u_n))$$

Since $\tilde u_{n+1}-\tilde u_n\in D_e(\mathcal{E})$, it follows from the above inequality and (2.10) that

$$\mathrm{Cap}_{(0)}(\tilde u_{n+1}-\tilde u_n>\varepsilon) \le \varepsilon^{-2}\mathcal{E}(u_{n+1}-u_n,T_{\varepsilon+\delta}(u_{n+1}-u_n))$$
$$= \varepsilon^{-2}\int_E T_{\varepsilon+\delta}(\tilde u_{n+1}-\tilde u_n)\,d(\mu_{n+1}-\mu_n)$$
$$\le (\varepsilon+\delta)\varepsilon^{-2}\mu(E\setminus H_n).$$

Taking $\varepsilon = 2^{-n}$ and letting $\delta\searrow 0$ we get

$$\mathrm{Cap}_{(0)}(\tilde u_{n+1}-\tilde u_n>2^{-n}) \le 2^{-2n},\quad n\ge 1. \tag{2.11}$$

By [15, Theorem 2.1.2], there exists a nest $\{G_k\}$ such that $\tilde u_n$ is continuous on G_k for all $k,n\ge 1$. Let $F_n = \bigcap_{k=n}^{\infty}(E\setminus U_k)\cap G_k$, where $U_k = \{\tilde u_{k+1}-\tilde u_k>2^{-k}\}$. From (2.11) it is clear that $\{F_n\}$ is a nest and $\tilde u$ defined q.e. as $\tilde u = \lim_{n\to\infty}\tilde u_n$ is quasi-continuous. Of course, $\tilde u$ is an m-version of u. □

Lemma 2.3. *Let $\mu \in \mathcal{M}_{0,b}$. Then for every $k \geq 0$, $T_k(G\mu) \in D_e(\mathcal{E})$ and*

$$\mathcal{E}(T_k(G\mu), T_k(G\mu)) \leq k\|\mu\|_{TV}. \tag{2.12}$$

Proof. By Lemma 2.2, $G\mu^+, G\mu^-$ are finite m-a.e., so $G\mu$ is well defined m-a.e. Let $\{F_n\}$ be a generalized nest for μ such that $\mu_n = \mathbf{1}_{F_n}\mu \in V'$, $n \geq 1$. Set $u_n = G\mu_n$, $u = G\mu$. Then $u_n \in D_e(\mathcal{E})$ and by (2.2) and (2.3),

$$\mathcal{E}(u_n, T_k(u_n)) = \int_E \widetilde{T_k(u_n)}(x)\,\mu_n(dx) \leq k\|\mu\|_{TV}.$$

By (2.10), $\mathcal{E}(T_k(u_n), T_k(u_n)) \leq \mathcal{E}(u_n, T_k(u_n))$. Hence

$$\mathcal{E}(T_k(u_n), T_k(u_n)) \leq k\|\mu\|_{TV}. \tag{2.13}$$

In particular, $\{T_k(u_n)\}_n$ is weakly relatively compact in V. Taking a subsequence if necessary, we can assume that $T_k(u_n) \to v$ weakly in V as $n \to \infty$. By the Banach-Saks theorem, there is a subsequence (n_l) such that the Cesàro mean $\{v_N := \frac{1}{N}\sum_{l=1}^N T_k(u_{n_l})\}$ converges strongly to v in V. Hence, by (2.1), $v_N \to v$ in $L^1(E; g\cdot m)$. On the other hand, $u_n \to u$ m-a.e., so $v_N \to T_k(u)$ m-a.e. Consequently, $T_k(u_n) \to T_k(u)$ weakly in V as $n \to \infty$. Therefore letting $n \to \infty$ in (2.13) yields (2.12). □

Lemma 2.4. *Let $\mu \in \mathcal{M}^+_{0,b}$, and let $\widetilde{G\mu}$ be a quasi-continuous m-version of $G\mu$. Then for every $\varepsilon > 0$,*

$$\mathrm{Cap}_{(0)}(\widetilde{G\mu} > \varepsilon) \leq \varepsilon^{-1}\|\mu\|_{TV}.$$

Proof. By Lemma 2.3, $T_k(\widetilde{G\mu}) \in D_e(\mathcal{E})$ and (2.12) holds true. By this and the 0-order version of [15, (2.1.10)], for all $\varepsilon, \delta > 0$ we have

$$\begin{aligned}\mathrm{Cap}_{(0)}(\widetilde{G\mu} > \varepsilon) = \mathrm{Cap}_{(0)}(T_{\varepsilon+\delta}(\widetilde{G\mu}) > \varepsilon) &\leq \varepsilon^{-2}\mathcal{E}(T_{\varepsilon+\delta}(\widetilde{G\mu}), T_{\varepsilon+\delta}(\widetilde{G\mu}))\\ &\leq \varepsilon^{-2}(\varepsilon+\delta)\|\mu\|_{TV},\end{aligned}$$

which implies the desired inequality. □

Lemma 2.5. *Assume that $\{\mu_n\} \subset \mathcal{M}^+_{0,b}$ and $\|\mu_n\|_{TV} \to 0$. Then, up to a subsequence, $\widetilde{G\mu_n} \to 0$ q.e.*

Proof. We can and do assume that $\|\mu_n\| \leq 2^{-2n}$, $n \geq 1$. Then by Lemma 2.4,

$$\mathrm{Cap}_{(0)}(\widetilde{G\mu_n} > 2^{-n}) \leq 2^{-n}.$$

Let $F = \bigcup_{n\geq 1}\bigcap_{k\geq n}\{\widetilde{G\mu_k} \leq 2^{-k}\}$. By the above inequality, $\mathrm{Cap}_{(0)}(E\setminus F) = 0$. This proves the lemma because by the definition of F, $\widetilde{G\mu_n} \to 0$ q.e. on F. □

Lemma 2.6. *There exists a strictly positive function $g \in \mathcal{B}(E)$ such that $\|Gg\|_\infty < \infty$.*

Proof. Since $\mathcal{E}$ is transient there exists a strictly positive $h \in L^1(E; m)$ such that $Gh < \infty$. By [15, Theorem 2.2.4], there exist a nest $\{F_n\}$ such that $\mathbf{1}_{F_n}\cdot h \in L^1(E; m) \cap V'$. Write $H_{n,k} = \{G(\mathbf{1}_{F_n}h) \leq k\}$ and $H_k = \{Gh \leq k\}$.

By [15, Lemma 2.2.4], $G(\mathbf{1}_{F_n\cap H_{n,k}}h)\le k$. Letting $n\to\infty$ yields $G(\mathbf{1}_{H_k}h)\le k$. Set $g=\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)}\mathbf{1}_{H_n}h$. Then

$$Gg=\sum_{n=0}^{\infty}\frac{1}{2^n(n+1)}G(\mathbf{1}_{H_n}h)\le\sum_{n=0}^{\infty}2^{-n},$$

which proves the lemma. □

Lemma 2.7. *For any positive $\eta\in L^1(E;m)$ such that $\|G\eta\|_\infty<\infty$ and any positive $\mu\in\mathcal{M}_{0,b}$,*

$$\langle\mu,\widetilde{G\eta}\rangle=\int_E\eta(x)G\mu(x)\,m(dx),\tag{2.14}$$

where $\widetilde{G\eta}$ is a quasi-continuous m-version of $G\eta$.

Proof. We first assume that $\mu\in\mathcal{M}_{0,b}\cap V'$ and $\eta\in L^1(E;m)\cap V'$. Then

$$\mathcal{E}(G\mu,G\eta)=\int_E\widetilde{G\eta}(x)\,\mu(dx),\qquad\mathcal{E}(G\eta,G\mu)=\int_E\eta(x)G\mu(x)\,m(dx).$$

Since $\mathcal{E}$ is symmetric, this implies (2.14). Now assume that $\mu\in\mathcal{M}_{0,b}$, $\eta\in L^1(E;m)$ and $G\eta$ is bounded. Let $\{F_n\}$ be a generalized nest such that $\eta_n=\mathbf{1}_{F_n}\cdot\eta\in L^1(E;m)\cap V'$ and $\mu_n=\mathbf{1}_{F_n}\cdot\mu\in\mathcal{M}_{0,b}\cap V'$. By what has already been proved,

$$\langle\mu_n,\widetilde{G\eta_n}\rangle=\int_E\eta(x)G\mu_n(x)\,m(dx),\quad n\ge1.$$

Letting $n\to\infty$ we get (2.14). □

In the rest of this section we assume that the absolutely continuity condition (1.9) is satisfied. Condition (1.9) was introduced by Meyer [27]. It is sometimes called condition (L) (see [12, p. 246]). By [15, Theorem 4.2.4], condition (1.9) is equivalent to (1.10). If (1.9) is satisfied, then for any $\alpha>0$ there exists a positive $\mathcal{B}(E)\otimes\mathcal{B}(E)$-measurable function $r_\alpha:E\times E\to\mathbb{R}$ such that $r_\alpha(x,y)=r_\alpha(y,x)$, $x,y\in E$, and for any $f\in\mathcal{B}^+(E)$,

$$G_\alpha f(x)=\int_E r_\alpha(x,y)f(y)\,m(dy)\quad\text{for}\quad m\text{-a.e. } x\in E.\tag{2.15}$$

Moreover, there exists a positive $\mathcal{B}(E)\otimes\mathcal{B}(E)$-measurable function $r:E\times E\to\mathbb{R}$ such that $r(x,y)=r(y,x)$, $x,y\in E$, and for any $f\in\mathcal{B}^+(E)$,

$$Gf(x)=\int_E r(x,y)f(y)\,m(dy),\quad\text{for}\quad m\text{-a.e. } x\in E.$$

In fact, $r(x,y)=\lim_{\alpha\downarrow0}r_\alpha(x,y)$ (see the remarks in [4, p. 256]).

Lemma 2.8. *Assume that (1.9) is satisfied. If $\mu\in\mathcal{M}_{0,b}$, then for m-a.e. $x\in E$,*

$$G\mu(x)=\int_E r(x,y)\,\mu(dy).\tag{2.16}$$

Proof. Let $\{F_n\}$ be a generalized nest such that $\mu_n = \mathbf{1}_{F_n} \cdot \mu \in \mathcal{M}_{0,b} \cap V'$. By [15, Exercise 4.2.2, Lemma 5.1.3], for any $\alpha > 0$ we have

$$G_\alpha \mu_n(x) = \int_E r_\alpha(x,y)\, \mu_n(dy)$$

for m-a.e. $x \in E$. Letting $\alpha \downarrow 0$ in the above equality yields (2.16) with μ replaced by μ_n. Then, using (2.6), (2.7) and the monotone convergence, we get (2.16) for μ. □

3. Existence and uniqueness of solutions

Throughout this section, we assume that $\mu \in \mathcal{M}_{0,b}$ and $f : E \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e. $f(\cdot, y)$ is measurable on E for each fixed $y \in \mathbb{R}$, and $f(x, \cdot)$ is continuous on $\mathbb{R}$ for each fixed $x \in E$.

Following [22] we adopt the following definition.

Definition 3.1. We say that $u : E \to \mathbb{R}$ is a solution of (1.1) (in the sense of duality) if

(a) $f(\cdot, u) \in L^1(E; m)$,
(b) for any $\eta \in L^1(E; m)$ such that $G|\eta|$ is bounded we have

$$\int_E u(x)\eta(x)\, m(dx) = \int_E f(x, u(x)) G\eta(x)\, m(dx) + \int_E \widetilde{G\eta}(x)\, \mu(dx). \tag{3.1}$$

Remark 3.2. If u is a solution of (1.1), then u has a quasi-continuous m-version, because then $u = G(f(\cdot, u) \cdot m + \mu)$ m-a.e. by Lemma 2.7, so the existence of a quasi-continuous m-version follows from Lemma 2.2.

Recall that an increasing sequence $\{F_n\}$ of closed subsets of E is called a generalized nest if $\mathrm{Cap}_{(0)}(K \setminus F_n) \to 0$ for any compact $K \subset E$.

Proposition 3.3. *Let u be a measurable function such that $f(\cdot, u) \in L^1(E; m)$. Then the following assertions are equivalent:*

(i) *u is a solution to (1.1).*
(ii) *$u = G(f(\cdot, u)) + G\mu$ m-a.e.*
(iii) *For any generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot (f(\cdot, u) \cdot m + \mu) \in \mathcal{M}_{0,b} \cap V'$ we have $u_n \to u$ m-a.e., where $u_n \in D_e(\mathcal{E})$ is the unique solution of the problem*

$$\mathcal{E}(u_n, \eta) = \langle \mathbf{1}_{F_n} f(\cdot, u) \cdot m, \eta \rangle + \langle \mathbf{1}_{F_n} \cdot \mu, \tilde{\eta} \rangle, \quad \eta \in D_e(\mathcal{E}) \tag{3.2}$$

(iv) *For some generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot (f(\cdot, u) \cdot m + \mu) \in \mathcal{M}_{0,b} \cap V'$ we have $u_n \to u$ m-a.e., where $u_n \in D_e(\mathcal{E})$ is the unique solution of (3.2).*
(v) *For any generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot (f(\cdot, u) \cdot m + \mu) \in \mathcal{M}_{0,b} \cap V'$ we have $\tilde{u}_n \to \tilde{u}$ q.e., where $u_n \in D_e(\mathcal{E})$ is the unique solution of (3.2).*
(vi) *For some generalized nest $\{F_n\}$ such that $\mathbf{1}_{F_n} \cdot (f(\cdot, u) \cdot m + \mu) \in \mathcal{M}_{0,b} \cap V'$ we have $\tilde{u}_n \to \tilde{u}$ q.e., where $u_n \in D_e(\mathcal{E})$ is the unique solution of (3.2).*

Proof. The equivalence of (i) and (ii) follows from Lemma 2.7. Obviously, (iii) implies (iv), (v) implies (iii) and (vi), and (vi) implies (iv). What is left ist to show that (ii) implies (v) and (iv) implies (ii). Let $\{F_n\}$ be a generalized nest for $f(\cdot,u)\cdot m+\mu$, and let

$$u_n = G(\mathbf{1}_{F_n} f(\cdot,u)) + G(\mathbf{1}_{F_n}\cdot\mu). \tag{3.3}$$

By the definition of $\{F_n\}$, $u_n \in V$. Moreover, by (2.2) and (2.3), u_n satisfies (3.2). If (ii) is satisfied, then $|\tilde u - \tilde u_n| \le \widetilde{G\mathbf{1}_{E\setminus F_n}|f(\cdot,u)|} + \widetilde{G\mathbf{1}_{E\setminus F_n}\cdot|\mu|}$ q.e. Hence, by Lemma 2.5, $\tilde u_n \to \tilde u$ q.e. Now assume (iv). By (2.3), since u_n solves (3.2), it is given by (3.3). Therefore letting $n\to\infty$ in (3.3) we get (ii). □

Remark 3.4. Let $f(\cdot,u)\in L^1(E;m)$. By Lemma 2.8 and Proposition 3.3, if (1.9) is satisfied, then u is a solution to (1.1) if and only if

$$u(x) = \int_E f(y,u(y))r(x,y)\,m(dy) + \int_E r(x,y)\,\mu(dy)$$

for m-a.e. $x\in E$.

Remark 3.5. In [23] the following definition of a solution of (1.1) is introduced: $u: E\to\mathbb{R}$ is a renormalized solution of (1.1) if

(a) $f(\cdot,u)\in L^1(E;m)$ and $T_k(u)\in D_e(\mathcal{E})$ for every $k>0$,
(b) there exists a sequence $\{\nu_k\}\subset\mathcal{M}_{0,b}(E)$ such that $\|\nu_k\|_{TV}\to 0$ as $k\to\infty$ and for every $k\in\mathbb{N}$ and every bounded $v\in D_e(\mathcal{E})$,

$$\mathcal{E}(T_k(u),v) = \langle f(\cdot,u)\cdot m+\mu,\tilde v\rangle + \langle \nu_k,\tilde v\rangle.$$

Note that in case of local operators, this is essentially [11, Definition 2.29]. By [22, Proposition 5.3] and [23, Theorem 3.5], u is a solution of (1.1) in the sense of Definition 3.1 if and only if it is a renormalized solution.

Lemma 3.6. (i) *Let u be a solution of (1.1) with f satisfying (1.2). Then for every $a>0$,*

$$\int_{\{|u|>a\}} |f(x,u(x))|\,m(dx) \le \|\mathbf{1}_{\{|\tilde u|>a\}}\cdot\mu\|_{TV}.$$

(ii) *Assume that f satisfies (1.6). If u_i, $i=1,2$, is a solution of (1.1) with μ replaced by $\mu_i\in\mathcal{M}_{0,b}$, then*

$$\|f(\cdot,u_1)-f(\cdot,u_2)\|_{L^1(E;m)} \le \|\mu_1-\mu_2\|_{TV}.$$

Proof. Let $\{F_n\}$ be a generalized nest such that $\mathbf{1}_{F_n}(|f(\cdot,u)|\cdot m+|\mu|)\in\mathcal{M}_{0,b}\cap V'$. For $n\ge 1$ we set $f_n=\mathbf{1}_{F_n}f(\cdot,u)$, $\mu_n=\mathbf{1}_{F_n}\cdot\mu$ and $u_n=G(f_n\cdot m+\mu_n)$. Then $u_n\in D_e(E)$. For $a>0$, $k\in\mathbb{N}$ we set

$$\psi_{a,k}(y) = \frac{k(y-a)^+}{1+k(y-a)^+} - \frac{k(y+a)^-}{1+k(y+a)^-},\quad y\in\mathbb{R}.$$

Since $\psi:=(1/k)\psi_{a,k}$ satisfies the assumptions of Lemma 2.1, $\psi_{a,k}(u_n)\in D_e(\mathcal{E})$ and

$$\mathcal{E}(u_n,\psi_{a,k}(u_n))\ge 0.$$

Let $\tilde{u}_n$ be a quasi-continuous m-version of u_n. Then $\psi_{a,k}(\tilde{u}_n)$ is a quasi-continuous m-version of $\psi_{a,k}(u_n)$. By Proposition 3.3,

$$\mathcal{E}(u_n,\psi_{a,k}(u_n))=\langle f_n\cdot m,\psi_{a,k}(u_n)\rangle+\langle\mu_n,\psi_{a,k}(\tilde{u}_n)\rangle.$$

Hence

$$\begin{aligned}-\int_E f_n(x)\psi_{a,k}(u_n(x))\,m(dx)&\le\int_E\psi_{a,k}(\tilde{u}_n(x))\,\mu_n(dx)\\&\le\int_{\{|\tilde{u}_n|>a\}}|\mu|(dx).\end{aligned}\tag{3.4}$$

By Proposition 3.3(v), $\tilde{u}_n\to\tilde{u}$ q.e. Therefore letting $n\to\infty$ in (3.4) and using the dominated convergence theorem we obtain

$$-\int_E f(x,u(x))\psi_{a,k}(u(x))\,m(dx)\le\int_{\{|\tilde{u}|>a\}}|\mu|(dx).$$

By (1.2) and the definition of $\psi_{a,k}$, $|f(\cdot,u)\psi_{a,k}(u)|=-f(\cdot,u)\psi_{a,k}(u)$. Hence

$$\int_E|f(x,u(x))||\psi_{a,k}(u(x))|\,m(dx)\le\int_{\{|\tilde{u}|>a\}}|\mu|(dx).$$

Letting $k\to\infty$ in the above inequality yields part (i) of the lemma. To get (ii), we observe that $v=u_1-u_2$ is a solution to the problem

$$-Lv=g(\cdot,v)+\mu_1-\mu_2$$

with $g(x,y)=f(x,y+u_2(x))-f(x,u_2(x))$. Since f satisfies (1.6), g satisfies (1.2). Therefore the desired inequality follows from part (i). □

Note that from Lemma 3.6(i) with $a=0$ the following absorption estimate follows:

$$\|f(\cdot,u)\|_{L^1(E;m)}\le\|\mu\|_{TV}.\tag{3.5}$$

Corollary 3.7. *If f satisfies (1.6), then there exists at most one solution to (1.1).*

Proof. Let u_1, u_2 be solutions of (1.1), and let $v=u_1-u_2$. By Lemma 3.6(ii), v is a solution of the problem $-Lv=0$. Hence $v=0$ m-a.e. by Proposition 3.3(ii). □

Proposition 3.8. *If u is a solution of (1.1) with f satisfying (1.2), then for every $k\ge0$, $T_k(u)\in D_e(\mathcal{E})$ and*

$$\mathcal{E}(T_k(u),T_k(u))\le 2k\|\mu\|_{TV}.\tag{3.6}$$

Proof. By Proposition 3.3(ii) and Lemma 2.3, $T_k(u)\in D_e(\mathcal{E})$ and

$$\mathcal{E}(T_k(u),T_k(u))\le k(\|f(\cdot,u)\|_{L^1(E;m)}+\|\mu\|_{TV}),$$

which when combined with (3.5) yields (3.6). □

Lemma 3.9. *Let $\mu\in\mathcal{M}_b$. If $|\mu|$ charges no set of capacity $\mathrm{Cap}_{(0)}$ zero, then for every $\varepsilon>0$ there exists $\delta>0$ such that for any Borel subset B of E, if $\mathrm{Cap}_{(0)}(B)\le\delta$, then $|\mu|(B)\le\varepsilon$.*

Proof. By the 0-order version of [15, Lemma 2.1.2] (see the remarks following [15, (2.1.14)]) and [15, Theorem A.1.2], $\mathrm{Cap}_{(0)}$ is a countably subadditive set function. Therefore the desired result follows from [31, Proposition 14.7]. □

Theorem 3.10. *Assume* (1.7). *If* f *satisfies* (1.2) *and* (1.5), *then there exists a solution of* (1.1). *Moreover, for every* $k \geq 0$, $T_k(u) \in D_e(\mathcal{E})$ *and* (3.6) *is satisfied.*

Proof. We divide the proof into two steps.

Step 1 We first assume that $\mu^+\mu^- \in \mathcal{M}_{0,b} \cap V'$. For a positive $g \in L^2(E;m) \cap V'$ set $f_n = \frac{ng}{1+ng} T_n(f)$, $n \in \mathbb{N}$. Under the hypothesis (1.5) the operator $A_n : V \to V'$ defined as $A_n(u) = -Lu - f_n(u)$ is pseudomonotone (see, e.g., [32, Section 2.1] for the definition). Indeed, it is clear that A_n maps bounded sets of V into bounded sets of V'. Next, suppose that $u_k \to u$ weakly in V. Then for any $v \in V$,

$$\begin{aligned}\liminf_{k\to\infty}\langle\langle -Lu_k, u_k - v\rangle\rangle &= \liminf_{k\to\infty}\mathcal{E}(u_k, u_k - v) = \liminf_{k\to\infty}\mathcal{E}(u_k, u_k) - \mathcal{E}(u_k, v)\\ &\geq \mathcal{E}(u,u) - \mathcal{E}(u,v) = \langle\langle -Lu, u - v\rangle\rangle.\end{aligned}$$

Furthermore, by (1.7), we can assume that $u_k \to u$ m-a.e. Consequently, we can assume that $f_n(\cdot, u_k) \to f_n(\cdot, u)$ in V'. Therefore, for any $v \in V$,

$$\begin{aligned}\lim_{k\to\infty}\langle\langle -f_n(\cdot,u_k), u_k - v\rangle\rangle &= \lim_{k\to\infty}(-f_n(\cdot,u_k), u_k - v)\\ &= \liminf_{k\to\infty}\int_E |f_n(x,u_k(x))u_k(x)|\, m(dx)\\ &\quad + \lim_{k\to\infty}\langle\langle f_n(\cdot,u_k), v\rangle\rangle\\ &\geq \int_E |f_n(x,u(x))u(x)|\, m(dx) + \langle\langle f_n(\cdot,u), v\rangle\rangle\\ &= (-f_n(\cdot,u), u - v) = \langle\langle -f_n(\cdot,u), u - v\rangle\rangle.\end{aligned}$$

Accordingly, A_n is pseudomonotone. Since by (1.2), for $u \in V$ we have $\langle\langle A_n u, u\rangle\rangle = \mathcal{E}(u,u) - (f_n(\cdot,u), u) \geq \mathcal{E}(u,u)$, the operator A_n is also coercive. Therefore A_n is surjective by standard result in the theory of pseudomonotone mappings (see, e.g., [32, Theorem 2.6]). Thus, there exists a weak solution $u_n \in D_e(\mathcal{E})$ of the equation

$$-Lu_n = f_n(\cdot, u_n) + \mu, \tag{3.7}$$

i.e. for any $v \in D_e(\mathcal{E})$,

$$\mathcal{E}(u_n, v) = \int_E f_n(x, u_n(x))v(x)\, m(dx) + \langle\langle \mu, v\rangle\rangle. \tag{3.8}$$

Taking u_n as a test function in (3.8) we get

$$\begin{aligned}\mathcal{E}(u_n,u_n) - \int_E f_n(x,u_n(x))u_n(x)\, m(dx) &= \langle\langle \mu, u_n\rangle\rangle\\ &\leq \|\mu\|_{V'}\mathcal{E}(u_n,u_n)^{1/2}.\end{aligned}\tag{3.9}$$

By (1.2) and (3.9),

$$\mathcal{E}(u_n,u_n)+\int_E |f_n(x,u_n(x))u_n(x)|\,m(dx)\le \|\mu\|^2_{V'},\quad n\ge 1. \tag{3.10}$$

By (1.7) and (3.10) there is $u\in D_e(\mathcal{E})$ and a subsequence (still denoted by n) such that $u_n\to u$ m-a.e. and weakly in V. Then, by the definition of f_n, $f_n(\cdot,u_n)\to f(\cdot,u)$ m-a.e. By (3.10), for any Borel subset B of E and $a>0$ we have

$$\begin{aligned}\int_B |f(x,u_n(x))|\,m(dx) &\le a^{-1}\int_{B\cap\{|u_n|>a\}} |f(x,u_n(x))u_n(x)|\,m(dx)\\ &\quad+\int_{B\cap\{|u_n|\le a\}} |f(x,u_n(x))|\,m(dx)\\ &\le a^{-1}\|\mu\|^2_{V'}+\int_B F_a(x)\,m(dx).\end{aligned}$$

From the above inequality and (1.5) we conclude that the sequence $\{f_n(\cdot,u_n)\}$ is equi-integrable and tight. Hence $f_n(\cdot,u_n)\to f(\cdot,u)$ in $L^1(E;m)$ by Vitali's convergence theorem (see, e.g., [14, Theorem 2.24]). Therefore letting $n\to\infty$ in (3.8) we see that

$$\mathcal{E}(u,v)=\int_E f(x,u(x))v(x)\,m(dx)+\langle\langle\mu,v\rangle\rangle \tag{3.11}$$

for any bounded $v\in D_e(\mathcal{E})$. Let $\eta\in L^1(E;m)$ be such that $\|G|\eta|\|_\infty<\infty$, and let $\{F_n\}$ be a generalized nest such that $\eta_n=\mathbf{1}_{F_n}\eta\in V'$. Then $G\eta_n$ is bounded and $G\eta_n\in V$. Therefore taking $v=G\eta_n$ as a test function in (3.11) we get

$$\int_E u\eta_n\,dm=\int_E f(x,u(x))G\eta_n(x)\,m(dx)+\int_E \widetilde{G\eta_n}(x)\,\mu(dx).$$

By Lemma 2.5, $\widetilde{G\eta_n}\to\widetilde{G\eta}$ as $n\to\infty$. Therefore letting $n\to\infty$ in the above equation we obtain (3.1). Thus u is a solution of (1.1). *Step 2* We now show how to dispense with the assumption that $\mu^+,\mu^-\in\mathcal{M}_{0,b}\cap V'$. By the 0-order version of [15, Theorem 2.2.4] (see the beginning of the proof of [15, Theorem 2.4.2(ii)]), there exists a generalized nest $\{F_n\}$ such that $\mu_n^{(+)}=\mathbf{1}_{F_n}\cdot\mu^+$, $\mu_n^{(-)}=\mathbf{1}_{F_n}\cdot\mu^-\in\mathcal{M}_{0,b}\cap V'$. Set $\mu_n=\mu_n^{(+)}-\mu_n^{(-)}$. By *Step 1*, there exists a solution $u_n\in D_e(\mathcal{E})$ of the equation

$$-Lu_n=f(\cdot,u_n)+\mu_n. \tag{3.12}$$

In particular,

$$\int_E u_n(x)\eta(x)\,m(dx)=\int_E f(x,u_n(x))G\eta(x)\,m(dx)+\int_E\widetilde{G\eta}(x)\,\mu(dx) \tag{3.13}$$

for any $\eta\in L^1(E;m)$ such that $G|\eta|$ is bounded. By Lemma 3.6, for any Borel subset B of E and $a>0$ we have

$$\int_B |f(x,u_n(x))|\,m(dx)=\int_{B\cap\{|u_n|\le a\}} |f(x,u_n(x))|\,m(dx)$$

$$+\int_{B\cap\{|u_n|>a\}}|f(x,u_n(x))|\,m(dx)$$
$$\le\int_B F_a(x)\,m(dx)+\|\mathbf{1}_{\{|u_n|>a\}}\cdot\mu\|_{TV}. \tag{3.14}$$

By Proposition 3.8,

$$\mathcal{E}(T_k(u_n),T_k(u_n))\le 2k\|\mu_n\|_{TV}, \tag{3.15}$$

whereas by the 0-order version of [15, (2.1.10)] and (3.19),

$$\mathrm{Cap}_{(0)}(\{|T_k(u_n)|>a\})\le a^{-2}\mathcal{E}(T_k(u_n),T_k(u_n)).$$

If $k>a$, then $\{|u_n|>a\}=\{|T_k(u_n)|>a\}$, so for any $k>a$,

$$\begin{aligned}\mathrm{Cap}_{(0)}(\{|u_n|>a\})&=\mathrm{Cap}_{(0)}(\{|T_k(u_n)|>a\})\\&\le a^{-2}\mathcal{E}(T_k(u_n),T_k(u_n))\le 2a^{-2}k\|\mu_n\|_{TV}.\end{aligned}$$

Hence

$$\mathrm{Cap}_{(0)}(\{|u_n|>a\})\le 2a^{-1}\|\mu\|_{TV}. \tag{3.16}$$

Let $\varepsilon>0$. As $|\mu|\in\mathcal{M}_{0,b}$, by Lemma 3.9 there exists $\delta>0$ such that $|\mu|(\{|u_n|>a\})\le\varepsilon/2$ if $\mathrm{Cap}_{(0)}(\{|u_n|>a\})\le\delta$. Hence, by (3.16), $|\mu|(\{|u_n|>a\})\le\varepsilon/2$ if $a=\delta^{-1}\|\mu\|_{TV}$. By (1.5) with $a=\delta^{-1}\|\mu\|_{TV}$, there is $\gamma>0$ such that $\int_B F_a(x)\,m(dx)<\varepsilon/2$ if $m(B)\le\gamma$. From this and (3.14) it follows that if $m(B)\le\gamma$, then $\int_B|f(x,u_n(x))|\,m(dx)\le\varepsilon$. Furthermore, by (1.5) and σ-finitness of m, there exists a Borel set $E_0\subset E$ such that $m(E_0)<\infty$ and $\int_{E\setminus E_0}F_a(x)\,m(dx)<\varepsilon/2$. Therefore taking $B=E\setminus E_0$ in (3.14) we get $\int_{E\setminus E_0}|f(x,u_n(x))|\,m(dx)\le\varepsilon$. This shows that the sequence $\{f(\cdot,u_n)\}$ is equi-integrable and tight. On the other hand, by (1.7) and (3.15), for each $k>0$ the sequence $\{T_k(u_n)\}_n$ is, up to a subsequence, convergent m-a.e., so using the diagonal argument, one can find a subsequence, still denoted by (n), such that $\{u_n\}$ converges m-a.e. to some u. Hence $f(\cdot,u_n)\to f(\cdot,u)$ m-a.e. Consequently, by Vitali's theorem, $f(\cdot,u_n)\to f(\cdot,u)$ in $L^1(E;m)$. Let $k>0$, and let g be a strictly positive function such that $\|Gg\|_\infty<\infty$ and $g\in L^1(E;m)$. Taking $\eta=g\mathbf{1}_{\{u_n\ge k\}}$ and $\eta=-g\mathbf{1}_{\{u_n\le -k\}}$ as test functions in (3.13) we obtain

$$\begin{aligned}\int_{\{|u_n|\ge k\}}|u_n(x)|g(x)\,m(dx)&\le-\int_E|f_n(x,u_n(x))|G(g\mathbf{1}_{\{|u_n|\ge k\}})(x)\,m(dx)\\&\quad+\int_E G(\widetilde{g\mathbf{1}_{\{|u_n|\ge k\}}})(x)\,\mu(dx).\end{aligned} \tag{3.17}$$

We already know that the sequence $\{f(\cdot,u_n)\}$ is equi-integrable and tight. Furthermore, the functions $G(\widetilde{g\mathbf{1}_{\{|u_n|\ge k\}}})$ is bounded q.e., and by Lemma 2.5, $G(\widetilde{g\mathbf{1}_{\{|u_n|\ge k\}}})\searrow 0$ q.e. as $k\to\infty$. Therefore from (3.17) it follows that the sequence $\{u_n\}$ is equi-integrable with respect to the finite measure $\nu=g\cdot m$. Since u_n is a solution of (3.12), for any $\eta\in L^1(E;m)$ such that $\|G|\eta|\|_\infty<\infty$ and any $k\ge 0$ we have

$$\begin{aligned}\int_E u_n(x)\eta(x)g_k(x)\,m(dx) &= \int_E f_n(x,u_n(x))G(\eta g_k)(x)\,m(dx)\\ &\quad+\int_E \mathbf{1}_{F_n}(x)\widetilde{G(\eta g_k)}(x)\,\mu(dx),\end{aligned} \tag{3.18}$$

where $g_k=\frac{kg}{1+kg}$. By what has already been proved, letting $n\to\infty$ in (3.18) yields

$$\begin{aligned}\int_E u(x)\eta(x)g_k(x)\,m(dx) &= \int_E f(x,u(x))G(\eta g_k)(x)\,m(dx)\\ &\quad+\int_E \mathbf{1}_{F}(x)\widetilde{G(\eta g_k)}(x)\,\mu(dx)\end{aligned}$$

with $F=\bigcup_{n=1}^\infty F_n$. Since $\mu(E\setminus F)=0$ by (2.6), letting $k\to\infty$ and using Lemma 2.5 shows that u is a solution to (1.1). □

Remark 3.11. (i) Let $\bar\mu=f(\cdot,0)\cdot m+\mu$, $\bar f(x,y)=f(x,y)-f(x,0)$. Then $\bar\mu\in\mathcal{M}_{0,b}$ and if f satisfies (1.5) and (1.6), then $\bar f$ satisfies (1.2) and (1.5). Furthermore, u is a solution of the problem $-Lu=\bar f(x,u)+\bar\mu$ if and only if it is a solution of (1.1). Therefore under (1.5) and (1.6) there exists a solution of (1.1).

(ii) If (1.5) and (1.6) are satisfied, then *Step 2* of the proof of Theorem 3.10 can be shortened. Indeed, by (1.6) and Lemma 3.6(ii),

$$\|f(\cdot,u_n)-f(\cdot,u_k)\|_{L^1(E;m)}\le\|\mu_n-\mu_k\|_{TV}.$$

Since

$$\begin{aligned}\|\mu_n-\mu\|_{TV}&\le\|\mu^+-\mathbf{1}_{F_n}\cdot\mu^+\|_{TV}+\|\mu^--\mathbf{1}_{F_n}\cdot\mu^-\|_{TV}\\ &=\mu^+(E\setminus F_n)+\mu^-(E\setminus F_n),\end{aligned}$$

we have

$$\limsup_{n\to\infty}\|\mu_n-\mu\|_{TV}\le\mu^+(E\setminus F)+\mu^-(E\setminus F)=0.$$

By the above, $\{f(\cdot,u_n)\}$ is convergent in $L^1(E;m)$. The rest of the proof runs as the proof of Theorem 3.10 (see the reasoning following the statement that $\{f(\cdot,u_n)\}$ is equi-integrable).

If (1.8) is satisfied, then the following Poincaré-type inequality holds true: there exists $c>0$ such that

$$\|u\|_{L^2(E;m)}\le c\mathcal{E}(u,u)^{1/2},\quad u\in D(\mathcal{E}) \tag{3.19}$$

(see [18, Corollary 2.5]). Hence, under (1.8), $D_e(\mathcal{E})=D(\mathcal{E})$ and the norms determined by $\mathcal{E}$ and $\mathcal{E}_1$ are equivalent. It follows in particular that under the assumptions of Theorem 3.10 the solutions of (1.3) belong to $D(\mathcal{E})$.

In general, solutions of (1.3) are not even locally integrable (see [22, Example 5.7]). Below we shall see that a simple condition guaranteeing their integrability is

$$\|G1\|_\infty<\infty.$$

This condition is sometimes expressed by saying that E is Green-bounded (see, e.g., [6,10]; note that in the case where problem (1.3) (resp. (1.4)) is considered, G is the Green function for Δ (resp. $\Delta^{\alpha/2}$) on D. The Green-bounded domain need not be bounded. For instance, if $L=\Delta$, then the infinite strip $\{(x,y)\in\mathbb{R}^2:|x|<a\}$ $(a>0)$ in $\mathbb{R}^2$ is Green-bounded (see [10, p. 39]).

Lemma 3.12. *If* (3.19) *is satisfied, then E is Green-bounded.*

Proof. For the constant c from (3.19) we set $\mathcal{E}^c(u,v)=\mathcal{E}(u,v)-\frac{1}{2c^2}(u,v)$, $u,v\in D(\mathcal{E})$. Then $(\mathcal{E}^c,D(\mathcal{E}))$ is a regular symmetric Dirichlet form on $L^2(E;m)$. Obviously,

$$\mathcal{E}(u,v)=\mathcal{E}^c(u,v)+\frac{1}{2c^2}(u,v),\quad u,v\in D(\mathcal{E}). \tag{3.20}$$

Let $(G^c_\alpha)_{\alpha>0}$ denote the resolvent associated with $(\mathcal{E}^c,D(\mathcal{E}))$. From (3.20) it follows that $G=G^c_{(2c^2)^{-1}}$. Hence $G1=G^c_{(2c^2)^{-1}}1\le 2c^2$ since $(\alpha G^c_\alpha)_{\alpha>0}$ is Markovian. □

Proposition 3.13. *Assume that E is Green-bounded. If u is a solution to* (1.1), *then $u\in L^1(E;m)$.*

Proof. To see this it is enough to consider an increasing sequence of compact sets $\{F_n\}$ such that $\bigcup_{n=1}^\infty F_n=E$, take $\eta_n=\mathbf{1}_{F_n}\mathrm{sign}(u)$ as test functions in (3.1), and use (3.5) and Fatou's lemma. □

4. Applications

In this section we provide some examples of local and nonlocal symmetric transient regular Dirichlet forms satisfying condition (1.7). Before proceeding, we make some general comments on conditions (1.8) and (1.9).

Since (1.8) implies (3.19), it is clear that (1.8) implies (1.7). That the absolute continuity condition (1.9) [or, equivalently, condition (1.10)] implies (1.7) follows from [20, Propositions 2.4 and 2.11]. We include a direct proof of this fact for completeness of exposition.

Proposition 4.1. *Condition* (1.9) *implies* (1.7).

Proof. Assume that $\{u_n\}\subset D_e(\mathcal{E})$ and $\sup_{n\ge1}\mathcal{E}(u_n,u_n)<\infty$. Choose $v\in D(\mathcal{E})$ such that $\|v\|_\infty<\infty$ and $v>0$ m-a.e., and for $k>0$ set $w^k_n=v\cdot T_ku_n$. By [15, Corollary 1.5.1], $w^k_n\in D_e(\mathcal{E})$ and

$$\mathcal{E}(w^k_n,w^k_n)\le\|v\|_\infty\mathcal{E}(u_n,u_n)+k\mathcal{E}(v,v). \tag{4.1}$$

Clearly

$$(w^k_n-\alpha G_\alpha w^k_n,w^k_n-\alpha G_\alpha w^k_n)=(w^k_n,w^k_n-\alpha G_\alpha w^k_n)+(\alpha G_\alpha w^k_n,\alpha G_\alpha w^k_n-w^k_n).$$

By [15, Lemma 1.3.4], $\alpha(w^k_n,w^k_n-\alpha G_\alpha w^k_n)\le\mathcal{E}(w^k_n,w^k_n)$ for every $\alpha>0$. Moreover,

$$\begin{aligned}(\alpha G_\alpha w^k_n,\alpha G_\alpha w^k_n-w^k_n)&=\alpha(G_\alpha w^k_n,\alpha G_\alpha w^k_n)-\mathcal{E}_\alpha(G_\alpha w^k_n,\alpha G_\alpha w^k_n)\\&=-\mathcal{E}(G_\alpha w^k_n,\alpha G_\alpha w^k_n)\le0.\end{aligned}$$

By the above estimates, $\|\alpha G_\alpha w_n^k - w_n^k\|^2_{L^2(E;m)} \le \alpha^{-1}\mathcal{E}(w_n^k, w_n^k)$ for $\alpha > 0$, which when combined with (4.1) shows that there is a constant $c(k, v)$ depending only on k and v such that

$$\|\alpha G_\alpha w_n^k - w_n^k\|^2_{L^2(E;m)} \le \alpha^{-1}c(k, v). \tag{4.2}$$

Since $\alpha G_\alpha 1 \le 1$, from (2.15) it follows that $\int_E r_\alpha(x, y)\, m(dy) \le \alpha^{-1}$. Hence $r_\alpha(x, \cdot) \in L^1(E; m)$ for every $x \in E$. Furthermore, since $\sup_{n\ge1} \|w_n^k\|_\infty \le k\|v\|_\infty < \infty$, there is a subsequence $(n') \subset (n)$ such that $\{w^k_{n'}\}$ converges weakly* in $L^\infty(E; m)$ to some $w \in L^\infty(E; m)$, i.e. $\int_E w^k_{n'}(x)\eta(x)\, m(dx) \to \int_E w(x)\eta(x)\, m(dx)$ for every $\eta \in L^1(E; m)$. In particular, for every $x \in E$,

$$\begin{aligned} G_\alpha w^k_{n'}(x) &= \int_E r_\alpha(x, y) w^k_{n'}(y)\, m(dy) \\ &\to \int_E r_\alpha(x, y) w(y)\, m(dy) = G_\alpha w(x). \end{aligned} \tag{4.3}$$

Since $|\alpha G_\alpha w_n^k| \le k\alpha G_\alpha|v|$, it follows from (4.3) that the sequence$\{\alpha G_\alpha w^k_{n'}\}$ converges in $L^2(E; m)$ for any fixed $\alpha > 0$, $k > 0$. This and (4.2) imply that there exists a subsequence $(n'') \subset (n')$ such that $\{w^k_{n''}\}$ converges in $L^2(E; m)$. Using the diagonal procedure one can find a further subsequence $(n''') \subset (n'')$ such that $\{u_{n'''}\}$ converges m-a.e. on E. □

Example 4.2. Let $D \subset \mathbb{R}^d$, $d \ge 1$, be a nonempty bounded open set, and let $a_{ij}{:}D \to \mathbb{R}$ be locally integrable functions such that $a_{ij}(x) = a_{ji}(x)$ for $x \in D$, $i, j = 1, \dots, d$, and for some $\lambda > 0$,

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2, \quad x \in D,\, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$

(i) (Dirichlet boundary conditions) The form defined by

$$\mathcal{E}(u, v) = \sum_{i,j=1}^d \int_D \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x) a_{ij}(x)\, dx, \quad u, v \in D(\mathcal{E}) \tag{4.4}$$

with $D(\mathcal{E}) = H^1_0(D)$ is a regular symmetric Dirichlet form on $L^2(D)$ (see, e.g., [15, Section 3.1]). The generator L of $(\mathcal{E}, D(\mathcal{E}))$ is of the form

$$Lu = \sum_{i,j=1}^d \frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big), \quad u \in D(L).$$

The form $(\mathcal{E}, D(\mathcal{E}))$ is transient by Poincaré's inequality, and (1.8) is satisfied by Rellich's theorem. Also note that by classical results (see [1]), assumption (1.10) [and hence (1.9)] is satisfied as well. Therefore Theorem 3.10 applies to the Dirichlet problem

$$-Lu = f(\cdot, u) + \mu \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{4.5}$$

Note that we impose no regularity assumption on the boundary ∂D of D. Note also that by Poincaré's inequality, $D_e(\mathcal{E}) = H^1_0(D)$. Consequently, $T_k(u) \in H^1_0(D)$ for every $k > 0$. Furthermore, since D is Green-bounded (see, e.g., [10, Theorem 1.17]), $u \in L^1(D; dx)$ by Proposition 3.13.

(ii) (Neumann boundary conditions) Assume additionally that ∂D is Lipschitz. Consider the form $\mathcal{E}$ defined by (4.4), but with domain $H^1(D)$. Then $(\mathcal{E}, H^1(D))$ is a regular symmetric Dirichlet form on $L^2(\bar{D}; dx)$ with $\bar{D} = D \cup \partial D$ (see [15, Example 4.5.3]), and clearly so is $(\mathcal{E}_\lambda, H^1(D))$ with $\lambda \geq 0$. Moreover, if $\lambda > 0$, then $(\mathcal{E}_\lambda, H^1(D))$ is transient because D is Green-bounded (see Lemma 3.12). The generator L^λ of $(\mathcal{E}_\lambda, H^1(D))$ is equal to $L - \lambda$, where L is the generator of $(\mathcal{E}, H^1(D))$. By Rellich's theorem, $H^1(D) \hookrightarrow L^2(D; dx)$ is compact, so the results of the paper apply to Eq. (1.1) with L replaced by the operator $L - \lambda$ defined above. A solution u to such equation can be viewed as a solution to the Neumann problem

$$-Lu = -\lambda u + f(\cdot, u) + \mu \quad \text{in } D, \qquad \frac{\partial u}{\partial (a \cdot \mathbf{n})} = 0 \quad \text{on } \partial D,$$

where $\mathbf{n}$ denotes the unit outward normal to ∂D.

Example 4.3. Assume that f satisfies (1.2) and (1.5). Let $\psi : \mathbb{R}^d \to [0, \infty)$ be a continuous negative definite function in the sense of Schoenberg (see [17, Chapter 3] for the definition). Denote by $H^{\psi,1}(\mathbb{R}^d)$ the space

$$H^{\psi,1}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \|u\|_{\psi,1} < \infty\},$$

where

$$\|u\|^2_{\psi,1} = \int_{\mathbb{R}^d} (1 + \psi(x)) |\hat{u}(x)|^2 \, dx$$

and $\hat{u}$ stands for the Fourier transform of u. It is known (see [15, Example 1.4.1] or [17, Example 4.1.28]) that $(\mathcal{E}, D(\mathcal{E}))$ defined as

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \hat{u}(x) \overline{\hat{v}(x)} \psi(x) \, dx, \quad u, v \in D(\mathcal{E}) := H^{\psi,1}(\mathbb{R}^d) \tag{4.6}$$

is a symmetric regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$. By [15, Example 1.5.2], it is transient if and only if

$$\frac{1}{\psi} \in L^1_{loc}(\mathbb{R}^d). \tag{4.7}$$

(i) Let ν_t, $t > 0$, be a probability measure on $\mathbb{R}^d$ such that $\hat{\nu}_t(x) = e^{-t\psi(x)}$, $x \in \mathbb{R}^d$. Then the semigroup $(P_t)_{t>0}$ associated with $\mathcal{E}$ has the form $P_t f(x) = \int_{\mathbb{R}^d} f(x+y) \, \nu_t(dy)$ for $f \in L^2(\mathbb{R}^d; dx) \cap \mathcal{B}_b(E)$ (see [15, Example 1.4.1]). It follows in particular that if ν_t are absolutely continuous with respect to the Lebesgue measure, then (1.10) is satisfied. For instance, this is the case when $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}^d$, with $\alpha \in (0, 2]$ (see [15, Example 1.4.1]). For such ψ, the operator corresponding to $\mathcal{E}$ is the fractional Laplacian $\Delta^{\alpha/2}$. If $\alpha < d$, then (4.7) is satisfied, so the form $\mathcal{E}$ is transient. Therefore Theorem 3.10 applies to the equation

$$-\Delta^{\alpha/2} = f(\cdot, u) + \mu \quad \text{in } \mathbb{R}^d$$

with $\alpha \in (0, 2 \wedge d)$. If f satisfies (1.2) and u is a solution to the above equation, then $T_k(u) \in D_e(\mathbb{R}^d)$ for any $k > 0$. For the characterisation of $D_e(\mathcal{E})$ see [15, Example 1.5.2]. Finally, let us note that some general conditions ensuring (1.10) are found in [33, Section 27].

(ii) Let $D \subset \mathbb{R}^d$ be a nonempty bounded open set, and let $(\mathcal{E}^D, D(\mathcal{E}^D))$ denote the part of $(\mathcal{E}, D(\mathcal{E}))$ on D, i.e.,

$$\begin{cases} D(\mathcal{E}^D) = \{u \in D(\mathcal{E}) : \tilde{u} = 0 \text{ q.e. on } \mathbb{R}^d \setminus D\}, \\ \mathcal{E}^D(u,v) = \mathcal{E}(u,v), \quad u,v \in D(\mathcal{E}^D) \end{cases} \tag{4.8}$$

(here $\tilde{u}$ denotes a quasi-continuous version of u). By [15, Theorem 4.4.3], $(\mathcal{E}^D, D(\mathcal{E}^D))$ is a symmetric regular Dirichlet form on $L^2(D; dx)$ and $D(\mathcal{E}^D) = H_0^{\psi,1}(D)$, where $H_0^{\psi,1}(D)$ denotes the closure of $C_c^\infty(D)$ in $H^{\psi,1}(\mathbb{R}^d)$. If (4.7) is satisfied, then the form $(\mathcal{E}^D, D(\mathcal{E}^D))$ is transient by [15, Theorem 4.4.4]. Let L denote the generator of $(\mathcal{E}, D(\mathcal{E}))$ and L^D denote the generator of $(\mathcal{E}^D, D(\mathcal{E}^D))$. By virtue of (4.8), the solution u of (1.1) with $E = D$ and operator L^D can be interpreted as a solution of the Dirichlet problem

$$-Lu = f(\cdot, u) + \mu \quad \text{in } D, \qquad u = 0 \quad \text{on } \mathbb{R}^d \setminus D. \tag{4.9}$$

By [17, Remark 3.10.6], the embedding of $V := H_0^{\psi,1}(D)$ (equipped with the norm $\|\cdot\|_{\psi,1}$) into $L^2(D; dx)$ is compact if and only if

$$\lim_{|\xi| \to \infty} \psi(\xi) = \infty. \tag{4.10}$$

Therefore, if (4.7) and (4.10) are satisfied, then by Theorem 3.10 there exists a solution u to (4.9). Since (1.8) implies (3.19), $D_e(\mathcal{E}^D) = D(\mathcal{E}^D)$. Consequently, $T_k(u) \in H_0^{\psi,1}(D)$ for $k > 0$.

For instance, (4.7) and (4.10) are satisfied for ψ defined as $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}^d$, with $\alpha \in (0, 2 \wedge d)$. Since then $L = \Delta^{\alpha/2}$, Eq. (1.1) with L^D can be interpreted as (1.4). Note also that, because D is bounded, it is Green-bounded (see, e.g., [6, (2.4)]). Therefore, by Proposition 3.13, if u is a solution to (1.4), then $u \in L^1(D; dx)$. Other examples of ψ satisfying (4.7) and (4.10) are found in [17, Chapter 3].

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Received: 22 December 2017.
Accepted: 7 July 2018.